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THE GROWTH OF GRADIENTS OF QC-MAPPINGS IN n-DIMENSIONAL EUCLIDEAN SPACE WITH BOUNDED LAPLACIAN

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Abstract. Here we review M. Mateljević's article [9], with some novelities. We focus on mappings between smooth domains which have bounded Laplacian. As an application, if these mappings are quasiconformal, we obtain some results on the behavior of their partial derivatives on the boundary. In the last part of this article, we announce one new result of the author of [9], which has been recently presented on Belgrade Seminary of Complex Analysis.

1. Introduction

In this article, we study quasiconformal mappings in the plane and space, which have a bounded Laplacian. As an application, we get some results which we can consider as spatial versions of Kellogg's theorem. This article is presentation of the part of the article [9]. The author of [9] pointed out that the ideas in that manuscript have been indicated in [6] in planar case and communicated at Workshop on Harmonic Mappings and Hyperbolic Metrics, Chennai, India, December 10–19, 2009 [16], see also paper cited here (in particular [15]) and the literature cited there. In [9], the author developed and proved some results announced and outlined in this communication. The main idea of this article is to present method of, so called, *Flattening the boundary*.

Also, in this article will be stated one new result of the author of [9], which can be regarded as a generalisation of series of previous results in this area. Namely, this result gives positive answer to the question weather qusaiconformal mapping between two $C^{1,\alpha}$ domains, which satisfies so called Laplacian-gradient inequality, is Lipshitz continuous. This

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was subject of interest on Belgrade Seminary of Complex analysis, where M. Mateljević proposed one proof of one more general statement, where answer to above question arises as a corollary.

We write $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, and by $|x|$ we denote Euclidean norm of vector x.

For $R > 0$, by $B(a, R)$ and $S(a, R)$ we denote the ball and the sphere in $\mathbb{R}^n$ with center in a of radius R. By $B(R)$ and $S(R)$ we denote $B(0, R)$ and $S(0, R)$. We use $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ for $B(1)$ and $S(1)$.

Let $\Omega \subset \mathbb{R}^n$, $\mathbb{R}_+ = [0, +\infty)$ and $f, g : \Omega \to \mathbb{R}_+$. If there is a positive constant c such that $f(x) \leqslant cg(x)$, $x \in \Omega$, we write $f \preceq g$ on Ω. If there is a positive constant c such that $\frac{1}{c}g(x) \leqslant f(x) \leqslant cg(x)$, $x \in \Omega$, we write $f \approx g$ (or $f \asymp g$) on Ω.

Let Ω be a domain in $\mathbb{R}^n$ and u a $C^2(\Omega)$ function. The Laplacian (linear) partial differential operator, denoted by Δ, is defined with

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \tag{1.1}$$

We say that function u is (Euclidean) harmonic in Ω if it satisfies *Laplace's equation*

$$\Delta u = 0.$$

Inhomogeneous form of Laplace's equation is called *Poisson's equation.* In this paper we will investigate the following Dirichlet's boundary value problem:

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ \ \ u = \varphi, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Laplace's equation has a radially symmetric solution r^{2-n} for $n > 2$ and $\log r$ for $n = 2$, r being the radial distance from some fixed point. Let us fix a point y in Ω and introduce the normalized *fundamental solution* for Laplace's equation:

$$\Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} \frac{1}{n(2-n)\omega_n} \cdot \frac{1}{|x-y|^{n-2}}, & \text{for } n > 2, \\ \frac{1}{2\pi} \log|x-y|, & \text{for } n = 2, \end{cases} \tag{1.3}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. By simple computation we have that, for every $1 \leqslant i \leqslant n$

$$\frac{\partial}{\partial x_i}\Gamma(x-y) = \frac{1}{n\omega_n} \cdot \frac{x_i - y_i}{|x-y|^n}, \tag{1.4}$$
$$\left|\frac{\partial}{\partial x_i}\Gamma(x-y)\right| \leqslant \frac{1}{n\omega_n} \cdot \frac{1}{|x-y|^{n-1}}.$$

It is convenient to introduce the inversion with respect to the sphere $S(R)$ of the point $y \neq 0$ as

$$J_R(y) = \frac{R^2}{|y|^2} y. \tag{1.5}$$

Sometimes we write y^* instead $J_R(y)$. It is important to notice that $J_R^{-1} = J_R$. Set

$$G_{1,R}(x,\xi) := \Gamma(|x-\xi|) \quad \text{and} \quad G_{2,R}(x,\xi) := -\left(\frac{|\xi|}{R}\right)^{2-n}\Gamma(|x-J_R(\xi)|).$$

