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ON THE BOUNDEDNESS OF q-HAUSDORFF OPERATORS ON q-HARDY SPACES

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Abstract. E. Liflyand and F. Móricz proved that the Hausdorff operator generated by a function $\varphi \in L^1(\mathbb{R})$ is a linear operator bounded on the real Hardy space $H^1(\mathbb{R})$ by using the classical Fourier transform and the Hilbert transform, they also proved that this operator commutes with Hilbert transform. In this work, we extend these results to the context of q-harmonic analysis associated with the q-Rubin's operator, we introduce the q-Hilbert transform on the real line, we study some of its main properties. Next, we define the q-Hardy spaces $\mathbb{H}^1_q(\mathbb{R}_q)$ by means of the q-Hilbert transforms, we finally study the q-Hausdorff operator and we prove the boundedness property and the commuting relation of this operator and q-Hilbert transform in q-Hardy spaces.

1. Introduction and Preliminaries

The Hausdorff operator is one of the important operators in harmonic analysis, it has a deep root in the study of the Fourier analysis. Particularly, a one-dimensional Hausdorff operator in Euclidean space is closely related to the summability of the classical Fourier series. The modern theory of Hausdorff operator started with the work of Siskakis in the complex analysis setting and with the work of Liflyand-Móricz in the Fourier transform setting [20], for more information about the background and the development of the Hausdorff operator one can refer to [7, 13, 15, 18, 19, 21] and references therein. For a locally integrable function φ on $(0, +\infty)$, the one-dimensional

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Hausdorff operator is defined in the integral form by

$$h_\varphi(f)(x) = \int_0^{+\infty} \frac{\varphi(t)}{t} f\left(\frac{x}{t}\right) dt.$$

It is worth pointing out that if the kernel function φ is taken appropriately, then the Hausdorff operator reduces to many classical operators in analysis such as the Hardy operator, the Cesáro operator, the Riemann-Liouville fractional integral operator and the Hardy-Littlewood average operator (see, e.g. [7, 13] and references therein).

Let $L^1(\mathbb{R})$, be the space of all Lebesgue measurable and integrable complex-valued functions defined on $\mathbb{R}$ endowed with the usual norm

$$\|f\|_1 := \int_{\mathbb{R}} |f(t)| dt < +\infty.$$

Then, we define the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R})$ by the following equality

$$\widehat{f}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(t) e^{-itx} dt, \quad \text{for all } x \in \mathbb{R}.$$

We also consider $\mathcal{H}(f)$ the Hilbert transform of a function $f \in L^1(\mathbb{R})$ in the form

$$\mathcal{H}(f)(x) := \frac{1}{\pi} \text{P. V.} \int_{\mathbb{R}} \frac{f(x-y)}{y} dy = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{+\infty} \frac{f(x-y) - f(x+y)}{y} dy, \tag{1.1}$$

for all $x \in \mathbb{R}$. We recall the definition of the Hardy space $H^1(\mathbb{R})$ as follows (see [20])

$$H^1(\mathbb{R}) := \left\{ f \in L^1(\mathbb{R}) : \mathcal{H}(f) \in L^1(\mathbb{R}) \right\},$$

endowed with the norm $||f||_{H^1} := ||f||_{L^1} + ||\mathcal{H}(f)||_{L^1}$. It has been proved in [2, 24] that $\mathcal{H}$ is a linear multiplier operator given by

$$\widehat{\mathcal{H}(f)}(t) = -i \operatorname{sgn}(t) \widehat{f}(t), \quad \text{for all } t \in \mathbb{R},$$

where $\operatorname{sgn}(t)$ equals $+1$ for $t > 0$, -1 for $t < 0$ and cancels at 0. This familiar identity is valid for all $f \in H^1(\mathbb{R})$ and plays a crucial role in this work.

In [20], E. Liflyand and F. Móricz studied the Hausdorff operators related to the Fourier transform on the real Hardy spaces $H^1(\mathbb{R})$ and proved the boundedness property and the commuting relations of these operators and Hilbert transforms. These results were recently obtained for the Dunkl-Hausdorff operator in [8]. For further details, the reader is invited to consult the following references [1,3,7,12,20,26].

In the present paper, by using some elements of the q-harmonic analysis associated to the q-Rubin's operator introduced in [22, 23], we attempt to explore the validity of these results in case of the q-Fourier transform. To describe our results, we introduce some necessary fundamental concepts of quantum analysis which will be used in this paper. For this purpose, we refer the reader to the book by G. Gasper and M. Rahman [10], for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions. We can also see the following references [4, 9–11, 25].

We mention that, in this paper, we will follow all the so-mentioned works which, for a convergence argument, they imposed that the parameter $0 < q < 1$ satisfies the condition

$$\frac{\log(1-q)}{\log(q)} \in 2\mathbb{Z}.$$

Throughout this paper, we assume $0 < q < 1$ and we denote

$$\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n : n \in \mathbb{Z}\} \quad \text{and} \quad \widehat{\mathbb{R}_q} = \mathbb{R}_q \cup \{0\}.$$

For $a \in \mathbb{C}$, the q-shifted factorials are defined by:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{l=0}^{n-1}(1-aq^l),\ n = 1,2\dots, \quad (a;q)_\infty = \prod_{l=0}^{+\infty}(1-aq^l).$$

We also denote for all $x \in \mathbb{C}$ and $n \in \mathbb{N}$

$$[x]_q = \frac{1-q^x}{1-q}, \quad [n]_q! = [1]_q \times [2]_q \times \cdots \times [n]_q = \frac{(q;q)_n}{(1-q)^n}, \quad [0]_q! = 1.$$

The q-Gamma function is given by (see [16])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \neq 0,-1,-2,\dots$$

It satisfies the following relations

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \quad \lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x), \quad \mathrm{Re}(x) > 0,$$

and tends to the classical Gamma function Γ when q tends to 1^-.

The q-derivative of a function f is given by

$$\mathcal{D}_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if } x \neq 0,$$

where $\mathcal{D}_q f(0) = f^{'}(0)$ provided $f^{'}(0)$ exists. We recall the following definition of the q-trigonometric functions. We define the q-cosine by

$$\cos(x;q^2) = \sum_{n=0}^{+\infty}(-1)^n q^{n(n+1)}\frac{x^{2n}}{[2n]_q!},$$

and the q-sine by

$$\sin(x;q^2) = \sum_{n=0}^{+\infty}(-1)^n q^{n(n+1)}\frac{x^{2n+1}}{[2n+1]_q!}.$$

So, we deduce the q-analogue of the classical exponential function by

$$e(ix;q^2) = \cos(x;q^2) + i\sin(x;q^2). \tag{1.2}$$

These three functions are entire on $\mathbb{C}$ and when q tends to 1^-, they tend to the corresponding classical ones pointwise and uniformly on compacts.

The following estimates are available for all $x \in \mathbb{R}_q$ (see [22])

$$|\cos(x;q^2)| \leq \frac{1}{(q;q)_\infty}, \quad |\sin(x;q^2)| \leq \frac{1}{(q;q)_\infty}.$$

So,

$$|e(ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{1.3}$$

The Rubin's q-differential operator is defined in [22, 23] by

$$\partial_q(f)(x) = \begin{cases} \frac{f(q^{-1}x)+f(-q^{-1}x)-f(qx)+f(-qx)-2f(-x)}{2(1-q)x}, & \text{if } x \neq 0, \\ \lim\limits_{x\to 0}\partial_q(f)(x) \text{ in } \mathbb{R}_q, & \text{if } x = 0. \end{cases}$$

Remark that if f is differentiable at x, then $\lim\limits_{q\to 1^-} \partial_q(f)(x) = f^{'}(x)$.

