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ALTERNATIVE APPROACH TO CERTAIN GENERALIZED (p,q)-FRACTIONAL FOURIER INTEGRAL TRANSFORM WITH AN ANGLE α

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Abstract. This article provides definitions and characteristics of (p,q)-analogues of a fractional Fourier integral transform, as well as their application to spaces of (p,q)-generalized functions. It also presents (p,q)-differential operators and (p,q)-convolution products to generate certain (p,q)-differentiable functions and classes of (p,q)-Boehmians. Furthermore, it demonstrates that the relevant (p,q)-analogues are linear and continuous across the spaces of construction. Additionally, it investigates the generalized (p,q)-fractional Fourier integral transforms, their characteristics, and inversion formulas on the aforementioned classes of (p,q)-Boehmians.

1. Introduction

The notion of obtaining q-analogous solutions without the need for limitations lies at the heart of the field of q-calculus theory. Jackson was the first to methodically establish the q-calculus theory [4]. He provided a general definition for the terms "q-integral" and "q-difference operator". Excellently, sets of non-differentiable functions, various classes of integral operators, orthogonal polynomials, and numerous classes of q-Bessel functions, q-hypergeometric functions, q-beta and q-gamma functions, and many more can all be handled with the help of the theory of q-calculus [5–8]. It is a key concept in several physical scientific domains, including conformal quantum mechanics, strings, mathematics and physics. In addition, it covers issues in quantum theory, mechanics, theory of relativity, orthogonal polynomials, hypergeometric functions, combinatorics, and number theory [29–38].

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In [39], Chakrabarti and Jagannathan introduced (p, q)-calculus as a generalized version of the q-calculus when $p = 1$. Consequently, in [6, 7, 10, 11, 41, 42], numerous concepts on (p, q)-integral, (p, q)-derivative, (p, q)-Taylor formulae, and the (p, q)-fundamental theorem of calculus have been the subject of comprehensive research and development by numerous scholars.

In [9], Sadjang looked into properties of the (p, q)-Laplace transform and solving certain (p, q)-difference equations. After that, he investigated the (p, q)-Sumudu transforms and provided traits that aid in the solution of (p, q)-difference equations. Furthermore, several authors developed thorough (p, q)-analogues of the Laplace and Sumudu transforms utilizing the (p, q)-Aleph function. In order to solve several (p, q)-differential equations, Jirakulchaiwong et al. [7] and some other authors [47–54] have recently created the (p, q)-Laplace-type integral transformations and discussed various applications.

Applied physics and engineering problems have made substantial use of generalized functions (distributions), which are continuous linear forms defined over sets of infinitely smooth functions [15]. Distributions are helpful in smoothing out discontinuous functions and describing physical occurrences as point charges. A structure in algebra that is comparable to the field of quotients outlines the modern generalized function space, referred to as the space of Boehmians [18, 19, 26]. Various Boehmian spaces are generated from the structure when they are applied to different function spaces when multiplication is regarded as a convolution [5, 42]. While building Boehmian spaces, delta sequences or the so-called approximating identities with declining support to the origin is necessary. This, in fact, lead to the uniqueness theorems, that are seen as uncertainty principle for the dynamics of Boehmians [44], see [20, 43, 45]. Nonetheless, because Boehmians have a definition in abstract algebraic notion, they permit various interpretations of those operators to constitute isomorphisms between the various Boehmian spaces.

In this article, we provide a brief overview of the (p, q)-calculus theory and the (p, q)-theory of Boehmians in Sections 2. In Section 3, we establish theorems on (p, q)-convolutions and derive some characteristics of the (p, q)-fractional Fourier transfrom. We discuss spaces of (p, q)-Boehmians in Section 4 and 5 and, in Section 6, several features of the generalized fractional integral operator are derived and some extensions are also examined.

2. (p, q)-Generalized Functions

Hereafter, we outline several common concepts and symbols arise from the (p, q)-calculus theory [40–42]. The definition of the (p, q)-analogue of the differential of a function ϑ, $d_{p,q}\vartheta(x) = \vartheta(px) - \vartheta(qx)$, $0 < q < p \leq 1$, is the first step in learning the (p, q)-theory. After saying this, we get the (p, q)-analogue of the ϑ derivative right away, which is known as the (p,q)-derivative,

$$(D_{p,q}\vartheta)(x) := \begin{cases} \frac{\vartheta(px)-\vartheta(qx)}{(p-q)x}, & x \neq 0, \\ \vartheta'(0), & x = 0. \end{cases} \tag{2.1}$$

If ϑ is differentiable, then $\lim_{p,q\to 1} D_{p,q}\vartheta(x) = \vartheta'(x)$. Given two functions, ϑ_1 and ϑ_2, the (p,q)-derivative of their product satisfies the following (p,q)-analog

$$D_{p,q}(\vartheta_1\vartheta_2)(x) = \vartheta_1(px)\, D_{p,q}\vartheta_2(x) + \vartheta_2(qx)\, D_{p,q}\vartheta_1(x). \tag{2.2}$$

Hereafter, $[r]_{p,q}$ and $[r]_{p,q}!$ will denote the (p,q)-analogues of numbers and those of factorials which are discussed by

$$[r]_{p,q} = \frac{p^r - q^r}{p-q} \quad \text{and} \quad [r]_{p,q}! = \prod\nolimits_{i=1}^{r} [i]_{p,q}, \quad [0]_{p,q} = 1, \tag{2.3}$$

respectively. While, the (p,q)-analogue of the power function $(t-v)^r_{p,q}$, $r = 0, 1, \ldots,$ $z, v \in \mathbb{R}$ is introduced by

$$(z-v)^r_{p,q} = \prod\nolimits_{i=0}^{r-1} \left(zp^i - vq^i\right).$$

Hence, for $r = 2$, we have

$$(t+v)^2_{p,q} = t^2 p + tv(p+q) + vq^2 \;\to pz^2 + 2pzv + pv^2, \quad \text{as } q \to p^-. \tag{2.4}$$

The (p,q)-integral of a function ϑ is defined by [4]

$$\int_0^x \vartheta(x)\, d_{p,q}x = (p-q)\, x \sum_0^{+\infty} \frac{q^i}{p^{i+1}} \vartheta\left(x \frac{q^i}{p^{i+1}}\right), \quad \left|\frac{p}{q}\right| > 1, \tag{2.5}$$

provided the sum converge absolutely for real number x. The (p,q)-integral in a generic interval $[a,b]$ is given by [4]

$$\int_a^b \vartheta(x)\, d_{p,q}x = \int_0^b \vartheta(x)\, d_{p,q}x - \int_0^a \vartheta(x)\, d_{p,q}x,$$

$$\int_a^b D_{p,q}\vartheta(x)\, d_{p,q}x = \vartheta(b) - \vartheta(a).$$

Definitions of the (p,q)-integral by parts for functions ϑ_1 and ϑ_2 are provided by (2.6)

$$\int_0^b \vartheta_1(px)\, D_{p,q}\vartheta_2(x)\, d_{p,q}x = \vartheta_2(b)\,\vartheta_1(b) - \vartheta_2(a)\,\vartheta_1(a) - \int_0^b \vartheta_2(qx)\, D_{p,q}\vartheta_1(x)\, d_{p,q}x.$$

Whereas the definitions of the two varieties of (p,q)-exponential functions are given in [10]

$$E_{p,q}(x) = \sum_{i=0}^{+\infty} q^{\frac{i(i-1)}{2}} \frac{x^i}{[i]_{p,q}!}, \quad x \in \mathbb{C}, \tag{2.7}$$

and

$$e_{p,q}(x) = \sum_{i=0}^{+\infty} p^{\frac{i(i-1)}{2}} \frac{x^i}{[i]_{p,q}!}, \quad |x| < 1. \tag{2.8}$$

When $p = 1$ is substituted in (2.7) and (2.8), the q-exponential functions E_p and e_p are obtained, respectively. The (p,q)-derivatives of the exponential function's (p,q)-analogues are also provided by

$$D_{p,q}e_{p,q}(rt) = re_{p,q}(rpt) \quad \text{and} \quad D_{p,q}E_{p,q}(rt) = rE_{p,q}(rqt). \tag{2.9}$$

Following our work in [46] we introduce the (p,q)-Boehmians as follows.

