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FITTED OPERATOR FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITION

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Abstract. This study presents a fitted operator numerical method for solving singularly perturbed boundary value problems with integral boundary condition. The stability and parameter uniform convergence of the proposed method are proved. To validate the applicability of the scheme, a model problem is considered for numerical experimentation and solved for different values of the perturbation parameter, ε and mesh size, h. The numerical results are tabulated in terms of maximum absolute errors and rate of convergence and it is observed that the present method is more accurate and ε-uniformly convergent for $h \geq \varepsilon$ where the classical numerical methods fails to give good result and it also improves the results of the methods existing in the literature.

1. Introduction

Boundary value problems with integral boundary conditions are an important class of problems which arise in various fields of applications such as electro-chemistry, thermo-elasticity, heat conduction, underground water flow and population dynamics, see, for example [12, 17, 19]. In fact, boundary value problems involving integral boundary conditions have received considerable attention in recent years [7, 9, 11] and [13]. For a discussion of existence and uniqueness results and for applications of problems with integral boundary conditions one can refer, [4–8], [10, 11] and the references therein. In [1, 2, 9, 11, 13, 16] it has been considered some approximating or numerical treatment aspects of this kind of problems. However, the methods or algorithms developed so far mainly concerned with the regular cases (i.e., when

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the boundary layers are absent). Boundary value problems with integral boundary conditions in which the leading derivative term is multiplied by a small parameter ε are called singularly perturbed problems with integral boundary conditions. The solutions of such types of problems manifest boundary layer phenomena where the solution changed abruptly. As a result, numerical analysis of singular perturbation cases has been far from trivial because of the boundary layer behavior of the solution. The solutions of the problems with boundary layer undergo rapid changes within very thin layers near the boundary or inside the problem domain [3], [13–15], [18] and hence classical numerical methods for solving such problems are unstable and fail to give good results when the perturbation parameter is small (i.e., for $h \geq \varepsilon$) [18]. Therefore, it is important to develop a numerical method that gives good results for small values of the perturbation parameter where others fails to give good result and convergent independent of the values of the perturbation parameter and the mesh sizes. Hence, this paper proposed a fitted operator numerical method that is simple, stable and uniformly convergent.

2. Statement of the Problem

Consider the following singularly perturbed problem with integral boundary condition

$$\varepsilon y''(x) + a(x)y'(x) = f(x), \quad 0 < x < l, \tag{2.1}$$

with the given conditions

$$y'(0) = \frac{\mu_0}{\varepsilon}, \tag{2.2}$$

$$\int_0^l b(x)y(x)dx = \mu_1, \tag{2.3}$$

where $0 < \varepsilon \ll 1$ is a positive parameter, $0 < a \leq a(x)$, $f(x)$, $b(x)$ are sufficiently smooth functions in the $[0, l]$ and μ_i $(i = 0, 1)$ are given constants. The function $y(x)$ has in general a boundary layer of thickness $O(\varepsilon)$ near $x = 0$.

In this paper, we analyze a fitted finite-difference scheme on uniform mesh for the numerical solution of the problem (2.1)–(2.3). Uniform convergence is proved in the discrete maximum norm. Finally, we formulate the algorithm for solving the discrete problem and give the illustrative numerical results.

3. Properties of Continuous Solution

The differential operator for the problem under consideration is given by

$$L_\varepsilon \equiv \varepsilon \frac{d^2}{dx^2} + \frac{d}{dx},$$

and it satisfies the following minimum principle for boundary value problems (BVPs). The following lemmas [15] are necessary for the existence and uniqueness of the solution and for the problem to be well-posed.

Lemma 3.1 (Continuous minimum principle)**.** *Assume that $v(x)$ is any sufficiently smooth function satisfying $v(0) \geq 0$ and $v(l) \geq 0$. Then $Lv(x) \leq 0$, for all $x \in \Omega = (0, l)$ implies that $v(x) > 0$, for all $x \in \Omega = [0, l]$.*

Proof. Let x^* be such that $v(x^*) = \min_{x\in[0,l]} v(x)$ and assume that $v(x^*) < 0$. Clearly $x^* \notin \{0, l\}$. Therefore, $v'(x^*) = 0$ and $v''(x^*) \geq 0$. Moreover, $Lv(x^*) = \varepsilon v''(x^*) + a(x^*)v'(x^*) \geq 0$, which is a contradiction. It follows that $v(x^*) > 0$ and thus $v(x) \geq 0$, for all $x \in [0, l]$. □

The uniqueness of the solution is implied by this minimum principle. Its existence follows trivially (as for linear problems, the uniqueness of the solution implies its existence). This principle is now applied to prove that the solution of (2.1)–(2.3) is bounded.