The following equality for the q-exponential function (see [23])

$$\partial_q e(itx;q^2) = ite(itx;q^2), \quad x \in \mathbb{R}_q. \tag{1.4}$$

holds, for every $t \in \mathbb{C}$.

The q-Jackson integrals are defined by (see [16, Sections 1.10 and 1.11])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{+\infty} q^n f(aq^n), \quad \int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

$$\int_0^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{+\infty} q^n f(q^n),$$

$$\int_{-\infty}^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{+\infty} q^n \left[f(q^n) + f(-q^n)\right],$$

provided the sums converge absolutely. In particular, for $a \in \mathbb{R}_q^+$,

$$\int_a^{+\infty} f(x)d_qx = (1-q)a\sum_{n=-\infty}^{-1} q^n f(aq^n).$$

Lemma 1.1. *The q-analogue of the integration theorem by a change of variable can be stated when $u(x) = \alpha x^\beta, \alpha \in \mathbb{C}$ as follows:*

$$\int_{u(a)}^{u(b)} f(u)d_qu = \int_a^b f(u(t))\mathcal{D}_{q^{1/\beta}}u(t)d_{q^{1/\beta}}t.$$

Proof. See [17]. □

The proof of Lemma 1.2 is straightforward: just use the definition of q-Jackson integrals.

Lemma 1.2. *If the stated integrals exist, then for all $a \in \mathbb{R}_q$, we have the following.*

- *If f is odd, then*
$$\int_{-a}^a f(t)d_qt = 0.$$
- *If f is even, then*
$$\int_{-a}^a f(t)d_qt = 2\int_0^a f(t)d_qt.$$

In what follows, let us fix some notations.

- $\mathcal{C}_q^p(\mathbb{R}_q)$ is the space of functions f, p times q-differentiable on $\widehat{\mathbb{R}_q}$ such that for all $0 \le n \le p$, $\partial_q^n f$ is continuous on $\widehat{\mathbb{R}_q}$.
- $\mathcal{S}_q(\mathbb{R}_q)$ is the space of functions f defined on $\mathbb{R}_q$ satisfying
$$P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q} |x^m \partial_q^n f(x)| < +\infty, \quad \text{for all } n, m \in \mathbb{N},$$
and
$$\lim_{x\to 0} \partial_q^n(f)(x) \quad (\text{in } \mathbb{R}_q) \quad \text{exists.}$$
- $L_q^p(\mathbb{R}_q)$, $p \in [1, +\infty]$, is the set of all functions defined on $\mathbb{R}_q$ such that
$$\|f\|_{q,p} = \begin{cases} \left(\int_{-\infty}^{+\infty} |f(x)|^p d_q x\right)^{1/p} < +\infty, & \text{if } 1 \le p < +\infty, \\ \sup_{x\in\mathbb{R}_q} |f(x)| < +\infty, & \text{if } p = +\infty. \end{cases}$$

Definition 1.1. The q-Fourier transform $\widehat{f}(\cdot; q^2)$ associated with the q-Rubin operator ∂_q is defined for every function f in $L_q^1(\mathbb{R}_q)$ by

$$\widehat{f}(x; q^2) = K \int_{-\infty}^{+\infty} f(t) e(-itx; q^2) d_q t, \tag{1.5}$$

for all $x \in \widehat{\mathbb{R}_q}$ where $K = (1+q)^{1/2}/2\Gamma_{q^2}(1/2)$.

It was shown in [23] that the q-Fourier transform $\widehat{f}(\cdot; q^2)$ verifies the following properties.

- If $f, \partial_q f \in L_q^1(\mathbb{R}_q)$, then
$$\widehat{\partial_q f}(x; q^2) = ix\widehat{f}(x; q^2), \quad \text{for all } x \in \widehat{\mathbb{R}_q}.$$
- If $f \in L_q^1(\mathbb{R}_q)$, then $\widehat{f}(\cdot; q^2) \in \mathcal{C}_q^0(\mathbb{R}_q)$ and we have

$$\|\widehat{f}(.; q^2)\|_{q,\infty} \le \frac{2K}{(q; q)_\infty} \|f\|_{q,1}. \tag{1.6}$$

Theorem 1.1. (i) *The q-Fourier transform $\widehat{f}(.; q^2)$ is an isomorphism from $L_q^2(\mathbb{R}_q)$ into itself and satisfies the following q-Plancherel formula:*

$$\|\widehat{f}(\cdot; q^2)\|_{q,2} = \|f\|_{q,2}, \quad \textit{for all } f \in L_q^2(\mathbb{R}_q). \tag{1.7}$$

(ii) *If $f \in L_q^1(\mathbb{R}_q)$ such that $\widehat{f}(.; q^2) \in L_q^1(\mathbb{R}_q)$, then the q-inversion formula holds and we have*

$$f(x) = K \int_{-\infty}^{+\infty} \widehat{f}(t; q^2) e(itx; q^2) d_q t, \quad \textit{for all } x \in \widehat{\mathbb{R}_q}.$$

The q-translation operator $\mathcal{T}_{q,x}$, $x \in \widehat{\mathbb{R}_q}$ is defined on $\mathcal{S}_q(\mathbb{R}_q)$ by (see [5, 22])

$$\mathcal{T}_{q,x}(f)(y) = K \int_{-\infty}^{+\infty} \widehat{f}(t; q^2) e(itx; q^2) e(ity; q^2) d_q t,$$
$$\mathcal{T}_{q,0}(f)(y) = f(y).$$

In particular the product formula

$$\mathcal{T}_{q,x}(e(it.;q^2))(y) = e(itx;q^2)e(ity;q^2), \quad x,y,t \in \mathbb{R}_q, \tag{1.8}$$

holds. It was shown in [8] that the q-translation operator can be also defined on $L_q^2(\mathbb{R}_q)$ and we have the following result.

Proposition 1.1. *For all $f \in L_q^2(\mathbb{R}_q)$, we have $\mathcal{T}_{q,x}f \in L_q^2(\mathbb{R}_q)$ and*

$$\|\mathcal{T}_{q,x}(f)\|_{q,2} \le \frac{2}{(q;q)_\infty}\|f\|_{q,2}, \quad \textit{for all } x \in \widehat{\mathbb{R}_q}.$$

Furthermore, the q-translation operator verifies the following properties.

Proposition 1.2. *For all $f \in L_q^1(\mathbb{R}_q)$, $t, x \in \mathbb{R}_q$, we have*

$$\mathcal{T}_{q,x}(f)(t) = \mathcal{T}_{q,t}(f)(x),$$

$$\int_{-\infty}^{+\infty} \mathcal{T}_{q,x}(f)(t)d_qt = \int_{-\infty}^{+\infty} f(t)d_qt,$$

$$\widehat{\mathcal{T}_{q,x}(f)}(t;q^2) = e(itx;q^2)\widehat{f}(t;q^2). \tag{1.9}$$

2. q-Hilbert Transform

In this section, we introduce the q-analogue of Hilbert transform and we study some of its main properties by using the q-harmonic analysis associated with the q-Rubin operator ∂_q.