Suppose that U is a linear space V's subspace. The products of any pair of elements $\theta \in (V, *_{p,q})$ and $\omega_1 \in (U, *_{p,q})$ are then allocated as $*_{p,q}$ and $*_{p,q}$ such that:

(i) $\omega_1, \omega_2 \in U$ implies $\omega_1 *_{p,q} \omega_2 \in U$, $\omega_1 *_{p,q} \omega_2 = \omega_2 *_{p,q} \omega_1$,

(ii) $\theta \in V$, $\omega_1, \omega_2 \in U$ implies $(\theta *_{p,q} \omega_1) *_{p,q} \omega_2 = \theta *_{p,q} (\omega *_{p,q} \omega_2)$,

(iii) $\theta_1, \theta_2 \in V$, $\omega_1 \in U$, $r \in \mathbb{R}$ implies $(\theta_1 + \theta_2) *_{p,q} \omega_1 = \theta_1 *_{p,q} \omega_1 + \theta_2 *_{p,q} \omega_1$, $r(\theta_1 *_{p,q} \omega_1) = (r\theta_1) *_{p,q} \omega_1$.

Let $\Delta(\mathbb{R})$ be a set consists of sequences from U. Then, if $\Delta(\mathbb{R})$ passes Δ_1 and Δ_2, it is a set of the so named delta sequences if the following hold.

(P_1) For $\theta_1, \theta_2 \in V$, $(\delta_n) \in \Delta(\mathbb{R})$ and $\theta_1 *_{p,q} \delta_n = \theta_2 *_{p,q} \delta_n$, we have $\theta_1 = \theta_2$ for all $n \in \mathbb{N}$.

(P_2) $(\omega_n), (\delta_n) \in \Delta(\mathbb{R})$ implies $(\omega_n *_{p,q} \psi_n) \in \Delta(\mathbb{R})$.

If $S = \{(\theta_n), (\omega_n), (\theta_n) \in V, (\omega_n) \in \Delta(\mathbb{R}), \text{ for all } n \in \mathbb{N}\}$, then $((\theta_n), (\omega_n))$ is a pair of two quotients of sequences in S if and only if

$$\theta_n *_{p,q} \omega_m = \theta_m *_{p,q} \omega_n, \tag{2.10}$$

for n and m being natural. The pairs $((\theta_n), (\omega_n))$ and $((\vartheta_n), (\delta_n))$ are equivalent pairs of quotients according to the notation $\sim$ if and only if

$$\theta_n *_{p,q} \delta_m = \vartheta_m *_{p,q} \omega_n, \tag{2.11}$$

for all natural numbers n and m. Accordingly, $\sim$ creates an equivalent relation on the set S, and as a result, $\frac{\theta_n}{\omega_n}$ represents an equivalence class called the (p,q)-Boehmian. The new space of those resulting Boehmians is denoted by B. Two (p,q)-Boehmians $\frac{\theta_n}{\delta_n}$ and $\frac{\vartheta_n}{\mu_n}$, $\vartheta_n, \theta_n \in V$ and $\mu_n, \theta_n \in \Delta$ for all natural n, can be added in B by the equation

$$\frac{\theta_n}{\delta_n} + \frac{\vartheta_n}{\mu_n} = \frac{\theta_n *_{p,q} \mu_n + \vartheta_n *_{p,q} \delta_n}{\delta_n *_{p,q} \mu_n}. \tag{2.12}$$

The (p,q)-Boehmian $\frac{\vartheta_n}{\mu_n}$, $\vartheta_n \in V$ and $\mu_n \in \Delta$, can be multiplied in B by a real number A as

$$A\frac{\vartheta_n}{\mu_n} = \frac{A\vartheta_n}{\mu_n}, \tag{2.13}$$

whereas the expansion of $*_{p,q}$ and $D^{\alpha}_{p,q}$ to the (p,q)-Boehmian B are expressed in the form

$$\frac{\theta_n}{\delta_n} *_{p,q} \frac{\vartheta_n}{\mu_n} = \frac{\theta_n *_{p,q} \vartheta_n}{\delta_n *_{p,q} \mu_n} \quad \text{and} \quad D^{\alpha}_{p,q}\frac{\vartheta_n}{\mu_n} = \frac{D^{\alpha}_{p,q}\vartheta_n}{\mu_n}, \quad \alpha \in \mathbb{R}. \tag{2.14}$$

The product $*_{p,q}$ can be enlarged to $B *_{p,q} V$ by the equation

$$\frac{\vartheta_n}{\mu_n} *_{p,q} \omega = \frac{\vartheta_n *_{p,q} \omega}{\mu_n}, \quad \omega \in V, \frac{\vartheta_n}{\mu_n} \in B. \tag{2.15}$$

Furthermore, if $\beta_n \in B$, then $\beta_n \xrightarrow{\delta} \beta$ in B, if there can be found a delta sequence (ϑ_n) such that

$$\lim_{n\to+\infty} \beta_n *_{p,q} \vartheta_k = \beta *_{p,q} \vartheta_k, \quad \text{in } V, \tag{2.16}$$

for $(\beta_n *_{p,q} \vartheta_k)$ and $(\beta *_{p,q} \vartheta_k) \in V$, $n, k \in \mathbb{N}$.

3. The (p,q)-Fractional Fourier Transforms and Convolution Theorems

The fractional Fourier integral operator usually extends the traditional Fourier integral operator into a fractional domain. Although this operator has been defined in a variety of ways, the most logical one has been to extend the idea of rotations across an angle $\pi/2$ in the conventional Fourier integral operator. The fractional Fourier integral operator correlates to a rotation over an angle α. In contrast, a rotation on the time frequency plane $\alpha = a\pi/2$, $a \in \mathbb{R}$, corresponds to the standard Fourier integral operator.

Definition 3.1 ([27])**.** Given a signal ϑ and an angle α, the fractional Fourier integral operator can be defined both explicitly and formally using the integral equation

$$F_\alpha(\vartheta)(w) = \int_{-\infty}^{+\infty} \vartheta(t) X_\alpha(t,w)\, dt, \tag{3.1}$$

such that

$$X_\alpha(t,w) = \begin{cases} \frac{c(\alpha)}{\sqrt{2\pi}} \exp\Big(ja(\alpha)\left((t^2+w^2) - 2b(\alpha)wt\right)\Big), & \alpha \neq n\pi, \\ \frac{e^{-jwt}}{\sqrt{2\pi}}, & \alpha = \frac{\pi}{2}, \end{cases} \tag{3.2}$$

where $a(\alpha)$, $b(\alpha)$ and $c(\alpha)$ are given by $\frac{\cot\alpha}{2}$, $\sec\alpha$, and $\sqrt{1-\cot\alpha}$, respectively.

From (3.2) and (3.1), the inversion formula has been retrieved as

$$\vartheta(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F_\alpha(\vartheta)(w) X_{-\alpha}(t,w)\, dw. \tag{3.3}$$

Nevertheless, the fractional Fourier integral operator satisfies the following formulas for specific values of α: $F_\pi(\vartheta)(w) = \vartheta(-w)$, $F_0(\vartheta)(w) = \vartheta(w)$ and $(\pi/2)(\vartheta)(w) = F(\vartheta)(w)$, where F is the conventional Fourier integral of ϑ. It does, in fact, have applications in the time filtering [28], optical systems [1], quantum mechanics [2], solving ordinary differential equations and some pattern recognitions [3, 16, 24]. A few characteristics are worth mentioning. It is noteworthy that many characteristics of this amazing integral include associativity, commutativity, index additivity, and linearity $F_{\alpha_1}F_{\alpha_2} = F_{\alpha_1+\alpha_2}$, $(F_{\alpha_1}F_{\alpha_2})F_{\alpha_3} = F_{\alpha_1}(F_{\alpha_2}F_{\alpha_3})$ of the fractional Fourier integral operator have been discussed in [21] (see also [12, 13, 15].