Lemma 3.2. *If y is the solution of the boundary value problem* (2.1)–(2.3) *and $y \in C^2(\Omega)$ then*

$$\|y(x)\| \leq \|f\| + \max\{|y_0|, |y_l|\},$$

where $k = 0, 1, 2, 3$ and $x \in [0, l]$.

Proof. We handle first the case when $k = 0$. Consider the barrier functions defined by

$$\psi^{\pm}(x) = [(l - x)\|f\| + \max\{|y_0|, |y_l|\}] \pm y(x),$$

when $x = 0$, we have

$$\psi^{\pm}(0) = \|f\|l + \max\{|y_0|, |y_l|\} \pm y(0) \geq 0, \quad \text{since} \quad \max\{|y_0|, |y_l|\}] \geq y(0).$$

When $x = l$, we have

$$\psi^{\pm}(l) = (l - l)\|f\| + \max\{|y_0|, |y_l|\} \pm y(l) \geq 0, \quad \text{since} \quad \max\{|y_0|, |y_l|\}] \geq y(l).$$

Now,

$$L_\varepsilon\psi^{\pm}(x) = \varepsilon(\psi^{\pm}(x))'' + (\psi^{\pm}(x))' = \|f\| + \max\{|y_0|, |y_l|\} \pm Ly(x) \leq 0.$$

Applying the minimum principle, we conclude that $\psi^{\pm}(x) \geq 0$, and therefore

$$\|y(x)\| \leq \|f\| + \max\{|y_0|, |y_l|\}. \qquad \square$$

The following lemma shows the bound for the derivatives of the solution.

Lemma 3.3. *Let y_ε be the solution of the continuous problem (P_ε). Then, for $k = 0, 1, 2, 3,$*

$$|y_\varepsilon^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k}\exp\left(\frac{-a}{\varepsilon}x\right)\right), \quad \textit{for all } x \in [0, l].$$

Proof. The homogeneous differential equation of (2.1) is

$$\varepsilon y''(x) + a(x)y'(x) = 0. \tag{3.1}$$

The characteristic equation of (3.1) becomes

$$\varepsilon m^2 + am = 0 \Rightarrow m = 0 \quad \text{or} \quad m = \frac{-a}{\varepsilon}.$$

The asymptotic solution of (3.1) is given by

$$u(x) = A + B\exp\left(\frac{-a}{\varepsilon}x\right),$$

where A and B are arbitrary constants.

To get the k^{th} derivative of the asymptotic solution of the homogeneous part of (3.1)

$$\begin{aligned}
u'(x) =& C\varepsilon^{-1}\exp\left(\frac{-a}{\varepsilon}x\right),\\
u''(x) =& C\varepsilon^{-2}\exp\left(\frac{-a}{\varepsilon}x\right),\\
u'''(x) =& C\varepsilon^{-3}\exp\left(\frac{-a}{\varepsilon}x\right).
\end{aligned}$$

In general, for $k = 1, 2, 3$

$$u^{(k)}(x) = C\varepsilon^{-k}\exp\left(\frac{-a}{\varepsilon}x\right).$$

The reduced problem obtained from (2.1) takes the $a(x)v_0'(x) = f(x)$, where $v_0(0) = y_0$ and has the solution

$$\begin{aligned}
v_0(x) =& y_0 + \int_0^x \frac{f(t)}{a(t)}dt \le |y_0| + \int_0^x \left|\frac{f(t)}{a(t)}\right|dt\\
\le& C + \left|\frac{f(\zeta)}{a(\zeta)}\right| \int_0^x dt \le C + \left|\frac{f(\zeta)}{a(\zeta)}\right| x, \quad x \in (0, l),\\
\le& C,
\end{aligned}$$

from the assumptions on a and f, it is clear that for $k = 0, 1, 2, 3$

$$|v_0^{(k)}(x)| \le C, \quad \text{for all } x \in [0, l].$$

So, from the relation $y_\varepsilon = v_0 + u$ we have $y_\varepsilon^{(k)} = v_0^{(k)} + u^{(k)}$ and from the relation of triangular inequality

$$|y_\varepsilon^{(k)}| \le |v_0^{(k)}| + |u^{(k)}| \le C + C\varepsilon^{-k}\exp\left(\frac{-a}{\varepsilon}x\right) \le C\left(1 + \varepsilon^{-k}\exp\left(\frac{-a}{\varepsilon}x\right)\right).$$

Therefore, it is well accepted that the solution of (2.1) has a boundary layer near $x = 0$ and its derivatives satisfy

$$|y_\varepsilon^{(k)}(x)| \le C\left(1 + \varepsilon^{-k}\exp\left(\frac{-a}{\varepsilon}x\right)\right), \quad \text{for all } x \in [0, l]. \qquad \square$$

4. Formulation of the Method

Consider the homogeneous differential equation with constant coefficient $\varepsilon y''(x) + ay'(x) = 0$ whose solution is given by

$$y(x) = A + B\exp\left(\frac{-a}{\varepsilon}x\right), \tag{4.1}$$

where A and B are constants which will be determined depending on the given conditions. Now, dividing the interval $[0, l]$ into N equal parts with constant mesh length $h = \frac{l}{N}$, we obtain $x_i = x_0 + ih$, for $i = 1, 2, \ldots, N$, where $x_0 = 0$, $x_N = l$.