For $f \in \mathcal{S}_q(\mathbb{R}_q)$, we define the q-analogue of Hilbert transform under the q-Rubin operator ∂_q, also called the q-Hilbert transform operator $\mathcal{H}_q$ by the principal value of the singular integral

$$\mathcal{H}_q(f)(x) := a_q \text{ P.V.} \int_{-\infty}^{+\infty} \frac{\mathcal{T}_{q,x}(f)(-t)}{t} d_qt, \tag{2.1}$$

where $a_q = (1+q)/2q\Gamma_{q^2}^2(1/2)$ and the principal value integral is defined to be $\lim\limits_{\varepsilon\to 0^+}\int_{|t|>\varepsilon}$.

Remark 2.1. (1) Letting q tends to 1^- subject to the formula (2.1), gives, at least formally, the classical Hilbert transform given by the formula (1.1).

(2) In view of formula (1.9), the function $\mathcal{T}_{q,x}(f) \in \mathcal{S}_q(\mathbb{R}_q)$ and one can write

$$\mathcal{H}_q(f)(x) = a_q \lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{+\infty} \frac{\mathcal{T}_{q,x}(f)(-t) - \mathcal{T}_{q,x}(f)(t)}{t} d_qt. \tag{2.2}$$

Let now us calculate the q-Hilbert transform of some basic functions, for this, we need the following proposition.

Proposition 2.1. *The following identities are valid*

$$\int_{-\infty}^{0} \frac{\sin(t;q^2)}{t} d_qt = \int_{0}^{+\infty} \frac{\sin(t;q^2)}{t} d_qt = \frac{\Gamma_{q^2}^2(1/2)}{1+q^{-1}}. \tag{2.3}$$

Proof. See [14, Proposition 5.3]. □

Proposition 2.2. *The following identities are fulfilled for all $x \in \mathbb{R}_q$:*

$$\mathcal{H}_q(\cos(\cdot;q^2))(x) = \sin(x;q^2), \tag{2.4}$$

$$\mathcal{H}_q(\sin(\cdot;q^2))(x) = -\cos(x;q^2). \tag{2.5}$$

Proof. Using the linearity of the q-translation and (1.2), we conclude that

$$\mathcal{T}_{q,x}(e(i.;q^2))(t) = \mathcal{T}_{q,x}(\cos(.;q^2))(t) + i\mathcal{T}_{q,x}(\sin(.;q^2))(t). \tag{2.6}$$

On other hands, it follows from (1.2) again that

$$\begin{aligned} e(ix;q^2)\cdot e(it;q^2) = &\cos(x;q^2)\cos(t;q^2) - \sin(x;q^2)\sin(t;q^2) \\ &+ i\left[\cos(x;q^2)\sin(t;q^2) + \sin(x;q^2)\cos(t;q^2)\right]. \end{aligned} \tag{2.7}$$

Using the product formula (1.8) and combining the relations (2.6) and (2.7), we have

$$\begin{aligned} \mathcal{T}_{q,x}(\cos(\cdot;q^2))(t) &= \cos(x;q^2)\cos(t;q^2) - \sin(x;q^2)\sin(t;q^2), \\ \mathcal{T}_{q,x}(\sin(\cdot;q^2))(t) &= \cos(x;q^2)\sin(t;q^2) + \sin(x;q^2)\cos(t;q^2). \end{aligned}$$

Therefore, by using the fact that $\cos(\cdot;q^2)$ is even and $\sin(\cdot;q^2)$ is odd, it follows from the definition of q-Hilbert transform (2.2) and Proposition 2.1 that

$$\begin{aligned} \mathcal{H}_q(\cos(\cdot;q^2))(x) &= a_q \lim_{\varepsilon\downarrow 0}\int_\varepsilon^{+\infty} \frac{\mathcal{T}_{q,x}(\cos(\cdot;q^2))(-t) - \mathcal{T}_{q,x}(cos(\cdot;q^2))(t)}{t} d_qt \\ &= 2a_q \sin(x;q^2)\int_0^{+\infty}\frac{\sin(t;q^2)}{t}d_qt \\ &= \sin(x;q^2). \end{aligned}$$

Similarly we have

$$\begin{aligned} \mathcal{H}_q(\sin(\cdot;q^2))(x) &= a_q \lim_{\varepsilon\downarrow 0}\int_\varepsilon^{+\infty} \frac{\mathcal{T}_{q,x}(\sin(\cdot;q^2))(-t) - \mathcal{T}_{q,x}(\sin(\cdot;q^2))(t)}{t} d_qt \\ &= -2a_q \cos(x;q^2)\int_0^{+\infty}\frac{\sin(t;q^2)}{t}d_qt \\ &= -\cos(x;q^2). \end{aligned}$$

Thus, Proposition 2.2 is proved. □

Remark 2.2. From (2.4) and (2.5), we have

$$\mathcal{H}_q(e(i\cdot;q^2))(x) = -ie(ix;q^2), \quad x \in \mathbb{R}_q.$$

Now, we are giving a very important property for the q-Hilbert transform under the q-Rubin operator, it has been proved in [24] and also in [2] that the Hilbert transform is a multiplier operator. Now, in the following theorem, we give a q-Fourier version of this result.

Theorem 2.1. *Let $f \in \mathcal{S}_q(\mathbb{R}_q)$. Then, the q-Hilbert transform is a multiplier operator with*

$$\widehat{\mathcal{H}_q(f)}(t;q^2) = -i\ \mathrm{sgn}(t)\widehat{f}(t;q^2), \tag{2.8}$$

for all $t \in \mathbb{R}_q$.

Proof. For $\xi \in \mathbb{R}_q$, we put

$$\mathcal{G}_{q,\xi}(\varepsilon, A) = \int_\varepsilon^A \frac{e(-it\xi;q^2) - e(it\xi,q^2)}{t} d_qt, \quad (\varepsilon, A) \in \mathbb{R}_q^+ \times \mathbb{R}_q^+.$$

In view of (1.2), we get

$$\begin{aligned}\mathcal{G}_{q,\xi}(\varepsilon, A) &= \int_\varepsilon^A \frac{e(-it\xi;q^2) - e(it\xi;q^2)}{t} d_qt = -2i \int_\varepsilon^A \frac{\sin(t\xi;q^2)}{t} d_qt \\ &= -2i\ \mathrm{sgn}(\xi) \int_{\varepsilon\xi}^{A\xi} \frac{\sin(t;q^2)}{t} d_qt.\end{aligned}$$

Consequently, by (2.3), we have

$$\lim_{\substack{\varepsilon \to 0 \\ A \to +\infty}} \mathcal{G}_{q,\xi}(\varepsilon, A) = -2i\ \mathrm{sgn}(\xi) \frac{\Gamma_{q^2}^2(1/2)}{1+q^{-1}} = \frac{-i\,\mathrm{sgn}(\xi)}{a_q}.$$

Furthermore, since

$$\mathcal{T}_{q,x}(f)(t) = \int_{-\infty}^{+\infty} \widehat{\mathcal{T}_{q,x}(f)}(\xi;q^2) e(it\xi;q^2) d_q\xi,$$

then, using Fubini's theorem and dominated convergence theorem, we get

$$\begin{aligned}\mathcal{H}_q(f)(x) &= a_q \int_0^{+\infty} \int_{-\infty}^{+\infty} \widehat{\mathcal{T}_{q,x}(f)}(\xi;q^2) \left(\frac{e(-it\xi;q^2) - e(it\xi;q^2)}{t} d_q\xi \right) d_qt \\ &= a_q \lim_{\substack{\varepsilon \to 0 \\ A \to +\infty}} \int_{-\infty}^{+\infty} \widehat{\mathcal{T}_{q,x}(f)}(\xi;q^2) \mathcal{G}_{q,\xi}(\varepsilon, A) d_q\xi \\ &= a_q \lim_{\substack{\varepsilon \to 0 \\ A \to +\infty}} \int_{-\infty}^{+\infty} e(ix\xi;q^2)\widehat{f}(\xi;q^2) \mathcal{G}_{q,\xi}(\varepsilon, A) d_q\xi \\ &= -i \int_{-\infty}^{+\infty} \mathrm{sgn}(\xi)\widehat{f}(\xi;q^2) e(ix\xi;q^2) d_q\xi.\end{aligned}$$

By the uniqueness of q-Fourier transform, we get exactly (2.8). □

Some direct application of the last theorem include the following.