Definition 3.2. Let ϑ be a signal. Then, we define the (p,q)-analogue of the fractional Fourier transform with an angle α of ϑ of the first-type as

$$F_{p,q,\alpha}(\vartheta)(w) = \int_{-\infty}^{+\infty} \vartheta(t)\, X_{p,q,\alpha}(t,w)\, d_{p,q}t, \tag{3.4}$$

such that

$$K_{p,q,\alpha}(t,w) = \begin{cases} \frac{c(\alpha)}{\sqrt{2\pi}} e_{p,q}\Big(jpa(\alpha)\left(\left(t^2+pw^2\right) - 2pb(\alpha)\, wt\right)\Big), & \alpha \neq n\pi, \\ \frac{e_{p,q}^{-jpwt}}{\sqrt{2\pi}}, & \alpha = \frac{\pi}{2}, \end{cases} \tag{3.5}$$

where $a(\alpha)$, $b(\alpha)$ and $c(\alpha)$ are given by $\frac{\cot\alpha}{2}$, $\sec\alpha$, and $\sqrt{1-\cot\alpha}$, respectively.

Definition 3.3. Let ϑ be a signal. Then, we define the (p,q)-analogue of the fractional Fourier transform of the signal ϑ of the second-type as

$$\check{F}_{p,q,\alpha}(\vartheta)(w) = \int_{-\infty}^{+\infty} \vartheta(t)\, \check{X}_{p,q,\alpha}(t,w)\, d_{p,q}t, \tag{3.6}$$

such that

$$\check{X}_{p,q,\alpha}(t,w) = \begin{cases} \frac{c(\alpha)}{\sqrt{2\pi}} E_{p,q}\Big(jqa(\alpha)\left(\left(t^2+pw^2\right) - 2qb(\alpha)\, wt\right)\Big), & \alpha \neq n\pi, \\ \frac{E_{p,q}^{-jqwt}}{\sqrt{2\pi}}, & \alpha = \frac{\pi}{2}, \end{cases} \tag{3.7}$$

where $a(\alpha)$, $b(\alpha)$ and $c(\alpha)$ are given by $\frac{\cot\alpha}{2}$, $\sec\alpha$ and $\sqrt{1-\cot\alpha}$, respectively.

In order to prove (p,q)-convolution theorems for the preceding (p,q)-analogues, we first propose a convolution product restricted to the condition $q \to p^-$ as follows.

Definition 3.4. The (p,q)-convolution product between two signals ϑ and ϑ_0 and an angle α is given by

$$\left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(t) = \int_{-\infty}^{+\infty} \vartheta(z)\, \vartheta_0(t-z)\, \psi_{p,q}(t,z)\, d_{p,q}z, \tag{3.8}$$

where $\psi_{p,q}(t,z) = e_{p,q}(2pqjz(z-t)\, a(\alpha))$.

Theorem 3.1. *Let $a(\alpha)$, $b(\alpha)$ and $c(\alpha)$ be given by $\frac{\cot\alpha}{2}$, $\sec\alpha$ and $\sqrt{1-\cot\alpha}$, respectively. Then, we have*

$$F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(w) = \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jp^2 a(\alpha)\, w^2\right) \left(F_{p,q,\alpha}\vartheta\right)(w) \left(F_{p,q,\alpha}\vartheta_0\right)(w). \tag{3.9}$$

Proof. By aid of the definition of $F_{p,q,\alpha}$ (3.4), the definition of $K_{p,q,\alpha}$ (3.5) and the product $*_{p,q}^{\alpha}$ (3.8) , we are led to write

$$F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(w) = \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(t)\, X_{p,q,\alpha}(t,w)\, d_{p,q}t$$

$$
\begin{aligned}
&= \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(t) \\
&\quad \times e_{p,q}\left(jpa(\alpha)\left(\left(t^2+w^2\right)-2pb(\alpha)wt\right)\right) d_{p,q}t \\
&= \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \vartheta(z)\vartheta_0(t-z)\psi_{p,q}(t,z) K_{p,q,\alpha}(t,w) d_{p,q}t \\
&= \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \vartheta(z)\vartheta_0(t-z) e_{p,q}\left(jpa(\alpha)t^2\right) \\
&\quad \times e_{p,q}\left(jp^2 a(\alpha)w^2\right) e_{p,q}\left(-jp^2(wt)a(\alpha)b(\alpha)\right) \\
&\quad \times \psi_{p,q}(t,z) d_{p,q}t d_{p,q}z.
\end{aligned}
$$

By using the fact $\psi_{p,q}(t,z) = e_{p,q}(2pqjz(z-t)a(\alpha))$, $a(\alpha)b(\alpha) = (\sec\alpha)(\cot\alpha) = \csc\alpha$, we get

$$
\begin{aligned}
F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(w) &= \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \vartheta(z)\vartheta_0(t-z) e_{p,q}\left(jp\frac{\cot\alpha}{2}t^2\right) \\
&\quad \times e_{p,q}\left(jp^2 a(\alpha)w^2\right) e_{p,q}\left(-jp^2(wt)\csc\alpha\right) \\
&\quad \times e_{p,q}\left(jp^2 z(z-t)\cot\alpha\right) d_{p,q}t d_{p,q}z. \qquad (3.10)
\end{aligned}
$$

The alteration of variables $v = t - z$ results in

$$
\begin{aligned}
F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(w) &= \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \vartheta(z)\vartheta_0(v) e_{p,q}\left(jpa(\alpha)(v+z)^2\right) \\
&\quad \times e_{p,q}\left(p^2 a(\alpha)w^2\right) e_{p,q}\left(-jp^2(w(v+z)\csc\alpha)\right) \\
&\quad \times e_{p,q}\left(-jp^2 zv\cot\alpha\right) d_{p,q}z d_{p,q}v. \qquad (3.11)
\end{aligned}
$$

Hence, by moving the items around in (3.11) and (2.4) we obtain

$$
\begin{aligned}
F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(w) &= \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \vartheta(z)\vartheta_0(v) e_{p,q}\left(jpa(\alpha)\left(pz^2+2pzv+pv^2\right)\right) \\
&\quad \times e_{p,q}\left(p^2 a(\alpha)w^2\right) e_{p,q}\left(-jp^2(w(v+z)\csc\alpha)\right) \\
&\quad \times e_{p,q}\left(-jp^2 zv\cot\alpha\right) d_{p,q}z d_{p,q}v. \\
&= \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \vartheta(z)\vartheta_0(v) e_{p,q}\left(jp^2 a(\alpha)z^2\right) \\
&\quad \times e_{p,q}\left(jp^2 a(\alpha)2zv\right) e_{p,q}\left(jp^2 a(\alpha)v^2\right) \\
&\quad \times e_{p,q}\left(p^2 a(\alpha)w^2\right) e_{p,q}\left(-jp^2(wva(\alpha)b(\alpha))\right) \\
&\quad \times e_{p,q}\left(-jp^2(wza(\alpha)b(\alpha))\right) e_{p,q}\left(-2jp^2 zva(\alpha)\right) d_{p,q}z d_{p,q}v.
\end{aligned}
$$

Therefore, we establish that

$$
F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(w) = \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \vartheta(z) e_{p,q}\left(jp^2 a(\alpha)z^2\right) e_{p,q}\left(jp^2 a(\alpha)w^2\right)
$$

$$\times e_{p,q}\left(-jp^2wz\left(\csc\alpha\right)\right)e_{p,q}\left(-jp^2wv\left(\csc\alpha\right)\right)d_{p,q}z. \tag{3.12}$$

Multiplying (3.12) by $\frac{c(\alpha)}{\sqrt{2\pi}}e_{p,q}\left(jp^2a\left(\alpha\right)w^2\right)$ yields

$$\begin{aligned}&F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha}\vartheta_0\right)(w)\frac{c\left(\alpha\right)}{\sqrt{2\pi}}e_{p,q}\left(jp^2a\left(\alpha\right)w^2\right)\\ =&\frac{c\left(\alpha\right)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\vartheta\left(z\right)e_{p,q}\left(jp^2\left(\cot\alpha\right)z^2\right)\\ &\times e_{p,q}\left(jp^2a\left(\alpha\right)w^2\right)e_{p,q}\left(-jp^2wz\left(\csc\alpha\right)\right)d_{p,q}z\frac{c\left(\alpha\right)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\vartheta_0\left(v\right)e_{p,q}\left(jp^2a\left(\alpha\right)v^2\right)\\ &\times e_{p,q}\left(jp^2a\left(\alpha\right)w^2\right)e_{p,q}\left(-jp^2wv\left(\csc\alpha\right)\right)d_{p,q}v.\end{aligned}$$