To demonstrate the procedure, we consider (2.1), at discrete nodes x_i

$$\varepsilon y_i''(x) + a_i(x) y_i'(x) = f_i. \tag{4.2}$$

Approximating (4.2) by central difference approximations, we obtain:

$$\varepsilon \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + a_i \frac{y_{i+1} - y_{i-1}}{2h} = f_i. \tag{4.3}$$

Under the assumption that f_i is bounded, introducing the fitting parameter σ onto the higher order difference approximation of (4.3), multiply both sides by h and evaluating its limit gives

$$\sigma = -\frac{\rho a \lim\limits_{h \longrightarrow 0} (y_{i+1} - y_{i-1})}{2 \lim\limits_{h \longrightarrow 0} (y_{i+1} - 2y_i + y_{i-1})}, \tag{4.4}$$

where $\rho = \frac{h}{\varepsilon}$.

Evaluating (4.1) at each nodal point x_i, we obtain

$$\begin{cases} \lim\limits_{h \to 0} y_i = A + B \exp(-ai\rho), \\ \lim\limits_{h \to 0} y_{i+1} = A + B \exp(-ai\rho) \exp(-a\rho), \\ \lim\limits_{h \to 0} y_{i-1} = A + B \exp(-ai\rho) \exp(a\rho), \end{cases} \tag{4.5}$$

$$\sigma = -\frac{\rho a \lim\limits_{h \to 0} (y_{i+1} - y_{i-1})}{2 \lim\limits_{h \to 0} (y_{i+1} - 2y_i + y_{i-1})} = \frac{a\rho}{2} \coth\left(\frac{a\rho}{2}\right). \tag{4.6}$$

Hence, from (4.3) and (4.6), we get

$$\frac{\varepsilon\sigma}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \frac{a_i}{2h}(y_{i+1} - y_{i-1}) = f_i.$$

This can be rewritten as three term recurrence relation

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \ldots, N-1, \tag{4.7}$$

where

$$\begin{cases} E_i = \dfrac{\varepsilon\sigma}{h^2} - \dfrac{a_i}{2h}, \\ F_i = \dfrac{-2\varepsilon\sigma}{h^2}, \\ G_i = \dfrac{\varepsilon\sigma}{h^2} + \dfrac{a_i}{2h}, \\ H_i = f_i. \end{cases}$$

Since the problem has no Dirichlet boundary conditions, we apply the following two cases, to obtain two equations at each end point.

For $i = 0$, (4.7) becomes

$$E_0 y_{-1} + F_0 y_0 + G_0 y_1 = H_0. \tag{4.8}$$

Here, in (4.8) the term is out of the domain, so that using (2.2) we have

$$y'(0) = \frac{y_1 - y_{-1}}{2h} \Rightarrow y_{-1} = y_1 - 2hy'(0). \tag{4.9}$$

Substituting (4.9) into (4.8) gives

$$F_0 y_0 + (E_0 + G_0) y_1 = H_0 + 2hE_0 y'(0). \tag{4.10}$$

For $i = N$ (Simpson's rule) suppose $b(x)y(x)$ is a function defined on the interval $[0, l]$ and let x_i be a uniform partition of with step length h. The composite Simpson's rule approximates the integral of $b(x)y(x)$ by

(4.11)

$$\int_0^l b(x)y(x)dx = \frac{h}{3}\left[b(0)y(0) + b(l)y(l) + 2\sum_{i=1}^{N-1} b(x_{2i})y(x_{2i}) + 4\sum_{i=1}^{N} b(x_{2i-1})y(x_{2i-1})\right].$$

Using the integral boundary condition given in condition in (2.3), (4.11) can be written as

$$\frac{h}{3}\left[b(0)y(0) + b(l)y(l) + 2\sum_{i=1}^{N-1} b(x_{2i})y(x_{2i}) + 4\sum_{i=1}^{N} b(x_{2i-1})y(x_{2i-1})\right] = \mu_1. \tag{4.12}$$

Therefore, the problem in (2.1) with the given boundary conditions (2.2) and (2.3), can be solved using the schemes in (4.7), (4.10) and (4.12) which gives $N \times N$ system of algebraic equations.

5. Uniform Convergence Analysis

In this section, we need to show the discrete scheme in (4.7), (4.10) and (4.12) satisfy the discrete minimum principle, uniform stability estimates, and uniform convergence.