We define the Green function for *Dirichlet's problem* on the ball $B(R)$ as

$$\overline{g}_R(x,\xi) := G_{1,R}(x,\xi) + G_{2,R}(x,\xi).$$

The Green function G_Ω for the Dirichlet's problem on the domain Ω in $\mathbb{R}^n$ is chosen to satisfy

$$G_\Omega(x,y) = 0, \quad \text{for } x \in \partial\Omega.$$

For more information about Green functions, see Section 5. The Poisson kernel for the ball B_R is defined by

$$P_R(x,\xi) = \frac{R^2-|x|^2}{n\omega_n R|x-\xi|^n}.$$

When $R=1$, we omit R from the notation. Let us introduce the *Poisson's integral*

$$P[\varphi](x) := \int_{\mathbb{S}^{n-1}} P_R(x,y)\varphi(y)\,\mathrm{d}\sigma(y),$$

and the *Green potential*

$$G[f](x) := \int_{\mathbb{B}^n} \overline{g}_R(x,y)f(y)\,\mathrm{d}\nu(y).$$

2. Gradient Estimate of the Green Potential

We will give a short proof of an important result from [3].

Theorem 2.1. *Assume $u : \overline{B}(R) \to \mathbb{R}$ is continuous, belongs to $C^2(B(R))$, $u=\varphi$ on $S(R)$ and $f = \Delta u$ is bounded and locally Hölder continuous on $B(R)$. Then*

$$u(x) = P_R[\varphi](x) + G_R[f](x). \tag{2.1}$$

Lemma 2.1 ([9]). *If f is a bounded function on $\mathbb{B}^n$, then the partial derivatives of $G[f]$ are continuous on $\overline{\mathbb{B}^n}$.*

Proof. Let us define $u(x) = \int_{\mathbb{B}^n} \overline{g}(x,y)f(y)\,\mathrm{d}\nu(y)$. Then, for every $1 \leqslant i \leqslant n$, we have that

$$\begin{aligned}\frac{\partial}{\partial x_i}u(x) &= \int_{\mathbb{B}^n} \frac{\partial}{\partial x_i}\overline{g}(x,y)f(y)\,\mathrm{d}\nu(y)\\ &= \int_{\mathbb{B}^n} \frac{\partial}{\partial x_i}G_1(x,y)f(y)\,\mathrm{d}\nu(y) + \int_{\mathbb{B}^n} \frac{\partial}{\partial x_i}G_2(x,y)f(y)\,\mathrm{d}\nu(y).\end{aligned}$$

If we define

$$I_{i,1}(x) := \int_{\mathbb{B}^n} \frac{\partial}{\partial x_i}G_1(x,y)f(y)\,\mathrm{d}\nu(y) \quad \text{and} \quad I_{i,2}(x) := \int_{\mathbb{B}^n} \frac{\partial}{\partial x_i}G_2(x,y)f(y)\,\mathrm{d}\nu(y),$$

we have that, for $k=1,2$,

$$I_{i,k}(x) = \int\limits_{|y|\leqslant 1/2} \frac{\partial}{\partial x_i}G_k(x,y)f(y)\,\mathrm{d}\nu(y) + \int\limits_{1/2<|y|\leqslant 1} \frac{\partial}{\partial x_i}G_k(x,y)f(y)\,\mathrm{d}\nu(y).$$

Finally, let us introduce the notation

$$I_{i,k,1}(x)=\int\limits_{|y|\leqslant 1/2}\frac{\partial}{\partial x_i}G_k(x,y)f(y)\,\mathrm{d}\nu(y)\quad\text{ and }\quad I_{i,k,2}(x)=\int\limits_{1/2<|y|\leqslant 1}\frac{\partial}{\partial x_i}G_k(x,y)f(y)\,\mathrm{d}\nu(y).$$

Let us prove that $I_{i,2}$ is continuous on the $\overline{\mathbb{B}^n}$. The proof that $I_{i,2}$ is continuous on the closed unit ball is analogous. It will be suffice to prove that $I_{i,2,k}$ is continuous for $k=1,2$.