Hence, this yields

$$F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha}\vartheta_0\right)(w)\frac{c\left(\alpha\right)}{\sqrt{2\pi}}e_{p,q}\left(jp^2a\left(\alpha\right)w^2\right)=\left(F_{\alpha}\vartheta\right)(w)\left(F_{\alpha}\vartheta_0\right)(w). \tag{3.13}$$

Multiplying both sides of (3.13) by $\frac{\sqrt{2\pi}}{c(\alpha)}$ yields

$$F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha}\vartheta_0\right)(w)\,e_{p,q}\left(jp^2a\left(\alpha\right)w^2\right)=\frac{\sqrt{2\pi}}{c\left(\alpha\right)}\left(F_{\alpha}\vartheta\right)(w)\left(F_{\alpha}\vartheta_0\right)(w). \tag{3.14}$$

Once again, upon multiplying (3.14) by $e_{p,q}\left(-jp^2a\left(\alpha\right)w^2\right)$ reveals

$$F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha}\vartheta_0\right)(w)=\frac{\sqrt{2\pi}}{c\left(\alpha\right)}e_{p,q}\left(-jp^2a\left(\alpha\right)w^2\right)\left(F_{p,q,\alpha}\vartheta\right)(w)\left(F_{p,q,\alpha}\vartheta_0\right)(w). \tag{3.15}$$

The proof is finished. □

Theorem 3.2. *Let $a(\alpha)$, $b(\alpha)$ and $c(\alpha)$ be given by $\frac{\cot\alpha}{2}$, $\sec\alpha$ and $\sqrt{1-\cot\alpha}$, respectively. Then, we obtain*

$$\check{F}_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha}\vartheta_0\right)(w)=\frac{\sqrt{2\pi}}{c\left(\alpha\right)}E_{p,q}\left(-jq^2a\left(\alpha\right)w^2\right)\left(\check{F}_{p,q,\alpha}\vartheta\right)(w)\left(\check{F}_{p,q,\alpha}\vartheta_0\right)(w), \tag{3.16}$$

as $q\to p^-$.

Due to [13] and Zemanian [45] as well, we offer the definition that follows.

Definition 3.5. An infinitely (p,q)-differentiable complex-valued signal ϑ is in the class $S_{r.p,q}^{v}(\mathbb{R})$ if and only if

$$\gamma_{r,p,q}\left(\vartheta\right)=\sup_{t\in\mathbb{R}}\left|t^rD_{p,q}^{v}\vartheta\left(t\right)\right|<+\infty, \tag{3.17}$$

where r and v are constants.

Alternatively, (3.17) can usually be written as

$$J_{r,v}\left(\vartheta\right)=\sup_{t\in\mathbb{R}}\left|\left(1+|t|^2\right)^{\frac{r}{2}}D_{p,q}^{v}\vartheta\left(t\right)\right|<+\infty,\quad r,v\in\mathbb{N}.$$

We denote by the subspace of $S^{v}_{r.p,q}(\mathbb{R})$ of those functions of compact supports over $\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}}\left|D^{v}_{p,q}\vartheta(t)\right| < +\infty,$$

is denoted by $D^{v}_{r,p,q}(\mathbb{R})$.

Definition 3.6. By $S^{\alpha,v}_{r,p,q}(\mathbb{R})$ we denote the set of all signals ϑ such that

$$\gamma^{\alpha}_{r,v.p,q}(\vartheta) = \sup_{x\in\mathbb{R}}\left|x^{r}\Delta^{v,x}_{p,q}\vartheta(x)\right| < +\infty,$$

where

$$\Delta^{v,x}_{p,q} = \left(D^{v,x}_{p,q} - jp^{2}\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}(\cot\alpha)\,x\right). \tag{3.18}$$

Based on the previous analysis, we arrive to the following conclusion.

Proposition 3.1. *Let $K_{p,q,\alpha}(x,\xi)$ be the kernel function of the fractional Fourier transform. Then, we have*

$$\Delta^{v,x}_{p,q}\left(X_{p,q,\alpha}(x,\xi)\right) = \left(-jp^{2}\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}\xi\csc\alpha\right)^{v} X_{p,q,\alpha}(x,\xi)\,, \quad v\in\mathbb{N}_{0}.$$

Proof. Through the use of (p,q)-differentiation rules we obtain

$$\begin{aligned}
D^{v,x}_{p,q}X_{p,q,\alpha}(x,\xi) =& D^{v,x}_{p,q}\left(\frac{c(\alpha)}{\sqrt{2\pi}}e_{p,q}\left(jpa(\alpha)\left(\left(x^{2}+p\xi^{2}\right)-2pb(\alpha)\xi x\right)\right)\right)\\
=&\frac{c(\alpha)}{\sqrt{2\pi}}jpa(\alpha)\left(px+qx-2pb(\alpha)\xi\right)\\
&\times e_{p,q}\left\{jpa(\alpha)\left(\left(x^{2}+p\xi^{2}\right)-2pb(\alpha)\xi x\right)\right\}\\
=&\frac{c(\alpha)}{\sqrt{2\pi}}jpa(\alpha)\left(px+px-2pb(\alpha)\xi\right)\\
&\times e_{p,q}\left(jpa(\alpha)\left(\left(x^{2}+p\xi^{2}\right)-2pb(\alpha)\xi x\right)\right)\\
=&\frac{c(\alpha)}{\sqrt{2\pi}}jpa(\alpha)\left(2px-2pb(\alpha)\xi\right)\\
&\times e_{p,q}(jpa(\alpha)\left(\left(x^{2}+p\xi^{2}\right)-2pb(\alpha)\xi x\right)\\
=&\frac{c(\alpha)}{\sqrt{2\pi}}jpa(\alpha)\left(2px-2pb(\alpha)\xi\right)X_{p,q,\alpha}(x,\xi)\,.
\end{aligned}$$

Consequently, if $K_{p,q,\alpha} = K_{p,q,\alpha}(x,\xi)$, then the equation above can be simply expressed as

$$D^{v,x}_{p,q}K_{p,q,\alpha} - \frac{c(\alpha)}{\sqrt{2\pi}}jpa(\alpha)\,2pxK_{p,q,\alpha} = \frac{c(\alpha)}{\sqrt{2\pi}}jpa(\alpha)\left(-2pb(\alpha)\xi\right)K_{p,q,\alpha}. \tag{3.19}$$

Hence, by motivating the equation we obtain

$$\left(D_{p,q}^{v,x}-jp^{2}\frac{c\left(\alpha\right)}{\sqrt{2\pi}}a\left(\alpha\right)2x\right)X_{p,q,\alpha}\left(x,\xi\right)=-\frac{c\left(\alpha\right)}{\sqrt{2\pi}}jp^{2}a\left(\alpha\right)2b\left(\alpha\right)\xi X_{p,q,\alpha}\left(x,\xi\right). \tag{3.20}$$

Therefore, taking into account the real values of $a\left(\alpha\right), b\left(\alpha\right)$ and $c\left(\alpha\right)$ we establish that

$$\begin{aligned}\left(D_{p,q}^{v,x}-jp^{2}\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}\left(\cot\alpha\right)x\right)X_{p,q,\alpha}\left(x,\xi\right)&=\left(-jp^{2}\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}\left(\cot\alpha\right)\xi\sec\alpha\right)\\&\quad\times X_{p,q,\alpha}\left(x,\xi\right)\\&=\left(-jp^{2}\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}\xi\csc\alpha\right)X_{p,q,\alpha},\end{aligned}$$

where $X_{p,q,\alpha}=X_{p,q,\alpha}\left(x,\xi\right)$. Continuing this process, assuming $X_{p,q,\alpha}=X_{p,q,\alpha}\left(x,\xi\right)$, indeed gives

$$\left(D_{p,q}^{v,x}-\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}jp^{2}\left(\cot\alpha\right)x\right)^{v}X_{p,q,\alpha}=\left(-\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}jp^{2}\xi(\csc\alpha)\right)^{v}X_{p,q,\alpha}. \tag{3.21}$$

This finishes the proof. □

Proposition 3.2. *The* (p,q)*-derivative of the* (p,q)*-analogue* $F_{p,q,\alpha}$ *is given by*

$$\Delta_{p,q}^{v,\xi}\left(F_{p,q,\alpha}\vartheta\right)\left(\xi\right)=F_{p,q,\alpha}\left(\left(-jp^{2}\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}x\left(\csc\alpha\right)\right)^{v}\vartheta\left(x\right)\right)\left(\xi\right).$$