Lemma 5.1 (Discrete Minimum Principle)**.** *Let v_i be any mesh function that satisfies $v_0 \geq 0$, $v_N \geq 0$ and $L_\varepsilon v_i \leq 0$, $i = 1, 2, 3, \ldots, N-1$, then $v_i \geq 0$ for $i = 0, 1, 2, \ldots, N$.*

Proof. The proof is by contradiction. Let j be such that $v_j = \min_i v_i$ and suppose that $v_j \leq 0$. Clearly, $j \notin \{0, N\}$, $v_{j+1} - v_j \geq 0$ and $v_j - v_{j-1} \leq 0$.

Therefore,

$$\begin{aligned}
L_\varepsilon v_j =& \varepsilon\left[\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}\right] + a_j\left[\frac{v_{j+1} - v_{j-1}}{2h}\right] \\
=& \frac{\varepsilon}{h^2}[v_{j+1} - 2v_j + v_{j-1}] + \frac{a_j}{2h}[v_{j+1} - v_{j-1}] \\
=& \frac{\varepsilon}{h^2}[(v_{j+1} - v_j) - (v_j - v_{j-1})] + \frac{a_j}{2h}[(v_{j+1} - v_j) + (v_j - v_{j-1})] \\
\geq& 0,
\end{aligned}$$

where the strict inequality holds if $v_{j+1} - v_j > 0$. This is a contradiction and therefore $v_j \geq 0$. Since j is arbitrary, we have $v_i \geq 0$, $i = 0, 1, 2, \ldots, N$. From the discrete minimum principle we obtain an ε-uniform stability property for the operator L_ε^N. □

Lemma 5.2 (Uniform stability estimate)**.** *If ϕ_j is any mesh function such that*

$$\phi_0 = \phi_N = 0,$$

then

$$|\phi_j| \le \frac{1}{a} \max_{1\le i\le N-1} |L_\varepsilon^N \phi_i|, \quad j = 0, 1, 2, \ldots, N.$$

Proof. As in [21], we introduce two mesh functions ψ_j^+, ψ_j^- defined by

$$\psi_j^\pm = \left(\frac{1}{a} \max_{1\le i\le N-1} |L_\varepsilon^N \phi_i|\right) \pm \phi_j.$$

It follows that

$$\begin{aligned}\psi^\pm(0) &= \left(\frac{1}{a} \max_{1\le i\le N-1} |L_\varepsilon^N \phi_i|\right) \pm \phi_0\\ &= \frac{1}{a} \max_{1\le i\le N-1} |\varepsilon\delta^2\phi_i + a_i D^+\phi_i| \pm \phi_0\\ &= \frac{1}{a} \max_{1\le i\le N-1} |\varepsilon\delta^2\phi_i + a_i D^+\phi_i|\\ &\ge 0\end{aligned}$$

and

$$\begin{aligned}\psi^\pm(N) &= \left(\frac{1}{a} \max_{1\le i\le N-1} |L_\varepsilon^N \phi_i|\right) \pm \phi_N\\ &= \frac{1}{a} \max_{1\le i\le N-1} |\varepsilon\delta^2\phi_i + a_i D^+\phi_i| \pm \phi_N\\ &= \frac{1}{a} \max_{1\le i\le N-1} |\varepsilon\delta^2\phi_i + a_i D^+\phi_i|\\ &\ge 0,\end{aligned}$$

and for all $j = 1, 2, \ldots, N-1$,

$$L_\varepsilon^N \psi_j^\pm = \left(\frac{1}{a} \max_{1\le i\le N-1} |L_\varepsilon^N \phi_i|\right) \pm L_\varepsilon^N \phi_j \le 0.$$

From discrete minimum principle, if $\psi_0 \ge 0, \psi_N \ge 0$ and $L_\varepsilon^N \psi_j \le 0$, for all $0 < j < N$, then $\psi_j^\pm \ge 0$, $0 \le j \le N$. □

We provide above the discrete operator L_ε^N satisfy the minimum principle. Next we analyze the uniform convergence analysis.

Theorem 5.1 (Uniform Convergence)**.** *The numerical solution y^h of (P_ε^h) and the exact solution y of (P_ε) satisfying ε-uniform error estimates, if there exist a positive integer N_0 and positive numbers C and P, all independent of N and ε, such that for all $N \ge N_0$, $|y^h - y|_\Omega^h \le Ch^2$.*

Proof. Consider the convection-diffusion problem of a linear singularly perturbed two-point boundary value problem of the form

$$\varepsilon y''(x) + a(x)y'(x) = f(x), \quad x \in \Omega = (0, l). \tag{5.1}$$

Now, introducing a variable fitting factor (4.6), $\sigma_i = \frac{a_i\rho_i}{2}\coth(\frac{a_i\rho_i}{2})$, in our scheme, we obtain

$$\frac{\sigma_i}{\rho_i}(y_{i+1} - 2y_i + y_{i-1}) + \left(\frac{y_{i+1} - y_{i-1}}{2}\right) = hf_i, \quad \rho_i = \frac{h}{\varepsilon}. \tag{5.2}$$