Let us consider the function $I_{i,2,1}$ and assume that $|y|<1/2$. Then for all $x\in\overline{\mathbb{B}^n}$, we have that $|x-y^*|\geqslant 1$. Now, using (1.4) we can check that

$$\left|\frac{\partial}{\partial x_i}G_2(x,y)f(y)\right|\preceq\frac{1}{|y|^{n-2}}\cdot\frac{1}{|x-y^*|^{n-1}}\preceq\frac{1}{|y|^{n-2}}.$$

This means that, for every $x_0\in\overline{\mathbb{B}^n}$,

$$\lim_{x\to x_0}\int\limits_{|y|\leqslant 1/2}\frac{\partial}{\partial x_i}G_2(x,y)f(y)\,\mathrm{d}\nu(y)=\int\limits_{|y|\leqslant 1/2}\frac{\partial}{\partial x_i}G_2(x_0,y)f(y)\,\mathrm{d}\nu(y),$$

by Lebesgue dominance convergence theorem. This precisely means that function $I_{i,2,1}$ is continuous at the point x_0.

Now, we need to investigate continuity of the function $I_{i,2,2}$ on the closed unit ball. After introduction change of variable $y=J(z)$, where $J_J(z)$ denotes Jacobian determinant of the mapping J defined as in (1.5) we get

$$I_{i,2,2}(x)=\int\limits_{1<|z|<2}\frac{\partial}{\partial x_i}G_2(x,z^*)f(z^*)J_J(z)\,\mathrm{d}\nu(z).\tag{2.2}$$

After introducing change of variables $x-z=u$ in the integral on the right side of (2.2) we get

$$I_{i,2,2}(x)=\int\limits_{1<|u-x|<2}\frac{\partial}{\partial x_i}G_2(x,(u-x)^*)f((u-x)^*)J_J(u-x)\,\mathrm{d}\nu(u).$$

Again, after using formula (1.4) we get

$$\left|\frac{\partial}{\partial x_i}G_2(x,(u-x)^*)f((u-x)^*)J_J(u-x)\right|\preceq K_2(u):=|u-x|^{n-2}f((u-x)^*)J_J(u-x)\frac{1}{|u|^{n-1}}.$$

Since the function C defined as $C(z):=|z|^{n-2}f(z^*)J_J(z)$ is bounded for $1<|z|<2$, we have that the function C_1, $C_1(u):=C(u-x)$ is bounded on $1<|u-x|<2$ and

$$|K_2(x,u)|\preceq\frac{1}{|u|^{n-1}},\quad\text{for } 1<|u-x|<2.\tag{2.3}$$

If we define the function

$$H(x,u)=\begin{cases}\frac{\partial}{\partial x_i}G_2(x,(u-x)^*)f((u-x)^*)J_J(u-x), & 1<|x-u|<2,\\ 0, & |u|<3, |x-u|<1, |x-u|>2,\end{cases}$$

we have that $I_{i,2,2}(x) = \int\limits_{|u|<3} H(x,u)\,\mathrm{d}\nu(u)$. Using (2.3) we get that

$$\lim_{x\to x_0} \int\limits_{|u|<3} H(x,u)\,\mathrm{d}\nu(u) = \int\limits_{|u|<3} H(x_0,u)\,\mathrm{d}\nu(u), \quad \text{for every } |x_0| \leqslant 1,$$

by Lebesgue dominance convergence theorem, q.e.d. □

3. Local C^2-coordinate Method Flattening the Boundary

Let Ω be open subset of $\mathbb{R}^n$ and $C^k(\Omega)$ the set of functions having all derivatives of order less then or equal to k continuous in Ω. Next, let $C^k(\overline{\Omega})$ be the set of functions in $C^k(\Omega)$ all of whose derivatives of order less than or equal to k have continuous extensions to $\overline{\Omega}$.

Let $x_0 \in D$, where D is bounded subset of $\mathbb{R}^n$ and f is function defined on D. For $0 < \alpha < 1$, we say that f is *Hölder continuous* with exponent α at x_0, if

$$\sup_{x\in D} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} < +\infty.$$

When $\alpha = 1$, we say that f is Lipschitz-continuous at x_0.

Suppose that D is not necessarily bounded. We say that f is *uniformly Hölder continuous* with exponent α in D if

$$\sup_{\substack{x,y\in D,\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < +\infty, \quad 0 < \alpha < 1.$$

Let Ω be an open set in $\mathbb{R}^n$ and k a non-negative integer. The *Hölder spaces* $C^{k,\alpha}(\Omega)$ and $C^{k,\alpha}(\overline{\Omega})$ are defined as the subspaces of $C^k(\Omega)$, resp. $C^k(\overline{\Omega})$, consisting of functions whose k-th order partial derivatives are locally Hölder continuous (uniformly Hölder continuous) with exponent α in Ω. By $\mathrm{Lip}(\Omega)$ we denote class of function which are Lipshitz continuous on the set Ω.