Proposition 2.3. *Let $f \in \mathcal{S}_q(\mathbb{R}_q)$. Then, the following equality is valid for the q-Hilbert transform $\mathcal{H}_q$:*

$$\mathcal{H}_q(\mathcal{H}_q f) = -f. \tag{2.9}$$

Proof. Let $t \in \mathbb{R}_q$, then by using (2.8), we get

$$\widehat{\mathcal{H}_q(\mathcal{H}_q f)}(t;q^2) = -i\ \mathrm{sgn}(t)\widehat{\mathcal{H}_q f}(t;q^2) = -i\ \mathrm{sgn}(t) \times -i\ \mathrm{sgn}(t)\widehat{f}(t;q^2) = -\widehat{f}(t;q^2).$$

Hence, by uniqueness of q-Fourier transform, we have the result. □

Note that, from (2.9), we can see that $\mathcal{H}_{q,\alpha}$ is a unitary operator in $L^2_q(\mathbb{R}_q)$ and we have $\mathcal{H}_q^2 = -\mathcal{I}$, where $\mathcal{I}$ denotes the identity operator in $L^2_q(\mathbb{R}_q)$. From this, the inverse q-Hilbert transform operator can be written symbolically as $\mathcal{H}_q^{-1} = -\mathcal{H}_q$, and so,

$$f(x) = -a_q\ \mathrm{P.V.}\int_{-\infty}^{+\infty} \frac{\mathcal{T}_{q,x}(\mathcal{H}_q f)(-t)}{t} d_q t.$$

Theorem 2.2. (i) *The q-Hilbert transform $\mathcal{H}_q$ is an isomorphism from $\mathcal{S}_q(\mathbb{R}_q)$ onto itself. Moreover, for all $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have the following isometric property*

$$\|\mathcal{H}_q(f)\|_{q,2} = \|f\|_{q,2}. \tag{2.10}$$

(ii) *The q-Hilbert transform can be uniquely extended to an isometric isomorphism on $L^2_q(\mathbb{R}_q)$.*

Proof. Let $f \in \mathcal{S}_q(\mathbb{R}_q)$. From (1.3) and (1.4), we can easily see that the q-Fourier transform leaves $\mathcal{S}_q(\mathbb{R}_q)$ invariant and by taking into account the relation (1.9), we conclude that $\mathcal{T}_{q,x}(f)$ leaves $\mathcal{S}_q(\mathbb{R}_q)$ invariant, too. Therefore, $\mathcal{H}_q(f) \in \mathcal{S}_q(\mathbb{R}_q)$ and the application $\mathcal{H}_q : \mathcal{S}_q(\mathbb{R}_q) \to \mathcal{S}_q(\mathbb{R}_q)$ is well defined. The bijectivity of $\mathcal{H}_q$ follows immediately from relation (2.9). On the other hand, in view of Theorem 2.1 and q-Plancherel formula (1.7), we have

$$\begin{aligned}
\|\mathcal{H}_q(f)\|^2_{q,2} = \|\widehat{\mathcal{H}_q(f)}(.;q^2)\|^2_{q,2} &= \int_{-\infty}^{+\infty} |\widehat{\mathcal{H}_q f}(t;q^2)|^2 d_q t \\
&= \int_{-\infty}^{+\infty} |-i\ \mathrm{sgn}(t)\widehat{f}(t;q^2)|^2 d_q t = \int_{-\infty}^{+\infty} |\widehat{f}(t;q^2)|^2 d_q t \\
&= \|\widehat{f}(.;q^2)\|^2_{q,2} = \|f\|^2_{q,2}.
\end{aligned}$$

Then we have (2.10). The result of (ii) follows from (i) and the density of $\mathcal{S}_q(\mathbb{R}_q)$ in $L^2_q(\mathbb{R}_q)$ (see [4]). □

Let $\mathcal{D}_s$, $s > 0$, $s \in \mathbb{R}_q$, be the dilation operator defined by $\mathcal{D}_s f(x) = f(sx)$ and $\mathcal{R}$ be the reflection operator $\mathcal{R}f(x) = f(-x)$. Then, the following properties are fulfilled for the q-Hilbert transform $\mathcal{H}_q$.

Proposition 2.4. *The q-Hilbert transform $\mathcal{H}_q$ commutes with q-translations $\mathcal{T}_{q,x}$, $x \in \mathbb{R}_q$, positive dilations $\mathcal{D}_s$ and anticommutes with the reflection $\mathcal{R}$, that means*

$$\mathcal{H}_q(\mathcal{T}_{q,x} f) = \mathcal{T}_{q,x}(\mathcal{H}_q f), \tag{2.11}$$

$$\mathcal{H}_q(\mathcal{D}_s f) = \mathcal{D}_s(\mathcal{H}_q f), \tag{2.12}$$

$$\mathcal{H}_q(\mathcal{R} f) = -\mathcal{R}(\mathcal{H}_q f), \tag{2.13}$$

for all $f \in L^2_q(\mathbb{R}_q)$.

Proof. Let $t \in \mathbb{R}_q$, according to the formula (1.9) and Theorem 2.1, we have

$$\begin{aligned}\widehat{\mathcal{T}_{q,x}\mathcal{H}_q f}(t;q^2) &= e(itx;q^2)\widehat{\mathcal{H}_q f}(t;q^2) = -i\ \mathrm{sgn}(t)e(itx;q^2)\widehat{f}(t;q^2)\\ &= -i\ \mathrm{sgn}(t)\widehat{\mathcal{T}_{q,x}f}(t;q^2) = \widehat{\mathcal{H}_q\mathcal{T}_{q,x}f}(t;q^2).\end{aligned}$$

By injectivity of q-Fourier transform, we get (2.11). To prove (2.12), let $f \in L_q^2(\mathbb{R}_q)$, then $\mathcal{D}_s f \in L_q^2(\mathbb{R}_q)$ and by a simple change of variable, we can see that

$$\widehat{\mathcal{D}_s f}(t;q^2) = s^{-1}\widehat{f}(t/s;q^2), \quad t,s \in \mathbb{R}_q, s>0.$$