Proof. By taking into account (3.18) and Definition 3.2, we get

$$\begin{aligned}\Delta_{p,q}^{v,\xi}\left(F_{p,q,\alpha}\vartheta\right)\left(\xi\right)&=\int_{-\infty}^{+\infty}\vartheta\left(x\right)\Delta_{p,q}^{v,\xi}X_{p,q,\alpha}\left(x,\xi\right)d_{p,q}x\\&=\int_{-\infty}^{+\infty}\vartheta\left(x\right)\left(-jp^{2}\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}x\left(\csc\alpha\right)\right)^{v}X_{p,q,\alpha}\left(x,\xi\right)d_{p,q}x\\&=\int_{-\infty}^{+\infty}\left(\left(-jp^{2}\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}x\left(\csc\alpha\right)\right)^{v}\vartheta\left(x\right)\right)X_{p,q,\alpha}\left(x,\xi\right)d_{p,q}x\\&=F_{p,q,\alpha}\left(\left(-jp^{2}\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}x\left(\csc\alpha\right)\right)^{v}\vartheta\left(x\right)\right)\left(\xi\right).\end{aligned}$$

This ends the proof. □

Theorem 3.3. $F_{p,q,\alpha}:S_{r.p,q}^{v}\left(\mathbb{R}\right)\rightarrow S_{r,p,q}^{\alpha,v}\left(\mathbb{R}\right)$ *is linear and continuous.*

Proof. Linearity of $F_{p,q,\alpha}$ is obvious. To prove continuity, let $\left(\vartheta_{n}\right)\in S_{r,p,q}^{\alpha,v}\left(\mathbb{R}\right)$. Then, by Proposition 3.2, we have

$$\Delta_{p,q}^{v,\xi}\left(F_{p,q,\alpha}\vartheta_{n}\right)\left(\xi\right)=F_{p,q,\alpha}\left(\left(-\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}jp^{2}x\left(\csc\alpha\right)\right)^{v}\vartheta_{n}\left(x\right)\right)\left(\xi\right). \tag{3.22}$$

Therefore, (3.18) gives

$$\sup_{\xi\in\mathbb{R}}\left|\xi^{\beta}\Delta_{p,q}^{v,\xi}\left(F_{p,q,\alpha}\vartheta_n\right)(\xi)\right|=\sup_{\xi\in\mathbb{R}}\left|\xi^{\beta}F_{p,q,\alpha}\left(\left(-\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}jp^2x\left(\csc\alpha\right)\right)^{v}\vartheta_n\left(x\right)\right)(\xi)\right|.$$

Since $\vartheta_n\in S_{r,p,q}^{v}(\mathbb{R})$ we have $\left(-jp^2\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}x\left(\csc\alpha\right)\right)^{v}\vartheta_n(x)\in S_{r,p,q}^{v}(\mathbb{R})$. Hence, it follows that

$$F_{p,q,\alpha}\left(\left(-jp^2\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}x\left(\csc\alpha\right)\right)^{v}\vartheta_n\left(x\right)\right)(\xi)\in S_{r,p,q}^{v}(\mathbb{R}). \tag{3.23}$$

Therefore, if $\vartheta_n(x)\to 0$ as $n\to+\infty$, then

$$\sup_{\xi\in\mathbb{R}}\left|\xi^{\beta}F_{p,q,\alpha}\left(\left(-jp^2\frac{\sqrt{1-\cot\alpha}}{\sqrt{2\pi}}x\left(\csc\alpha\right)\right)^{v}\vartheta_n\right)(\xi)\right|\to 0,\quad \text{as } n\to+\infty. \tag{3.24}$$

This completes the proof of our result. □

4. The (p,q)-Space $\beta_{r,p,q}^{v}\left(S_{r,p,q}^{v}(\mathbb{R}),\left(D_{r,p,q}^{v}(\mathbb{R}),*_{p,q}^{\alpha}\right),*_{p,q}^{\alpha},\Delta_{p,q}^{a(\alpha)}\right)$

In the present section, it is aimed to establish the space of (p,q)-Boehmians with the sets $S_{r,p,q}^{v}(\mathbb{R}), D_{r,p,q}^{v}(\mathbb{R}), *_{p,q}^{\alpha}$ and $\Delta_{p,q}^{a(\alpha)}$. Therefore, we introduce the new class of (p,q)-delta sequences as follows.

Definition 4.1. Let $\Delta_{p,q}^{a(\alpha)}$ denote the set of sequences (δ_n) of $D_{r,p,q}^{v}(\mathbb{R})$ such that the following hold:

$$\int_{-\infty}^{+\infty}\delta_n(t)\,e_{p,q}^{jpa(\alpha)t^2}d_{p,q}t=1,\quad \text{for all } n\in\mathbb{N}, \tag{4.1}$$

$$\int_{-\infty}^{+\infty}\left|\delta_n(t)\right|e_{p,q}^{jpa(\alpha)t^2}d_{p,q}t<A, \tag{4.2}$$

$$\max\nolimits_{|t|\geq\delta}\left|\delta_n(t)\right|\to 0,\qquad \text{as } n\to+\infty \text{ for all } \delta>0, \tag{4.3}$$

where A is a real number.

We now prove the subsequent theorem.

Theorem 4.1. *The collection* $\left(\Delta_{p,q}^{a(\alpha)},*_{p,q}^{\alpha}\right)$ *forms a collection of (p,q)-delta sequences.*

Proof. We show $\left(\delta_n *_{p,q}^{\alpha}\gamma_n\right)\in\Delta_{\beta,p,q}^{\tilde{\beta}}$ for all $(\delta_n),(\gamma_n)\in\Delta_{p,q}^{a(\alpha)}$. As (4.2) and (4.3) have simple proofs, it is sufficient to demonstrate that (3.24) holds. Upon using the theorem of (p,q)-convolutions at $w=0$, we derive

$$F_{p,q,\alpha}\left(\delta_n *_{p,q}^{\alpha}\vartheta_0\right)(0)=\frac{\sqrt{2\pi}}{c(\alpha)}\left(F_{p,q,\alpha}\delta_n\right)(0)\left(F_{p,q,\alpha}\gamma_n\right)(0). \tag{4.4}$$

Therefore, by applying (3.4) and multiplying by $\frac{c(\alpha)}{\sqrt{2\pi}}$ we obtain

$$\frac{c(\alpha)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\delta_n *_{p,q}^{\alpha}\gamma_n\right)(t)\,e_{p,q}\left(jpa(\alpha)t^2\right)d_{p,q}t$$

$$
\begin{aligned}
&= \frac{c(\alpha)}{\sqrt{2\pi}} \cdot \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta_n(t)\, e_{p,q}\left(jpa(\alpha)\, t^2\right) d_{p,q}t \\
&\times \left(\frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \gamma_n(t)\, e_{p,q}\left(jpa(\alpha)\, t^2\right) d_{p,q}t \right). \qquad (4.5)
\end{aligned}
$$

Thus, employing (4.1) for γ_n and δ_n, we get

$$\int_{-\infty}^{+\infty} \left(\delta_n *_{p,q}^{\alpha} \gamma_n\right)(t)\, e_{p,q}\left(jpa(\alpha)\, t^2\right) d_{p,q}t = 1.$$

The proof is completely finished. □

Theorem 4.2. *The discussed product $*_{p,q}^{\alpha}$ is commutative in $S_{r.p,q}^{v}(\mathbb{R})$, i.e., $\vartheta *_{p,q}^{\alpha} \vartheta_0 = \vartheta_0 *_{p,q}^{\alpha} \vartheta$.*

Proof. According to the (p,q)-convolution theorem, we can say that

$$
\begin{aligned}
F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(w) &= \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jp^2 a(\alpha)\, w^2\right) \left(F_{p,q,\alpha}\vartheta\right)(w) \left(F_{p,q,\alpha}\vartheta_0\right)(w) \\
&= \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jp^2 a(\alpha)\, w^2\right) \left(F_{p,q,\alpha}\vartheta_0\right)(w) \left(F_{p,q,\alpha}\vartheta\right)(w) \\
&= F_{p,q,\alpha}\left(\vartheta_0 *_{p,q}^{\alpha} \vartheta\right)(w). \qquad (4.6)
\end{aligned}
$$