Multiply both sides of (5.2) by $2\rho_i$ and rearranging, we get

$$-E_i y_{i-1} + F_i y_i - G_i y_{i+1} = H_i, \tag{5.3}$$

where

$$\begin{cases} E_i = 2\sigma_i - \rho_i, \\ F_i = 4\sigma_i, \\ G_i = 2\sigma_i + \rho_i, \\ H_i = -2\rho_i h f_i. \end{cases}$$

Consider the given problem on two distinct meshes with step sizes h and $k = \frac{h}{2}$ which implies the following relations. For the mesh size h

$$\rho_1 = \frac{h}{\varepsilon}, \quad E_1 = 2\sigma_1 - \rho_1, \quad F_1 = 4\sigma_1, \quad G_1 = 2\sigma_1 + \rho_1, \quad \sigma_1 = \frac{\rho_1}{2}\coth\left(\frac{\rho_1}{2}\right).$$

For the mesh size k,

$$\rho_2 = \frac{k}{\varepsilon} = \frac{\rho_1}{2}, \quad E_2 = 2\sigma_2 - \rho_2, \quad F_2 = 4\sigma_2, \quad G_2 = 2\sigma_2 + \rho_2, \quad \sigma_2 = \frac{\rho_1}{4}\coth\left(\frac{\rho_1}{4}\right).$$

For the operator we have

$$L_\varepsilon^h y_{ih}^h = -Ey_{i-1} + Fy_i - Gy_{i+1} = H_i. \tag{5.4}$$

Now, consider the given problem on two mesh sizes h and k of (5.4) as

$$L_\varepsilon^h y_{ih}^h = -E_1 y_{i-1} + F_1 y_i - G_1 y_{i+1} = H_{ih}, \tag{5.5}$$

$$L_\varepsilon^h y_{2ih}^{\frac{h}{2}} = -E_2 y_{2i-2} + F_2 y_{2i} - G_2 y_{2i+2} = H_{2ih}, \tag{5.6}$$

where

$$\begin{aligned} E_1 &= 2\sigma_1 - \rho_1, \quad F_1 = 4\sigma_1, \quad G_1 = 2\sigma_1 + \rho_1, \\ E_2 &= 2\sigma_2 - \rho_2, \quad F_2 = 4\sigma_2, \quad G_2 = 2\sigma_2 + \rho_2. \end{aligned}$$

Similarly, using the second mesh size k, we have

$$L_\varepsilon^{\frac{h}{2}} y_{2i\frac{h}{2}}^{\frac{h}{2}} = -E_2 y_{2i-1} + F_2 y_{2i} - G_2 y_{2i+1} = H_{2i\frac{h}{2}}, \tag{5.7}$$

$$L_\varepsilon^{\frac{h}{2}} y_{(2i+1)\frac{h}{2}}^{\frac{h}{2}} = -E_2 y_{2i} + F_2 y_{2i+1} - G_2 y_{2i+2} = H_{(2i+1)\frac{h}{2}}, \tag{5.8}$$

$$L_\varepsilon^{\frac{h}{2}} y_{(2i-1)\frac{h}{2}}^{\frac{h}{2}} = -E_2 y_{2i-2} + F_2 y_{2i-1} - G_2 y_{2i} = H_{(2i-1)\frac{h}{2}}. \tag{5.9}$$

To eliminate y_{2i+1} using (5.7) and (5.8), we have

$$-G_2^2 y_{2i+2} - F_2 E_2 y_{2i-1} + (F_2^2 - G_2 E_2) y_{2i} = F_2 H_{2ik} + G_2 H_{(2i+1)k}.$$

Thus, we have the values of y_{2i+2} as

$$y_{2i+2}=\frac{-F_2E_2}{G_2^2}y_{2i-1}+\frac{(F_2^2-G_2E_2)}{G_2^2}y_{2i}-\frac{F_2}{G_2^2}H_{2ik}-\frac{1}{G_2}H_{(2i+1)k}. \tag{5.10}$$

Also,to eliminate y_{2i-1} using (5.7) and (5.9),we have

$$-E_2^2y_{2i-2}+(F_2^2-E_2G_2)y_{2i}-F_2G_2y_{2i+1}=F_2H_{2ik}+E_2H_{(2i-1)K}.$$

Thus, we have the value of y_{2i-2} as

$$y_{2i-2}=\frac{(F_2^2-E_2G_2)}{E_2^2}y_{2i}-\frac{-F_2G_2}{E_2^2}y_{2i+1}-\frac{F_2}{E_2^2}H_{2ik}-\frac{1}{E_2}H_{(2i-1)k}. \tag{5.11}$$