From this and (2.8), we have

$$\begin{aligned}\widehat{\mathcal{H}_q\mathcal{D}_s f}(t;q^2) &= -i\ \mathrm{sgn}(t)\widehat{\mathcal{D}_s f}(t;q^2) = -i\ \mathrm{sgn}(t)s^{-1}\widehat{f}(t/s;q^2)\\ &= s^{-1}(-i\ \mathrm{sgn}(t/s)\widehat{f}(t/s;q^2)) = s^{-1}\widehat{\mathcal{H}_q f}(t/s;q^2) = \widehat{\mathcal{D}_s\mathcal{H}_q f}(t;q^2).\end{aligned}$$

By uniqueness of q-Fourier transform, we get (2.12) and it is equally obvious that

$$\mathcal{H}_q(\mathcal{D}_s f) = -\mathcal{D}_s(\mathcal{H}_q f), \quad \text{for } s<0.$$

From this, (2.13) is proved in the case $s=-1$. □

3. The q-Hausdorff Operators in q-Hardy Spaces

To give the main results of this work, we first introduce the q-analogue of Hardy spaces using the q Hilbert transform on the image under the q-Fourier transform. Next, we prove the boundedness of q-Hausdorff operator by using the closed graph theorem and the fact that if $f \in L_q^1(\mathbb{R}_q)$ satisfies

$$\widehat{f}(t;q^2) = 0, \quad \text{for } t<0 \text{ (or } t>0), t \in \mathbb{R}_q,$$

then $f \in \mathbb{H}_q^1(\mathbb{R}_q)$.

Let $f \in L_q^1(\mathbb{R}_q)$ and $x \in \widehat{\mathbb{R}_q}$. Since for all $z,t \in \mathbb{R}_q$, $|e(-itz;q^2)| \le \frac{2}{(q;q)_\infty}$, and $t \mapsto e(itx;q^2)$ is in $L_q^1(\mathbb{R}_q)$, then

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|f(z)e(itx;q^2)e(-itz;q^2)\right|d_qtd_qz\\ \le&\frac{2}{(q;q)_\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(z)|\left|e(itx;q^2)\right|d_qtd_qz\\ =&\frac{2}{(q;q)_\infty}\|f\|_{q,1}\left\|e(ix\cdot;q^2)\right\|_{q,1} < +\infty.\end{aligned}$$

Therefore, we can define the q-Hilbert transform on $L_q^1(\mathbb{R}_q)$ as

$$\mathcal{H}_q(f)(x) = -i\int_{-\infty}^{+\infty}\mathrm{sgn}(t)\widehat{f}(t;q^2)e(itx;q^2)d_qt.$$

So, the q-Hardy space $\mathbb{H}_q^1(\mathbb{R}_q)$ is an important subspace of $L_q^1(\mathbb{R}_q)$ defined by

$$\mathbb{H}_q^1(\mathbb{R}_q) := \left\{f \in L_q^1(\mathbb{R}_q) :\ \mathcal{H}_q(f) \in L_q^1(\mathbb{R}_q)\right\},$$

endowed with the norm

$$||f||_{\mathbb{H}^1_q} := ||f||_{q,1} + ||\mathcal{H}_q(f)||_{q,1}. \tag{3.1}$$

This space is a Banach algebra under pointwise addition, scalar multiplication and convolution. It is much better adapted to problems arising in the theory of q-harmonic analysis.

Furthermore, it follows from (2.8) that, if $f \in \mathbb{H}^1_q(\mathbb{R}_q)$, then

$$\widehat{\mathcal{H}_q f}(t; q^2) = -i \ \mathrm{sgn}(t)\widehat{f}(t; q^2) \tag{3.2}$$

and

$$\mathcal{H}_q(\mathcal{H}_q f)(t) = -f(t), \quad t \in \mathbb{R}_q. \tag{3.3}$$

From (3.3), we can easily see that $\mathcal{H}_q$ is a bounded operator from $\mathbb{H}^1_q(\mathbb{R}_q)$ into itself. In particular, if $f \in \mathbb{H}^1_q(\mathbb{R}_q)$, then $\mathcal{H}_q(f) \in \mathbb{H}^1_q(\mathbb{R}_q)$ and we have

$$||\mathcal{H}_q(f)||_{\mathbb{H}^1_q} = ||f||_{\mathbb{H}^1_q}.$$

In what follows, we give some auxiliary results interesting in themselves.

Proposition 3.1. *If f belongs to $\mathbb{H}^1_q(\mathbb{R}_q)$, then*

$$\int_{-\infty}^{+\infty} f(t)d_qt = 0. \tag{3.4}$$

Proof. Since $f \in \mathbb{H}^1_q(\mathbb{R}_q)$, then by using the definition of q-Hardy space, we have $f \in L^1_q(\mathbb{R}_q)$ and $\mathcal{H}_q(f) \in L^1_q(\mathbb{R}_q)$, by using the relation (1.6) and the fact that

$$\|\widehat{f}(\cdot; q^2)\|_{q,\infty} \leq \frac{4K}{(q,q)_\infty}\|f\|_{q,1} < +\infty,$$
$$\|\widehat{\mathcal{H}_q f}(\cdot; q^2)\|_{q,\infty} \leq \frac{4K}{(q,q)_\infty}\|\mathcal{H}_q(f)\|_{q,1} < +\infty,$$

we have $\widehat{f}(\cdot; q^2), \widehat{\mathcal{H}_q f}(\cdot; q^2) \in \mathcal{C}^0_q(\mathbb{R}_q)$. From (3.2), it follows that

$$\widehat{\mathcal{H}_q f}(0; q^2) = -i\widehat{f}(0; q^2) = i\widehat{f}(0; q^2).$$

So, $\widehat{f}(0; q^2) = 0$, which gives (3.4). □

Proposition 3.2. *If f belong to $L^1_q(\mathbb{R}_q)$ and satisfies:*

$$\widehat{f}(t; q^2) = 0, \quad \textit{for } t > 0 \textit{ and } t \in \mathbb{R}_q, \tag{3.5}$$

or

$$\widehat{f}(t; q^2) = 0, \quad \textit{for } t > 0 \textit{ and } t \in \mathbb{R}_q, \tag{3.6}$$

then $f \in \mathbb{H}^1_q(\mathbb{R}_q)$.

Proof. It follows from (2.8) and (3.5) that

$$\widehat{\mathcal{H}_q f}(t;q^2) = -i\ \mathrm{sgn}(t)\widehat{f}(t;q^2) = i\widehat{f}(t;q^2).$$

By uniqueness of q-Fourier transform, we get

$$\mathcal{H}_q(f)(t) = if(t), \tag{3.7}$$

for almost all $t \in \mathbb{R}_q$, therefore the result follows from (3.7). A similar proof is obtained from (3.6). □

Proposition 3.3. *If $f \in \mathbb{H}_q^1(\mathbb{R}_q)$, then there are two functions f_1 and f_2 both in $\mathbb{H}_q^1(\mathbb{R}_q)$ such that $f = f_1 + f_2$ and*

$$\begin{aligned}\widehat{f_1}(t;q^2) =& 0, \quad \text{for } t > 0 \text{ and } t \in \mathbb{R}_q,\\ \widehat{f_2}(t;q^2) =& 0, \quad \text{for } t > 0 \text{ and } t \in \mathbb{R}_q.\end{aligned}$$

Proof. We set

$$f_1(t) = \frac{f(t) + i\mathcal{H}_q(f)(t)}{2} \quad \text{and} \quad f_2(t) = \frac{f(t) - i\mathcal{H}_q(f)(t)}{2}.$$