The given result is obtained by using the inverse $F_{p,q,\alpha}$ transform. □

Theorem 4.3. *Let $\vartheta, \vartheta_0, h \in S_{r.p,q}^{v}(\mathbb{R})$. Then, $\vartheta *_{p,q}^{\alpha} \left(\vartheta_0 *_{p,q}^{\alpha} h\right) = \left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right) *_{p,q}^{\alpha} h$.*

Proof. By applying Theorem 4.2, we have

$$
\begin{aligned}
&F_{p,q,\alpha}\left(\vartheta *_{p,q}^{\alpha} \left(\vartheta_0 *_{p,q}^{\alpha} h\right)\right)(w) \\
&= \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jp^2 a(\alpha)\, w^2\right) \left(F_{p,q,\alpha}\vartheta\right)(w) \left(F_{p,q,\alpha}\left(\vartheta_0 *_{p,q}^{\alpha} h\right)\right)(w) \\
&= \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jp^2 a(\alpha)\, w^2\right) \left(F_{p,q,\alpha}\vartheta\right)(w) \\
&\quad \times \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jp^2 a(\alpha)\, w^2\right) \left(F_{p,q,\alpha}\vartheta_0\right)(w) \left(F_{p,q,\alpha}h\right)(w) \\
&= \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jp^2 a(\alpha)\, w^2\right) \left(F_{p,q,\alpha}\vartheta *_{p,q}^{\alpha} \vartheta_0\right)(w) \left(F_{p,q,\alpha}h\right)(w) \\
&= F_{p,q,\alpha}\left(\left(\vartheta *_{p,q}^{\alpha} \vartheta_0\right) *_{p,q}^{\alpha} h\right)(w).
\end{aligned}
$$

This ends the proof. □

Simple calculations can be used to obtain a proof for the following two theorems.

Theorem 4.4. *If* $\theta \in D^{v}_{r,p,q}(\mathbb{R})$ *and* $\vartheta, \vartheta_n, \vartheta_0 \in S^{v}_{r,p,q}(\mathbb{R})$, *such that* $\vartheta_n \to \vartheta$, *as* $n \to +\infty$, *then*

(i) $(\vartheta + \vartheta_0) *^{\alpha}_{p,q} \theta = \vartheta *^{\alpha}_{p,q} \theta + \vartheta_0 *^{\alpha}_{p,q} \theta$,

(ii) $\vartheta_n *^{\alpha}_{p,q} \theta \to \vartheta *^{\alpha}_{p,q} \theta$ *as* $n \to +\infty$,

(iii) $\eta\left(\vartheta *^{\alpha}_{p,q} \theta\right) = \left(\eta\vartheta *^{\alpha}_{p,q} \theta\right)$, *for every complex number* η.

Theorem 4.5. *If* $\vartheta \in S^{v}_{r,p,q}(\mathbb{R})$ *and* $\vartheta_0 \in D^{v}_{r,p,q}(\mathbb{R})$, *then* $\vartheta *^{\alpha}_{p,q} \vartheta_0 \in S^{v}_{r,p,q}(\mathbb{R})$.

Theorem 4.6. *Let* $(\omega_n) \in \Delta^{a(\alpha)}_{p,q}$, $\vartheta \in S^{v}_{r,p,q}(\mathbb{R})$. *Then,* $\vartheta *^{\alpha}_{p,q} \omega_n \to \vartheta$ *as* $n \to +\infty$.

Proof. By aid of (3.17), we get

$$\left|t^r D^{v}_{p,q}\left(\vartheta *^{\alpha}_{p,q} \omega_n - \vartheta\right)(t)\right| = \left|t^r D^{v}_{p,q}\left(\int_{-\infty}^{\infty} \vartheta_z(t) - \vartheta(t)\right)\omega_n(z)\, d_{p,q}z\right|$$
$$\leq \int_C M\left|t^r D^{v}_{p,q}(\vartheta_z - \vartheta)(t)\right| d_{p,q}z \to 0,$$

as $n \to +\infty$, where $\vartheta(t-z) = \vartheta_z(t)$, C is a compact subset of $\mathbb{R}$ containing the support of (ω_n), for all $n \in \mathbb{N}$, and $|\omega_n| \leq A$, A is a constant.

The proof is ended. □

The space $\beta^{v}_{r,p,q}\left(\beta^{v}_{r,p,q} \equiv \beta^{v}_{r,p,q}\left(S^{v}_{r,p,q}(\mathbb{R}), \left(D^{v}_{r,p,q}(\mathbb{R}), *^{\alpha}_{p,q}\right), *^{\alpha}_{p,q}, \Delta^{a(\alpha)}_{p,q}\right)\right)$ of Boehmians is defined. The sequences (ϑ_n, ω_n) and (θ_n, μ_n) in $\beta^{v}_{r,p,q}$ are equivalent, $(\vartheta_n, \omega_n) \sim (\theta_n, \mu_n)$, if

$$\vartheta_n *^{\alpha}_{p,q} \mu_m = \theta_m *^{\alpha}_{p,q} \omega_n, \quad \text{for all } m, n \in \mathbb{N}. \tag{4.7}$$

Indeed, $\sim$ expresses an equivalence relation on $\beta^{v}_{r,p,q}$. The equivalence class in $\beta^{v}_{r,p,q}$ containing (ϑ_n, ω_n) is denoted as

$$\frac{\vartheta_n}{\omega_n} \tag{4.8}$$

and is a (p,q)-Boehmian. The following is an embedding between $S^{v}_{r.p,q}(\mathbb{R})$ and $\beta^{v}_{r,p,q}$,

$$y \to \frac{y *^{\alpha}_{p,q} \omega_n}{\omega_n}, \tag{4.9}$$

for all m and $n \in \mathbb{N}$. If $\frac{\vartheta_n}{\omega_n} \in \beta^{v}_{r,p,q}$ and $\varepsilon \in \beta^{v}_{r,p,q}$, then we have $\left(\frac{\vartheta_n}{\omega_n}\right) *^{\alpha}_{p,q} \varepsilon = \frac{\vartheta_n *^{\alpha}_{p,q} \varepsilon}{\omega_n}$.

5. The (p,q)-space $\beta^{\alpha,v}_{r,p,q}\left(S^{\alpha,v}_{r,p,q}(\mathbb{R}), \left(D^{\alpha,v}_{r,p,q}(\mathbb{R}), \circ^{q}_{p}\right), \circ^{q}_{p}, \tilde{\Delta}^{a(\alpha)}_{p,q}\right)$

To define the ultra space of (p,q)-Boehmians, let $S^{\alpha,v}_{r,p,q}(\mathbb{R})$ and $D^{\alpha,v}_{r.p,q}(\mathbb{R})$ be the collection of $F_{p,q,\alpha}$ of $S^{v}_{r.p,q}(\mathbb{R})$ and $D^{v}_{r.p,q}(\mathbb{R})$, respectively. By similar technique, let $\tilde{\Delta}^{a(\alpha)}_{p,q}$ be the collection of fractional Fourier transforms of $F_{p,q,\alpha}$ of all sequences in $\Delta^{a(\alpha)}_{p,q}$. Then, an operation on $S^{\alpha,v}_{r,p,q}$ is defined by

$$\left(U \circ^{q}_{p} V\right)(w) = \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jpa(\alpha)\, w^2\right) U(w)\, V(w). \tag{5.1}$$

Consequently, the following theorem can be easily proved by using this foundation.

Theorem 5.1. *Let $U, U_n, H, V \in S^{\alpha,v}_{r,p,q}(\mathbb{R})$, $U_n \to U$ as $n \to +\infty$ and $Y \in D^{v}_{r.p,q}$. Then, the identities listed below hold.*
(i) $(U+V) \circ^q_p Y = U \circ^q_p Y + V \circ^q_p Y$.
(ii) $U_n \circ^q_p Y \to U \circ^q_p Y$ *as* $U_n \to U$ *as* $n \to +\infty$.
(iii) $U \circ^q_p V = V \circ^q_p U$.
(iv) $U \circ^q_p \left(V \circ^q_p H\right) = \left(U \circ^q_p V\right) \circ^q_p H$.
(v) $\eta \left(U \circ^q_p V\right) = \left(\eta U \circ^q_p V\right)$, $\eta \in \mathbb{C}$.