Substituting both values of y_{2i+2} and y_{2i-2} from (5.10) and (5.11) into (5.6)

$$\begin{aligned}L_\varepsilon^h y_{2ih}^{\frac{h}{2}}=&-E_2y_{2i-2}+F_2y_{2i}-G_2y_{2i+2}\\=&-E2\left\{\frac{(F_2^2-E_2G_2)}{E_2^2}y_{2i}-\frac{F_2G_2}{E_2^2}y_{2i+1}-\frac{F_2}{E_2^2}H_{2ik}-\frac{1}{E_2}H_{(2i-1)k}\right\}\\&+F_2y_{2i}-G_2\left\{\frac{-F_2E_2}{G_2^2}y_{2i-1}+\frac{(F_2^2-G_2E_2)}{G_2^2}y_{2i}-\frac{F_2}{G_2^2}H_{2i}-\frac{1}{G_2}H_{(2i+1)}\right\},\end{aligned}$$

(5.12)

$$\begin{aligned}L_\varepsilon^h y_{2ih}^{\frac{h}{2}}=&\left\{F_2-\frac{F_2^2-E_2G_2}{E_2}-\frac{F_2-E_2G_2}{G_2}\right\}y_{2i}+\frac{F_2E_2}{G_2}y_{2i-1}\\&+\frac{F_2G_2}{E_2}y_{2i+1}+\left\{\frac{F_2}{E_2}+\frac{F_2}{G_2}\right\}H_{2i}+H_{2i-1}+H_{2i+1}.\end{aligned}$$

Using (5.5) and (5.12)

$$\begin{aligned}&\left|L_\varepsilon^h y_i^h-L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right|\\=&\left|L_\varepsilon^h(y_i^h-y_{2i}^{\frac{h}{2}})(ih)\right|\\=&\left|-E_1y_{i-1}+F_1y_1-G_1y_{i+1}-\left\{F_2-\frac{F_2^2-E_2G_2}{E_2}-\frac{F_2-E_2G_2}{G_2}\right\}y_{2i}\right.\\&\left.-\frac{F_2E_2}{G_2}y_{2i-1}-\frac{F_2G_2}{E_2}y_{2i+1}+\left\{\frac{F_2}{E_2}+\frac{F_2}{G_2}\right\}H_{2i}-(H_{2i-1}+H_{2i+1})\right|,\end{aligned} \tag{5.13}$$

(5.14)

$$\begin{aligned}\left|L_\varepsilon^h y_i^h-L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right|=&\left|-\left\{F_2-\frac{F_2^2-E_2G_2}{E_2}-\frac{F_2-E_2G_2}{G_2}\right\}y_{2i}-\frac{F_2E_2}{G_2}y_{2i-1}\right.\\&\left.-\frac{F_2G_2}{E_2}y_{2i+1}-\left\{-1+\frac{F_2}{E_2}+\frac{F_2}{G_2}\right\}H_i-\left(H_{i-\frac{h}{2}}+H_{i+\frac{h}{2}}\right)\right|.\end{aligned}$$

Using Taylor series expansion up to third term, we have the following

$$
\begin{cases}
y_{2i+h} = y_{i+\frac{h}{2}} = y_i + \dfrac{h}{2}y_i' + \dfrac{h^2}{8}y_i'' + \dfrac{h^3}{48}y_i''' + O(h^4), \\
H_{i+\frac{h}{2}} = H_i + \dfrac{h}{2}H_i' + \dfrac{h^2}{8}H_i'' + \dfrac{h^3}{48}H_i''' + O(h^4), \\
y_{2i-h} = y_{i-\frac{h}{2}} = y_i - \dfrac{h}{2}y_i' + \dfrac{h^2}{8}y_i'' - \dfrac{h^3}{48}y_i''' + O(h^4), \\
H_{i-\frac{h}{2}} = H_i - \dfrac{h}{2}H_i' + \dfrac{h^2}{8}H_i'' - \dfrac{h^3}{48}H_i''' + O(h^4).
\end{cases} \tag{5.15}
$$

Now, substituting the expanded parts of (5.15) into (5.14), we get

$$
\begin{aligned}
\left|L_\varepsilon^h y_i^h - L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right| = \Bigg| & \left\{ - F_2 + \frac{F_2^2 - E_2G_2}{E_2} + \frac{F_2^2 - E_2G_2}{E_2} - \frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2} \right\} y_i \\
& + \left\{ 1 - \frac{F_2}{E_2} - \frac{F_2}{G_2} - 2 \right\} H_i + \frac{h}{2}\left\{ \frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2} \right\} y_i' \\
& + \frac{h^2}{8}\left\{ - \frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2} \right\} y_i'' \\
& - \frac{h^2}{4}H_i' + \frac{h^3}{48}\left\{ - \frac{F_2E_2}{G_2} - \frac{F_2G_2}{E_2} \right\} y_i''' \Bigg|.
\end{aligned}
$$