By (3.3), we have $f_1, f_2 \in \mathbb{H}_q^1(\mathbb{R}_q)$. Furthermore, assume that $t < 0$ with $t \in \mathbb{R}_q$ and according to the formula (3.2), we get

$$\begin{aligned}2\widehat{f_1}(t;q^2) &= \widehat{f}(t;q^2) + i\widehat{\mathcal{H}_q f}(t;q^2) = \widehat{f}(t;q^2) + i\left(-i\ \mathrm{sgn}(t)\widehat{f}(t;q^2)\right)\\ &= (1 + \mathrm{sgn}(t))\widehat{f}(t;q^2) = 0,\end{aligned}$$

and analogously, we have

$$2\widehat{f_2}(t;q^2) = (1 - \mathrm{sgn}(t))\widehat{f}(t;q^2) = 0, \quad \text{for } t > 0, t \in \mathbb{R}_q,$$

which completes the proof. □

In the following, we introduce the q-analogue of Hausdorff operator in q-Fourier analysis, also known as the q-Hausdorff operator. We define the q-Hausdorff operator $h_{q,\psi}$ of a function $f \in L_q^1(\mathbb{R}_q)$ generated by a function ψ belonging also to $L_q^1(\mathbb{R}_q)$ as follows:

$$\widehat{h_{q,\psi} f}(t;q^2) = \int_{-\infty}^{+\infty} \widehat{f}(tx;q^2)\psi(x)d_q x, \quad t \in \mathbb{R}_q, \tag{3.8}$$

provided the integral is convergent, where

$$h_{q,\psi} f(x) = \int_{-\infty}^{+\infty} \frac{\psi(t)}{|t|} f\left(\frac{x}{t}\right) d_q t, \quad x \in \mathbb{R}_q. \tag{3.9}$$

For $x > 0$, $x \in \mathbb{R}_q$, by a change of variables, one has

$$h_{q,\psi} f(x) = \int_{-\infty}^{+\infty} \frac{f(t)}{|t|} \psi\left(\frac{x}{t}\right) d_q t.$$

The existence of such a function $h_{q,\psi} f$ in $L_q^1(\mathbb{R}_q)$ will be clear from the proof of the following theorem.

Theorem 3.1. *Let f, ψ belong to $L_q^1(\mathbb{R}_q)$. Then, the q-Hausdorff operator $h_{q,\psi} : L_q^1(\mathbb{R}_q) \to L_q^1(\mathbb{R}_q)$ is a bounded operator with*

$$\|h_{q,\psi} f\|_{q,1} \leq \|f\|_{q,1} \|\psi\|_{q,1},$$

and the formula (3.8) *holds.*

Proof. Let $f, \psi \in L_q^1(\mathbb{R}_q)$, from (3.9), we have

$$|h_{q,\psi} f(x)| \leq \int_{-\infty}^{+\infty} \left|\frac{\psi(t)}{t}\right| \cdot \left| f\left(\frac{x}{t}\right)\right| d_q t.$$

Putting $\xi = x/t$ and by using Fubini's theorem, we have

$$\begin{aligned}
\|h_{q,\psi} f\|_{q,1} &= \int_{-\infty}^{+\infty} |h_{q,\psi} f(x)| d_q x \\
&= \int_{-\infty}^{+\infty} \left|\frac{\psi(t)}{t}\right| d_q t \int_{-\infty}^{+\infty} \left| f\left(\frac{x}{t}\right)\right| d_q x \\
&= \int_{0}^{+\infty} \frac{|\psi(t)|}{t} d_q t \int_{-\infty}^{+\infty} \left| f\left(\frac{x}{t}\right)\right| d_q x - \int_{-\infty}^{0} \frac{|\psi(t)|}{t} d_q t \int_{-\infty}^{+\infty} \left| f\left(\frac{x}{t}\right)\right| d_q x \\
&= \int_{0}^{+\infty} |\psi(t)| d_q t \int_{-\infty}^{+\infty} |f(\xi)| d_q \xi + \int_{-\infty}^{0} |\psi(t)| d_q t \int_{-\infty}^{+\infty} |f(\xi)| d_q \xi \\
&= \int_{-\infty}^{+\infty} |\psi(t)| d_q t \int_{-\infty}^{+\infty} |f(\xi)| d_q \xi \\
&= \|f\|_{q,1} \|\psi\|_{q,1},
\end{aligned}$$

which completes the proof of boundness of $h_{q,\psi}$ on $L_q^1(\mathbb{R}_q)$. Finally, to get (3.8), it follows from (1.5) and (3.9) that

$$\begin{aligned}
\widehat{h_{q,\psi} f}(t; q^2) =& K \int_{-\infty}^{+\infty} e(-itx; q^2) h_{q,\psi} f(x) d_q x \\
=& K \int_{-\infty}^{+\infty} e(-itx; q^2) d_q x \int_{-\infty}^{+\infty} \frac{\psi(\xi)}{|\xi|} f\left(\frac{x}{\xi}\right) d_q \xi \\
=& \int_{0}^{+\infty} \frac{\psi(\xi)}{\xi} d_q \xi \, K \int_{-\infty}^{+\infty} f\left(\frac{x}{\xi}\right) e(-itx; q^2) d_q x \\
& - \int_{-\infty}^{0} \frac{\psi(\xi)}{\xi} d_q \xi \, K \int_{-\infty}^{+\infty} f\left(\frac{x}{\xi}\right) e(-itx; q^2) d_q x \\
=& \int_{0}^{+\infty} \psi(\xi) d_q \xi \, K \int_{-\infty}^{+\infty} f(u) e(-it\xi u; q^2) d_q u \\
& + \int_{-\infty}^{0} \psi(\xi) d_q \xi \, K \int_{-\infty}^{+\infty} f(u) e(-it\xi u; q^2) d_q u \\
=& \int_{-\infty}^{+\infty} \psi(\xi) d_q \xi \, K \int_{-\infty}^{+\infty} f(u) e(-it\xi u; q^2) d_q u \\
=& \int_{-\infty}^{+\infty} \widehat{f}(t\xi; q^2) \psi(\xi) d_q \xi.
\end{aligned}$$

Then, Theorem 3.1 is proved. □

Theorem 3.2. *Let ψ belongs to $L^1_q(\mathbb{R}_q)$, such that*

$$\psi(t) = 0, \quad \textit{for } t < 0, t \in \mathbb{R}_q. \tag{3.10}$$

Then, $h_{q,\psi} : \mathbb{H}^1_q(\mathbb{R}_q) \to \mathbb{H}^1_q(\mathbb{R}_q)$ is a bounded operator.