Proof. The proofs for (i) and (ii) are simple since they resemble the proofs provided to the space $\beta^{v}_{r,p,q}\left(S^{v}_{r,p,q}(\mathbb{R}), \left(D^{v}_{r,p,q}(\mathbb{R}), *^{\alpha}_{p,q}\right), *^{\alpha}_{p,q}, \Delta^{a(\alpha)}_{p,q}\right)$.

The proof of (iii) Let $\vartheta, \vartheta_0 \in S^{v}_{r,p,q}(\mathbb{R})$ be such that $U = F_{p,q,\alpha}\vartheta$ and $V = F_{p,q,\alpha}\vartheta_0$, then by (5.1), we have

$$\begin{aligned}\left(U \circ^q_p V\right)(w) =& \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jpa(\alpha) w^2\right) U(w) V(w) \\ =& \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jpa(\alpha) w^2\right) \left(F_{p,q,\alpha}\vartheta\right)(w) \left(F_{p,q,\alpha}\vartheta_0\right)(w) \\ =& F_{p,q,\alpha}\left(\vartheta *^{\alpha}_{p,q} \vartheta_0\right)(w) \in S^{\alpha,v}_{r,p,q}(\mathbb{R}).\end{aligned}$$

Since $\vartheta *^{\alpha}_{p,q} \vartheta_0 = \vartheta_0 *^{\alpha}_{p,q} \vartheta$ it happens from (5.1) that

$$\left(U \circ^q_p V\right)(w) = F_{p,q,\alpha}\left(\vartheta_0 *^{\alpha}_{p,q} \vartheta\right)(w) = \left(V \circ^q_p U\right)(w) \in S^{\alpha,v}_{r,p,q}(\mathbb{R}). \tag{5.2}$$

While the proof of (v) is simple, the proof of (iv) is comparable to the proof of (iii).

This finishes the proof. ☐

Theorem 5.2. *Let $(\theta_n), (\varphi_n) \in \tilde{\Delta}^{a(\alpha)}_{p,q}$ and $U \in S^{\alpha,v}_{r,p,q}(\mathbb{R})$. Then, $\left(\theta_n \circ^q_p \varphi_n\right) \in \tilde{\Delta}^{a(\alpha)}_{p,q}$ and $\lim_{n\to+\infty} U \circ^q_p \theta_n = U$.*

Proof. Let $(\delta_n), (\psi_n) \in \Delta^{a(\alpha)}_{p,q}$ be such that $F_{p,q,\alpha}\delta_n = \theta_n$ and $F_{p,q,\alpha}\psi_n = \varphi_n$ for all $n \in \mathbb{N}$. Then, by (5.1) we have

$$\left(\theta_n \circ^q_p \varphi_n\right)(w) = \frac{\sqrt{2\pi}}{c(\alpha)} e_{p,q}\left(-jpa(\alpha) w^2\right) \theta_n(w) \varphi_n(w) = F_{p,q,\alpha}\left(\delta_n *^{\alpha}_{p,q} \psi_n\right)(w).$$

Hence, $\left(\theta_n *^{\alpha}_{p,q} \varphi_n\right)$ belongs to $\Delta^{a(\alpha)}_{p,q}$ since $\left(\delta_n *^{\alpha}_{p,q} \psi_n\right)$ belongs to $\Delta^{a(\alpha)}_{p,q}$. In a similar manner, the proof of the second part of the theorem can be derived.

The proof is therefore ended. ☐

The space $\beta^{\alpha,v}_{r,p,q}\left(S^{\alpha,v}_{r,p,q}(\mathbb{R}), \left(D^{\alpha,v}_{r.p,q}, \circ^q_p\right), \circ^q_p, \tilde{\Delta}^{a(\alpha)}_{p,q}\right)$ of ultraBoehmians is obtained. The $(F_{p,q,\alpha}\vartheta_n, F_{p,q,\alpha}\delta_n)$ and $(F_{p,q,\alpha}\theta_n, F_{p,q,\alpha}t_n)$ in $\beta^{\alpha,v}_{r,p,q}$ are equivalent if

$$F_{p,q,\alpha}\vartheta_n \circ^q_p F_{p,q,\alpha}t_m = F_{p,q,\alpha}\theta_m \circ^q_p F_{p,q,\alpha}\delta_n, \quad \text{for all } m, n \in \mathbb{N}.$$

In fact, $\sim$ establishes an equivalence relation on $\beta^{\alpha,v}_{r,p,q}$. An ultraBoehmian in $\beta^{\alpha,v}_{r,p,q}$ is written as

$$\frac{F_{p,q,\alpha}\vartheta_n}{F_{p,q,\alpha}\theta_n}, \tag{5.3}$$

where $(F_{p,q,\alpha}\vartheta_n) \in S^{\alpha,v}_{r,p,q}(\mathbb{R})$ and $(F_{p,q,\alpha}\theta_n) \in \tilde{\Delta}^{a(\alpha)}_{p,q}$. The equivalence class in $\beta^{\alpha,v}_{r,p,q}$ containing $(F_{p,q,\alpha}\vartheta_n, F_{p,q,\alpha}\delta_n)$ is denoted as $\frac{F_{p,q,\alpha}\vartheta_n}{F_{p,q,\alpha}\delta_n}$ and is called Boehmian. An embedding between $S^{\alpha,v}_{r,p,q}(\mathbb{R})$ and $\beta^{\alpha,v}_{r,p,q}$ is expressed as $y \to \frac{x \circ^q_p F_{p,q,\alpha}\delta_n}{F_{p,q,\alpha}\delta_n}$, for all $m,n \in \mathbb{N}$. If $\frac{F_{p,q,\alpha}\vartheta_n}{F_{p,q,\alpha}\delta_n} \in \beta^{\alpha,v}_{r,p,q}$ and $\varepsilon \in \beta^{\alpha,v}_{r,p,q}$, then

$$\left(\frac{F_{p,q,\alpha}\vartheta_n}{F_{p,q,\alpha}\delta_n}\right) \circ^q_p \varepsilon = \frac{F_{p,q,\alpha}\vartheta_n \circ^q_p \varepsilon}{F_{p,q,\alpha}\delta_n}.$$

Comparable ideas of addition, convergence, and scalar multiplication may be found in $\beta^{\alpha,v}_{r,p,q}$ and $\beta^{v}_{r,p,q}$.

Definition 5.1. Let $(\delta_n) \in \Delta^{a(\alpha)}_{p,q}$ and $(\vartheta_n) \in S^{v}_{r.p,q}(\mathbb{R})$. Then, the extended (p,q)-fractional Fourier operator $\breve{F}_{p,q,\alpha}$ of $\frac{\vartheta_n}{\delta_n}$ in $\beta^{v}_{r,p,q}$ can be given as

$$\breve{F}_{p,q,\alpha}\left(\frac{\vartheta_n}{\delta_n}\right) = \frac{F_{p,q,\alpha}\vartheta_n}{F_{p,q,\alpha}\delta_n}, \tag{5.4}$$

which indeed a member of $\beta^{\alpha,v}_{r,p,q}$.

6. Characteristics and an Inversion Formula for $\breve{F}_{p,q,\alpha}$

The generalized fractional Fourier operator $\breve{F}_{p,q,\alpha}$ is examined in this section along with some of its characteristics. To prove that $\breve{F}_{p,q,\alpha}$ is well-defined, we have the following theorem.