For simplicity, let re-write the above equation as

$$
\left|L_\varepsilon^h y_i^h - L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right| \leq |A|y_i + |B|H_i + \frac{h}{2}|D|y_i' + \frac{h^2}{8}|M|y_i'' + K + \frac{h^3}{48}|N|y_i''',
$$

where

$$
\begin{cases}
A = -F_2 + \dfrac{F_2^2 - E_2G_2}{E_2} + \dfrac{F_2^2 - E_2G_2}{E_2} - \dfrac{F_2E_2}{G_2} - \dfrac{F_2G_2}{E_2}, \\
B = -1 - \dfrac{F_2}{E_2} - \dfrac{F_2}{G_2}, \\
D = \dfrac{F_2E_2}{G_2} - \dfrac{F_2G_2}{E_2}, \\
M = -\dfrac{F_2E_2}{G_2} - \dfrac{F_2G_2}{E_2}, \\
K = -\dfrac{h^2}{4}H_i', \\
N = -\dfrac{F_2E_2}{G_2} - \dfrac{F_2G_2}{E_2}.
\end{cases}
$$

Now, when we evaluate the limit of each variables separatly using L'Hospital's rule

$$\lim_{\rho_1\to 0}|A|=\lim_{\rho_1\to 0}\left\{-F_2+\frac{F_2^2-E_2G_2}{E_2}+\frac{F_2^2-E_2G_2}{E_2}-\frac{F_2E_2}{G_2}-\frac{F_2G_2}{E_2}\right\}=0,$$

$$\lim_{\rho_1\to 0}|B|H_i=\left(\lim_{\rho_1\to 0}\left\{-1-\frac{F_2}{E_2}-\frac{F_2}{G_2}\right\}\right)\left(\lim_{\rho_1\to 0}-2\rho_1 hf\right)=0,$$

$$\lim_{\rho_1\to 0}|D|=\lim_{\rho_1\to 0}\left\{\frac{F_2E_2}{G_2}-\frac{F_2G_2}{E_2}\right\}\Rightarrow\lim_{\rho_1\to 0}\frac{h}{2}|D|=0,$$

$$\lim_{\rho_1\to 0}|K|=\frac{h^2}{4}\lim_{\rho_1\to 0}H'\leq C_1h^2,$$

$$\lim_{\rho_1\to 0}|M|=\lim_{\rho_1\to 0}\left\{-\frac{F_2E_2}{G_2}-\frac{F_2G_2}{E_2}\right\}=-8\Rightarrow\frac{h^2}{8}|M|y_i''\leq C_2h^2,$$

$$\lim_{\rho_1\to 0}|N|=\lim_{\rho_1\to 0}\left\{\frac{F_2E_2}{G_2}-\frac{F_2G_2}{E_2}\right\}=0\Rightarrow\frac{h^3}{48}|N|y_i'''=0.$$

Therefore,

$$\begin{aligned}\left|L_\varepsilon^h y_i^h-L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right|&\leq|A|y_i+|B|H_i+\frac{h}{2}|D|y_i'+\frac{h^2}{8}|M|y_i''+K+\frac{h^3}{48}|N|y_i'''\\&\leq 0+0+0+C_1h^2+C_2h^2+0\\&\leq(C_1+C_2)h^2\\&\leq Ch^2.\end{aligned}$$

□

Lemma 5.3. *For all $0<h<h_0$ and for all $\varepsilon>0$, assume that L^h is stable with stability constant C and that*

$$\max\left\{\left|\left(y^h-y^{\frac{h}{2}}\right)(0)\right|,\left|\left(y^h-y^{\frac{h}{2}}\right)(l)\right|\right\}+C\left|L^h\left(y^h-y^{\frac{h}{2}}\right)\right|\leq C_2h^p,$$

then

$$\left|\left(y^h-y^{\frac{h}{2}}\right)(x_i)\right|\leq C_2h^p,$$

where C_2 is independent of ε.

Since $\left|L_\varepsilon^h y_i^h-L_\varepsilon^h y_{2ih}^{\frac{h}{2}}\right|\leq Ch^2$, we conclude that $\max_{1\leq j\leq N-1}|y(x_j)-Y(x_j)|\leq Ch^2$.

6. Numerical Example and Results

To validate the established theoretical results, we perform numerical experiments using the model problems of the form in (2.1)–(2.3).