Proof. Assume that $\psi \in L^1_q(\mathbb{R}_q)$ such that condition (3.10) is satisfied. Let $f \in \mathbb{H}^1_q(\mathbb{R}_q)$. By Proposition 3.3, there are two functions f_1 and f_2 both in $\mathbb{H}^1_q(\mathbb{R}_q)$ such that $f = f_1 + f_2$. The q-Hausdorff operator is clearly linear, hence

$$h_{q,\psi} f = h_{q,\psi} f_1 + h_{q,\psi} f_2. \tag{3.11}$$

Furthermore, from (3.8) and by a change of variable, we get

$$\begin{aligned}\widehat{h_{q,\psi} f}(t; q^2) &= \int_{-\infty}^{+\infty} \widehat{f}(tx; q^2)\psi(x) d_q x = \frac{1}{t}\int_{-\infty}^{+\infty} \operatorname{sgn}(t)\widehat{f}(\xi; q^2)\psi(\xi/t) d_q\xi \\ &= \frac{1}{|t|}\int_{-\infty}^{+\infty} \widehat{f}(\xi; q^2)\psi(\xi/t) d_q\xi.\end{aligned}$$

According to Proposition 3.3, we have

$$\widehat{f_1}(\xi; q^2) = 0, \quad \text{for } \xi < 0, \tag{3.12}$$

and the condition (3.10) yields to

$$\psi(\xi/t) = 0, \quad \text{for } t < 0 \text{ and } \xi > 0. \tag{3.13}$$

Therefore, from (3.12) and (3.13), we conclude that

$$\widehat{h_{q,\psi} f_1}(t; q^2) = \frac{1}{|t|}\int_0^{+\infty} \widehat{f_1}(\xi; q^2)\psi(\xi/t) d_q\xi = 0, \quad \text{for } t < 0. \tag{3.14}$$

In the same way, we establish that

$$\widehat{h_{q,\psi} f_2}(t; q^2) = \frac{1}{|t|}\int_0^{+\infty} \widehat{f_2}(\xi; q^2)\psi(\xi/t) d_q\xi = 0, \quad \text{for } t > 0. \tag{3.15}$$

Finally, by Theorem 3.1, Proposition 3.2 and combining the formulas (3.14) and (3.15), both $h_{q,\psi} f_2$ and $h_{q,\psi} f_2$ belongs to $\mathbb{H}^1_q(\mathbb{R}_q)$. From (3.11) it follows that $h_{q,\psi} f \in \mathbb{H}^1_q(\mathbb{R}_q)$, which completes the proof. □

Remark 3.1. In the general case, we decompose $\psi \in L^1_q(\mathbb{R}_q)$ in the trivial way:

$$\psi(x) = \psi_1(x) + \psi_2(x),$$

where

$$\psi_1(x) = \psi(x)\chi_{(0,+\infty)}(x) \quad \text{and} \quad \psi_2(x) = \psi(x) - \psi_1(x),$$

$\chi_{(0,+\infty)}(x)$ being the indicator function on $(0, +\infty)$. Clearly, we have

$$h_{q,\psi} f = h_{q,\psi_1} f + h_{q,\psi_2} f, \quad f \in \mathbb{H}^1_q(\mathbb{R}_q),$$

and the special cases in Theorem 3.2 apply separately to $h_{q,\psi_1} f$ and $h_{q,\psi_2} f$, respectively. Therefore, we have $h_{q,\psi} f \in \mathbb{H}^1_q(\mathbb{R}_q)$ whenever $f \in \mathbb{H}^1_q(\mathbb{R}_q)$.

Theorem 3.3. *If ψ belongs to $L_q^1(\mathbb{R}_q)$, then the q-Hausdorff operator $h_{q,\psi} : \mathbb{H}_q^1(\mathbb{R}_q) \to \mathbb{H}_q^1(\mathbb{R}_q)$ is a bounded operator.*

Proof. In this proof, we shall apply the closed graph theorem, which says that a linear operator mapping a Banach space into another Banach space is bounded if and only if it is closed. Linearity of the q-Hausdorff operator h_{q,ψ_1} is obvious. We have to show that the graph of $h_{q,\psi}$ is closed. To this effect, assume that a sequence of functions $\{f_n : n = 1, 2, \ldots\}$, belonging to $\mathbb{H}_q^1(\mathbb{R}_q)$ exists, such that

$$f_n \to f \quad \text{and} \quad h_{q,\psi} f_n \to g, \quad \text{as } n \to +\infty,$$

with $f, g \in \mathbb{H}_q^1(\mathbb{R}_q)$. This implies that

$$||f_n - f||_{\mathbb{H}_q^1} \to 0 \quad \text{and} \quad ||h_{q,\psi} f_n - g||_{\mathbb{H}_q^1} \to 0.$$

From (3.1), the above expression yields

$$||f_n - f||_{q,1} + ||\mathcal{H}_q(f_n - f)||_{q,1} \to 0, \quad \text{as } n \to +\infty,$$

and

$$||h_{q,\psi} f_n - g||_{q,1} + ||\mathcal{H}_q(h_{q,\psi} f_n - g)||_{q,1} \to 0, \quad \text{as } n \to +\infty.$$

Hence,

$$||f_n - f||_{q,1} \to 0 \quad \text{and} \quad ||h_{q,\psi} f_n - g||_{q,1} \to 0, \quad \text{as } n \to +\infty.$$

By Theorem 3.1, we have $h_{q,\psi} : L_q^1(\mathbb{R}_q) \to L_q^1(\mathbb{R}_q)$ is a bounded operator and hence by the closed graph theorem its graph must be closed.

Since

$$(f_n, h_{q,\psi} f_n) \to (f, g), \quad \text{as } n \to +\infty.$$

Hence,

$$(f, g) \in graph(h_{q,\psi}),$$

whence we conclude that $h_{q,\psi} f = g$. This implies that the graph of $h_{q,\psi}$ is closed in $\mathbb{H}_q^1(\mathbb{R}_q)$. □

Theorem 3.4. *If ψ belongs to $L_q^1(\mathbb{R}_q)$, then for all $f \in \mathbb{H}_q^1(\mathbb{R}_q)$, we have*

$$\mathcal{H}_q(h_{q,\psi} f) = h_{q,\mathrm{sgn}(\cdot)\psi(\cdot)}(\mathcal{H}_q(f)).$$

Proof. Let us put

$$\widetilde{\psi}(t) = \mathrm{sgn}(t)\psi(t), \quad t \in \mathbb{R}_q,$$

and let $f \in \mathbb{H}_q^1(\mathbb{R}_q)$, then by Theorem 3.3, we have $h_{q,\psi} f \in \mathbb{H}_q^1(\mathbb{R}_q)$, thus $\mathcal{H}_q(h_{q,\psi} f) \in L_q^1(\mathbb{R}_q)$, then in view of formula (3.2) and the fact that the function signum verify

$$\mathrm{sgn}(t) \times \mathrm{sgn}(x) = \mathrm{sgn}(tx), \quad \text{for all } t, x \in \mathbb{R}_q,$$

we have conclude that

$$\begin{aligned}\mathcal{H}_q\widehat{(h_{q,\psi}f)}(t;q^2) &= -\,i\,\mathrm{sgn}(t)\widehat{h_{q,\psi}f}(t;q^2) = -i\,\mathrm{sgn}(t)\int_{-\infty}^{+\infty}\widehat{f}(tx;q^2)\psi(x)d_qx\\ &= -i\,\mathrm{sgn}(t)\int_{-\infty}^{+\infty}\mathrm{sgn}^2(x)\widehat{f}(tx;q^2)\psi(x)d_qx\\ &= \int_{-\infty}^{+\infty}-i\,\mathrm{sgn}(tx)\widehat{f}(tx;q^2)\widetilde{\psi}(x)d_qx\\ &= \int_{-\infty}^{+\infty}\widehat{\mathcal{H}_q f}(tx;q^2)\widetilde{\psi}(x)d_qx = h_{q,\widetilde{\psi}}\widehat{(\mathcal{H}_q f)}(t;q^2).\end{aligned}$$

Then, by uniqueness of the q-Fourier transform, we have the desired result. □

A consequence of the last theorem is to give another version of the proof of Theorem 3.3.