Theorem 6.1. *The generalized (p,q)-fractional Fourier operator $\breve{F}_{p,q,\alpha} : \beta^{v}_{r,p,q} \to \beta^{\alpha,v}_{r,p,q}$ is well-defined.*

Proof. If it is assumed that $\frac{\vartheta_n}{\delta_n} = \frac{\theta_n}{\varepsilon_n} \in \beta^{v}_{r,p,q}$. Then, the concept of quotients of sequences in $\beta^{v}_{r,p,q}$ establishes that $\vartheta_n *^{\alpha}_{p,q} \varepsilon_m = \theta_m *^{\alpha}_{p,q} \delta_n$, $m,n \in \mathbb{N}$. Thus, applying the analogue $F_{p,q,\alpha}$ reveals that

$$F_{p,q,\alpha}\left(\vartheta_n *^{\alpha}_{p,q} \varepsilon_m\right) = F_{p,q,\alpha}\left(\theta_m *^{\alpha}_{p,q} \delta_n\right), \quad m,n \in \mathbb{N}. \tag{6.1}$$

Therefore, according to the (p,q)-convolution theorem,

$$e_{p,q}\left(-jp^2 a(\alpha) w^2\right)(F_{p,q,\alpha}\vartheta_n)(F_{p,q,\alpha}\varepsilon_m) = e_{p,q}\left(-jp^2 a(\alpha) w^2\right)(F_{p,q,\alpha}\theta_m)(F_{p,q,\alpha}\delta_n). \tag{6.2}$$

Alternatively, this might be written as

$$(F_{p,q,\alpha}\vartheta_n) \circ^q_p (F_{p,q,\alpha}\varepsilon_m) = (F_{p,q,\alpha}\theta_m) \circ^q_p (F_{p,q,\alpha}\delta_n). \tag{6.3}$$

As a consequence of the idea of quotients and (6.3) in $\beta^{\alpha,v}_{r,p,q}$ imply that

$$\frac{F_{p,q,\alpha}\vartheta_n}{F_{p,q,\alpha}\delta_n}=\frac{F_{p,q,\alpha}\theta_n}{F_{p,q,\alpha}\varepsilon_n},\quad m,n\in\mathbb{N}.$$

Thus, it has been obtained that

$$\breve{F}_{p,q,\alpha}\left(\frac{\vartheta_n}{\delta_n}\right)=\breve{F}_{p,q,\alpha}\left(\frac{\theta_n}{\varepsilon_n}\right),\quad m,n\in\mathbb{N}.$$

This ends the proof. □

Both of the following theorems have simple proofs. Thus, details have been removed.

Theorem 6.2. *The operator* $\breve{F}_{p,q,\alpha}:\beta^{v}_{r,p,q}\to\beta^{\alpha,v}_{r,p,q}$ *is linear.*

Theorem 6.3. *Let* $\beta\in\beta^{v}_{r,p,q},\beta=0$. *Then,* $\breve{F}_{p,q,\alpha}(\beta)=0$.

Theorem 6.4. *Let* $\beta_0,\beta\in\beta^{v}_{r,p,q}$. *Then, we have*

$$\breve{F}_{p,q,\alpha}\left(\beta_0 *^{\alpha}_{p,q}\beta\right)(w)=\frac{c(\alpha)}{\sqrt{2\pi}}e_{p,q}\left(jp^2a(\alpha)w^2\right)\breve{F}_{p,q,\alpha}(\beta_0)\,\breve{F}_{p,q,\alpha}(\beta).$$

Proof. Let $\beta_0=\frac{\vartheta_n}{\delta_n}$, $\beta=\frac{\theta_n}{\varepsilon_n}\in\beta^{v}_{r,p,q}$ be given. Then, by employing $*^{\alpha}_{p,q}$ we get

$$\breve{F}_{p,q,\alpha}\left(\beta_0 *^{\alpha}_{p,q}\beta\right)=\breve{F}_{p,q,\alpha}\left(\frac{\vartheta_n *^{\alpha}_{p,q}\theta_n}{\delta_n *^{\alpha}_{p,q}\varepsilon_n}\right).$$

Hence, Theorem 3.1 reveals

$$\breve{F}_{p,q,\alpha}\left(\beta_0 *^{\alpha}_{p,q}\beta\right)(w)=\frac{c(\alpha)}{\sqrt{2\pi}}e_{p,q}\left(jp^2a(\alpha)w^2\right)\breve{F}_{p,q,\alpha}(\beta_0)\,\breve{F}_{p,q,\alpha}(\beta).$$

This ends the proof. □

Definition 6.1. Let $\beta_0\in\beta^{\alpha,v}_{r,p,q}$, $\beta_0=\frac{F_{p,q,\alpha}\vartheta_n}{F_{p,q,\alpha}\delta_n}$. Then, the inverse operator of $\breve{F}_{p,q,\alpha}$, $\left(\breve{F}_{p,q,\alpha}\right)^{-1}:\beta^{\alpha,v}_{r,p,q}\to\beta^{v}_{r,p,q}$, is presented as

$$\left(\breve{F}_{p,q,\alpha}\right)^{-1}(\beta_0)=\frac{\vartheta_n}{\delta_n},\tag{6.4}$$

for each $(\delta_n)\in\Delta^{a(\alpha)}_{p,q}$.

Theorem 6.5. *The inverse operator* $\left(\breve{F}_{p,q,\alpha}\right)^{-1}:\beta^{\alpha,v}_{r,p,q}\to\beta^{v}_{r,p,q}$ *is well-defined and linear.*

Proof. Let $\beta_0=\beta$ in $\beta^{\alpha,v}_{r,p,q}$, $\beta_0=\frac{F_{p,q,\alpha}\vartheta_n}{F_{p,q,\alpha}\delta_n}$, $\beta=\frac{F_{p,q,\alpha}\theta_n}{F_{p,q,\alpha}\varepsilon_n}$. Then,

$$F_{p,q,\alpha}\vartheta_n\circ^q_p F_{p,q,\alpha}\varepsilon_m=F_{p,q,\alpha}\theta_m\circ^q_p F_{p,q,\alpha}\delta_n,$$

for some $(\theta_n),(\vartheta_n)$ in $S^{v}_{r.p,q}(\mathbb{R})$. By aid of the (p,q)-convolution theorem (Theorem 3.1) we obtain

$$\breve{F}_{p,q,\alpha}\left(\vartheta_n *^{\alpha}_{p,q}\varepsilon_m\right)=\breve{F}_{p,q,\alpha}\left(\theta_m *^{\alpha}_{p,q}\delta_n\right),\quad m,n\in\mathbb{N}.$$

Therefore, by the benefit of the the inverse operator (6.4) we get $\vartheta_n *^{\alpha}_{p,q} \varepsilon_m = \theta_m *^{\alpha}_{p,q} \delta_n \, (m,n \in \mathbb{N})$. Therefore, the concept of (p,q)-qoutient of $\beta^{v}_{r,p,q}$, we find that

$$\frac{\vartheta_n}{\delta_n} = \frac{\theta_n}{\varepsilon_n}.$$

To establish linearity of the inversion $\left(\breve{F}_{p,q,\alpha}\right)^{-1}$, let $\beta_0 = \frac{F_{p,q,\alpha}\vartheta_n}{F_{p,q,\alpha}\delta_n}$, $\beta = \frac{F_{p,q,\alpha}\theta_n}{F_{p,q,\alpha}\varepsilon_n}$ be members in $\beta^{\alpha,v}_{r,p,q}$, then by addition of $\beta^{\alpha,v}_{r,p,q}$ and the (p,q)-convolution theorem we write

$$\left(\breve{F}_{p,q,\alpha}\right)^{-1}(\beta_0+\beta) = \left(\breve{F}_{p,q,\alpha}\right)^{-1}\left(\frac{F_{p,q,\alpha}\left((\vartheta_n) *^{\alpha}_{p,q} \varepsilon_n + \left(\theta_n *^{\alpha}_{p,q} \delta_n\right)\right)}{F_{p,q,\alpha}\left(\delta_n *^{\alpha}_{p,q} \varepsilon_n\right)}\right).$$

Therefore, considering the inversion formula we get

$$\left(\breve{F}_{p,q,\alpha}\right)^{-1}(\beta_0+\beta) = \frac{\vartheta_n *^{\alpha}_{p,q} \varepsilon_n + \theta_n *^{\alpha}_{p,q} \delta_n}{\delta_n *^{\alpha}_{p,q} \varepsilon_n}.$$

Hence, by aid of the (p,q)-addition in $\beta^{v}_{r,p,q}$, the proof of our theorem follows.

□

7. Conclusions

This study examined a new generalized post-quantum calculus theory and explained its fundamental concepts, including (p,q)-delta sequences, (p,q)-distribution spaces, (p,q)-Boehmian spaces, and a few other concepts. Additionally, we presented (p,q)-analogues to the fractional Fourier transform and developed several axioms that led to new generalized spaces of (p,q)-Boehmians. The extended fractional transform of a (p,q)-Boehmian is then demonstrated to be a well-defined (p,q)-Boehmian that meets several generalized properties. A few inversion formulas for the fractional Fourier transform are also discussed.

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