Definition 3.1. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ belongs to the class $C^{k,\alpha}$, where $0\leqslant\alpha\leqslant 1$, $k\in\mathbb{N}$, if its boundary belongs to the class $C^{k,\alpha}$, i.e., if for every point $x_0 \in \partial\Omega$ there exists a ball $B = B(x_0, r_0)$ and a mapping $\psi : B \to D$ such that (cf. [3, page 95])

(a) $\psi(B \cap \Omega) \subset \mathbb{R}^n_+$;
(b) $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$;
(c) $\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$.

We refer to ψ as a local coordinate diffeomorphism *flattening the boundary* in a neighborhood of x_0.

Proposition 3.1. *ψ is bi-Lipshitz on $B_1 \subset B$ if $k \geqslant 1$. Also, $\left|\frac{\partial^2}{\partial x_i \partial x_j}\psi\right|$, $1\leqslant i, j\leqslant n$, are bounded for $k\geqslant 2$.*

In [9] the following lemma is proved. This lemma is an improvement of a similar result from [5], where only boundedness of the first partial derivatives on $\mathbb{B}^n$ is concluded.

Lemma 3.1. ([9, Claim 6]) *Let* $u : \overline{\mathbb{B}^n} \to \mathbb{R}$ *be a solution to the following Dirichlet's problem*

$$\begin{cases} \Delta u = f, & \text{in } \mathbb{B}^n, \\ u = \varphi, & \text{on } \mathbb{S}^{n-1}, \end{cases} \tag{3.1}$$

where $f \in L^\infty(\mathbb{B}^n)$ *and* $\varphi \in C^{1,\alpha}(\mathbb{S}^{n-1})$, $0 < \alpha < 1$. *Then* $u \in C^1(\overline{\mathbb{B}^n})$.

Lemma 3.2. ([9, Local Gradient Lemma Version 1, Lemma 2.2]) *For* $x_0 \in \mathbb{S}^{n-1}$ *and* $0 < r_0 < 2$ *let* $V_0 = \mathbb{B}^n \cap B(x_0, r_0)$, $V(r) = \mathbb{B}^n \cap B(x_0, r)$ *for* $0 < r < r_0$ *and* $T_0 = \mathbb{S}^{n-1} \cap B(x_0, r_0)$. *If* $u \in C^2(V_0) \cap C(V_0 \cup T_0)$ *is such that* $\Delta u \in L^\infty(V_0)$ *and* $u \in C^{1,\alpha}(T_0)$, *then*

$$\nabla u \in L^\infty(V(r)), \quad \text{for all } 0 < r < r_0.$$

Lemma 3.2 can be regarded as a local version of Lemma 3.1.

4. Quasiconformal and Quasiregular Mappings

Let D, D' and Ω be a domains in $\mathbb{R}^n$. If $f : D \to D'$ and $y = f(x)$ we write $y_i = f_i(x)$, $1 \leqslant i \leqslant n$.

Definition 4.1. (1) Suppose that $f : D \to D'$ is a differentiable mapping at point $x \in D$. By $f'(x) : T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^n$ we denote the differential of the mapping f at point x, which can be identified with the matrix $\left(\frac{\partial}{\partial x_j} f_i(x)\right)$, and by $T_x\mathbb{R}^n$ we denote the tangent space at point x. We define

$$|f'(x)| = \max_{|h|=1} |f'(x)h| \quad \text{and} \quad l(f'(x)) = \min_{|h|=1} |f'(x)h|.$$

(N) A homeormophism $f : D \to D'$ satisfies the condition (N) if $m(A) = 0$ implies $m(f(A)) = 0$. Here, by $f(A)$ we denote direct image of the set A by function f.

(2) A homeormophism $f : D \to D'$ is a K-*quasiconformal* (in the analytic sense) if f is absolutely continuous on lines, f is differentable a.e. in D and $|f'(x)|^n \leqslant K|J(x, f)|$ a.e. on D.

(3) Let $f : \Omega \to \mathbb{R}^n$ be continuous. We say that f is *quasiregular* if

(a) f belongs to Sobolev space $W^n_{1,\mathrm{loc}}(\Omega)$;

(b) there exists K, $1 \leqslant K < +\infty$, such that

$$|f'(x)|^n \leqslant K J_f(x) \text{ a.e.} \tag{4.1}$$

The smallest K in (4.1) is called the *outer dilatation* $K_O(f)$.