Theorem 3.5. *If ψ belongs to $L^1_q(\mathbb{R}_q)$, then for all $f \in \mathbb{H}^1_q(\mathbb{R}_q)$, we have $h_{q,\psi}f \in \mathbb{H}^1_q(\mathbb{R}_q)$ and*

$$||h_{q,\psi}f||_{\mathbb{H}^1_q} \le ||f||_{\mathbb{H}^1_q}||\psi||_{q,1}.$$

Proof. Assume that $f \in \mathbb{H}^1_q(\mathbb{R}_q)$. Then, $\mathcal{H}_q(f) \in L^1_q(\mathbb{R}_q)$. Thus, in view of (3.1) and Theorem 3.1, we get

$$\begin{aligned}||h_{q,\psi}f||_{\mathbb{H}^1_q} &= ||h_{q,\psi}f||_{q,1} + ||\mathcal{H}_q(h_{q,\psi}f)||_{q,1} = ||h_{q,\psi}f||_{q,1} + ||h_{q,\widetilde{\psi}}(\mathcal{H}_q f)||_{q,1}\\ &\le ||f||_{q,1}||\psi||_{q,1} + ||\mathcal{H}_q f||_{q,1}||\widetilde{\psi}||_{q,1}\\ &= ||f||_{q,1}||\psi||_{q,1} + ||\mathcal{H}_q f||_{q,1}||\psi||_{q,1} = (||f||_{q,1} + ||\mathcal{H}_q f||_{q,1})||\psi||_{q,1}\\ &= ||f||_{\mathbb{H}^1_q}||\psi||_{q,1}.\end{aligned}$$

□

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References

[1] C. Abdelkefi and M. Rachdi, *Some results on the Hardy space H^1_k associated with the Dunkl operators*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **61** (2015), 201–218. https://doi.org/10.1007/s11565-015-0229-4

[2] B. Amri, A. Gasmi and M. Sifi, *Linear and bilinear multiplier operators for the Dunkl transform*, Mediterr. J. Math. **7** (2010), 503–521. https://doi.org/10.1007/s00009-010-0057-9

[3] J-Ph. Anker, N. Ben Salem, J. Dziubański and N. Hamda, *The Hardy space H^1 in the rational Dunkl setting*, Constr. Approx. **42** (2015), 93–128. https://doi.org/10.1007/s00365-014-9254-2

[4] N. Bettaibi and R. H. Bettaieb, *q-analogue of the Dunkl transform on the real line*, Tamsui Oxf. J. Math. Sci. **25**(2) (2007), 117–205. https://doi.org/10.48550/arXiv.0801.0069

[5] N. Bettaibi, K. Mezlini and M. El Guénichi, *On Rubin's harmonic analysis and its related positive definite functions*, Acta Math. Sci. **32**(5) (2012), 1851–1874. https://doi.org/10.1016/S0252-9602(12)60145-3

[6] M. M. Chaffar, N. Bettaibi and A. Fitouhi, *Sobolev type spaces associated with the q-Rubin's operator*, Matematiche **69** (2014), 37–56.

[7] J. Chen, D. Fan and S. Wang, *Hausdorff operators on Euclidean space*, Appl. Math. J. Chin. Univ. **28** (2014), 548–564. https://doi.org/10.1007/s11766-013-3228-1

[8] R. Daher and F. Saadi, *The Dunkl-Hausdorff operator is bounded on the real Hardy space* $H^1_\alpha(\mathbb{R})$, Integral Transforms Spec. Funct. **28**(11) (2019), 882–892. https://doi.org/10.1080/10652469.2019.1636236

[9] R. Daher and O. Tyr, *An analog of Titchmarsh's theorem for the q-Dunkl transform in the space* $L^2_{q,\alpha}(\mathbb{R}_q)$, J. Pseudo-Differ. Oper. Appl. **11** (2020), 1933–1949. https://doi.org/10.1007/s11868-020-00330-6

[10] R. Daher and O. Tyr, *Growth properties of the q-Dunkl transform in the space* $L^p_{q,\alpha}(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$, Ramanujan J. **57** (2022), 119–134. https://doi.org/10.1007/s11139-021-00387-x

[11] R. Daher and O. Tyr, *Modulus of smoothness and theorems concerning approximation in the space* $L^2_{q,\alpha}(\mathbb{R}_q)$ *with power weight*, Mediterr. J. Math. **18**(69) (2021), 1–18. https://doi.org/10.1007/s00009-021-01715-7

[12] J. Dziubański, *Riesz transforms characterizations of Hardy spaces* H^1 *for the rational Dunkl setting and multidimensional Bessel operators*, J. Geom. Anal. **199** (2016), 2639–2663. https://doi.org/10.1007/s12220-015-9642-2

[13] D. Fan and X. Lin, *Hausdorff operator on real Hardy spaces*, Analysis **34**(4) (2014), 319–337.

[14] A. Fitouhi and F. Bouzaffour, *The q-cosine Fourier transform and the q-heat equation*, Ramanujan J. **28** (2012) 443–461. https://doi.org/10.1007/s11139-012-9412-8

[15] P. Galanopoulos and A. G. Siskakis, *Hausdorff matrices and composition operators*, Ill. J. Math. **45** (2001) 757–773.

[16] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Application Vol 35, Cambridge Univ. Press, Cambridge, 1990.

[17] V. G. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, NewYork, 2002.

[18] E. Liflyand, *Hausdorff operators on Hardy spaces*, Eurasian Math. J. **4**(4) (2013), 101–141.

[19] E. Liflyand and A. Miyachi, *Boundedness of the Hausdorff operators in* H^p *spaces,* $0 < p < 1$, Studia Math. **194**(3) (2009), 279–292.

[20] E. Liflyand and F. Móricz, *The Hausdorff operator is bounded on the real Hardy space* $H^1(\mathbb{R})$, Proc. Amer. Math. Soc. **128** (2000), 1391–1396.

[21] A. Miyachi, *Boundedness of the Cesáro operator in Hardy spaces*, J. Fourier Anal. Appl. **128**(1) (2004), 83–92. https://doi.org/10.1007/s00041-004-8005-3

[22] R. L. Rubin, *A* q^2*-analogue operator for* q^2*-analogue Fourier analysis*, J. Math. Anal. App. **212** (1997), 571–582. https://doi.org/10.1006/jmaa.1997.5547

[23] R. L. Rubin, *Duhamel: Solutions of non-homogenous* q^2*-analogue wave equations*, Proc. Amer. Math. Soc. **135**(3) (2007), 777–785.

[24] S. Thangavelyu and Y. Xu, *Riesz transforms and Riesz potentials for the Dunkl transform*, J. Comp. Appl. Math. **199** (2007), 181–195. https://doi.org/10.1016/j.cam.2005.02.022

[25] O. Tyr and R. Daher, *On the q-Bessel transform of Lipschitz and Dini-Lipschitz functions on weighted space* $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, Kragujevac J. Math. **49**(6) (2025), 831–846. https://doi.org/10.46793/KgJMat2506.831T

[26] O. Tyr, F. Saadi and R. Daher, *On the generalized Hilbert transform and weighted Hardy spaces in q-Dunkl harmonic analysis*, Ramanujan J. **60** (2023), 95–122. https://doi.org/10.1007/s11139-022-00666-1

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