Example 6.1. Consider the model singularly perturbed boundary value problem

$$\varepsilon y''(x)+y'(x)=1,\quad 0<x<1,$$

subject to the boundary conditions

$$y'(0) = \frac{1}{\varepsilon} \quad \text{and} \quad \int_0^1 y(x)dx = \frac{1}{2}.$$

Having $y_j \equiv y_j^h$ (the approximated solution obtained via fitted operator finite difference method) for different values of h and ε, the maximum errors. Since the exact solution is not available, the maximum errors (denoted by E_ε^h) are evaluated using the double mesh principle [15] for fitted operator finite difference methods using formula

$$E_\varepsilon^h := \max_{0 \le j \le n} |y_j^h - y_{2i}^{2h}|.$$

Further, we will tabulate the ε-uniform error

$$E^N = \max_{0<\varepsilon\le 1} E_\varepsilon^h.$$

The numerical rate of convergence are computed using the formula [15]

$$r_\varepsilon^h := \frac{\log(E_\varepsilon^h) - \log(E_\varepsilon^{\frac{h}{2}})}{\log(2)}$$

and the ε-uniform rate of convergence is computed using

$$R^N = \frac{\log(E^h) - \log(E^{\frac{h}{2}})}{\log(2)}.$$

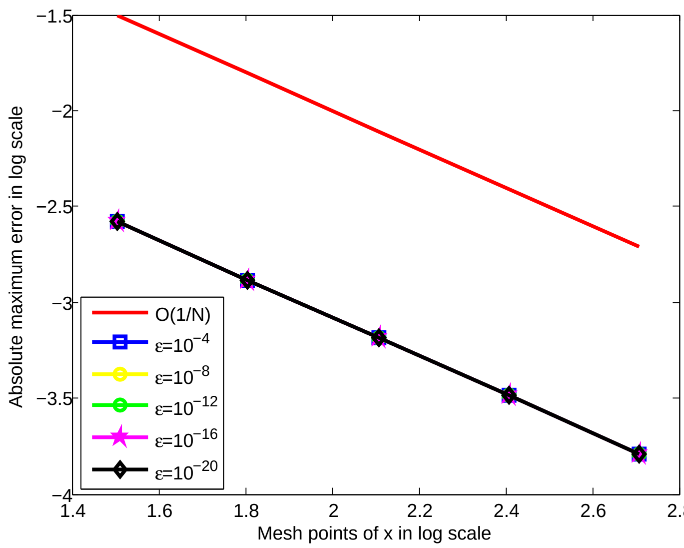

FIGURE 1. ε-uniform convergence with fitted operator in Log-Log scale

TABLE 1. Maximum absolute errors for different values of ε and mesh size, h with fitted parameter (WFP) and without fitted parameter (WOFP) for Example 6.1

ε	64	128	256	512	1024
WFP					
10^{-4}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-8}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-12}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-16}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-20}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
E^N	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
WOFP					
10^{-4}	8.2229e+03	1.8281e+03	4.1007e+02	8.7013e+01	1.7867e+01
10^{-8}	9.1177e+11	2.2841e+11	5.7162e+10	1.4297e+10	3.5745e+09
10^{-12}	9.1178e+19	2.2842e+19	5.7162e+18	1.4298e+18	3.5754e+17
10^{-16}	9.1083e+27	2.2845e+27	5.7140e+26	1.4301e+26	3.5756e+25
10^{-20}	5.7341e+37	7.1295e+36	8.8165e+35	1.0782e+35	2.2303e+34
E^N	5.7341e+37	7.1295e+36	8.8165e+35	1.0782e+35	2.2303e+34

TABLE 2. Maximum absolute errors and rate of convergence of Example 6.1 for different ε and mesh size h

ε	64	128	256	512	1024
WFP					
10^{-4}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-8}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-12}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-16}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
10^{-20}	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
E^N	2.6454e-03	1.3123e-03	6.5359e-04	3.2616e-04	1.6292e-04
R^N	1.7267	1.5726	1.4023	1.2526	

7. Discussion and Conclusion

This study introduces fitted operator numerical method for solving singularly perturbed boundary value problems with integral boundary conditions. The behavior of the continuous solution of the problem is studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. The numerical scheme is developed on uniform mesh by introducing the fitting operator

TABLE 3. ε-uniform Maximum absolute errors and ε-uniform rate of convergence for Example 6.1

ε	N=64	N=128	N=256	N=512
Present method				
E^N	0.0026454	0.0013159	0.00065359	0.00032616
R^N	1.0074	1.0096	1.0028	
Method in[20]				
E^N	0.0273271	0.0155869	0.00852830	0.00032616
R^N	0.81	0.87	0.97	

in to the higher order finite difference approximation used to replace the derivatives in the given differential equation. The stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, a model problem/example is considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see tables 1-3) and compared with the results of the previously developed numerical methods existing in the literature (Table 3). Further, the ε-uniform convergence of the method is shown by the log-log plot of the ε-uniform error (Figure 1). In a concise manner, the present method approximates the exact solution very well for reasonable value of the mesh size, $h \geq \varepsilon$, where existing classical numerical methods fails to give good results. Moreover, the method is convergent independent of the perturbation parameter ε and mesh size h and it improves the results of the methods developed so far for solving the problem under consideration.

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