If f is quasiregular, then

$$J_f(x) \leqslant K' l(f'(x))^n \text{ a.e.} \quad \text{for some } K',\ 1 \leqslant K' < +\infty. \tag{4.2}$$

The smallest K' in (4.2) is called the *inner dilatation* $K_I(f)$ and $K(f) = \max(K_O(f), K_I(f))$ is called the *maximal diletation* of f. If $K(f) \leqslant K$, then f is called K-*quasiregular*. Here, we will only state a few basic results.

(i_1) If mapping $f : D \to D'$ is a qc, then mapping f^{-1} is a qc and both satisfies the (N) condition.

(i_2) (Change of variables) If mapping $f : D \to D'$ is a qc, and A is a measurable subset of D, then the set $f(A)$ is a measurable, and

$$m(f(A)) = \int_A |J_f(x)| \, \mathrm{d}\nu(x).$$

Furthermore, $J_f(x) \neq 0$ almost everywhere in D.

(i_3) (Reshetnyak's main theorem) Every non-constant quasiregular map is discrete and open.

In [21], it can be seen that, when qc mapping f is differentiable at point x, only two possibilities can emerge. Either $J_f(x) \neq 0$ either $f'(x) = 0$. It can be checked in, for example [20], that, in case $J_f(x) \neq 0$, $|f'(x)|$ and $l(f'(x))$ can be regarded as the greatest and the least *singular values* of non-singular matrix $f'(x)$.

Proposition 4.1. *If $f : D \to D'$ is a quasiconformal mapping, we have that*

$$l(f'(x)) \leqslant |\nabla f_i(x)| \leqslant |f'(x)|.$$

Proof. Let $x \in D$ and $\nabla f_i(x) \neq 0$. Then we have that $\nabla f_i(x) = f'(x)^T e_i$, where $f'(x)^T$ is (Euclidean) tanspose of matrix $f'(x)$ and $e_i = (0 \dots, 0, 1, 0, \dots, 0)$ is i-th coordinate vector, $1 \leqslant i \leqslant n$. Since, both non-singular matrix and it's transpose have the same singular values, we conclude that $l(f'(x)) \leqslant |\nabla f(x)| \leqslant |f'(x)|$. □

Theorem 4.1. ([9, Theorem 2.1]) *Let $D \subset \mathbb{R}^n$ be a C^2 domain and $f : \mathbb{B}^n \to D$ a C^2 K-qc mapping. If $\Delta f \in L^\infty(\mathbb{B}^n)$, then $f \in \mathrm{Lip}(\mathbb{B}^n)$.*

Proof. Let D, D' be domains in $\mathbb{R}^n$ and $f : D \to D'$ and $h : D' \to \mathbb{R}$ be C^2 functions and set $\hat{h} = h \circ f$. If $y = f(x)$ and $f_k(x) := y_k, 1 \leqslant k \leqslant n$, the following formulas hold:

$$\frac{\partial}{\partial x_k}\hat{h} = \sum_{i=1}^{n} \frac{\partial h}{\partial y_i}\frac{\partial f_i}{\partial x_k}, \tag{4.3}$$

and

$$\Delta \hat{h} = \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} |\nabla f_i|^2 + 2 \sum_{1 \leqslant i < j \leqslant n} \frac{\partial^2 h}{\partial x_i \partial x_j} \langle \nabla f_i, \nabla f_j \rangle + \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} \Delta f_i. \tag{4.4}$$

Using Lemma 3.2, we get that $\tilde{f}_n$ is Lipshitz continuous in some neighbourhood V_0 of the point x_0. Next, Proposition 4.1 gives us that the whole function $\tilde{f}$ is Lipshitz continuous on V_0. Now, using Proposition 3.1 we get that function ψ is locally bi-Lipshitz, so $f = \psi^{-1} \circ f$ is Lipshitz continuous on V_0. From this we easily conclude that the function f is Lipshitz continuous on entire ball $\mathbb{B}^n$. See Figure 1. □

5. Further Results

Let D be a domain in $\mathbb{R}^n$ and $s : D \to R$. If

$$|\Delta s| \leqslant a|\nabla s|^2 + b, \quad \text{on } D,$$

then we say that s satisfies a, b - Laplacian-gradient inequality on D.

In [9] the author also studied the growth of gradient of mappings which satisfy certain PDE equations (or inequalities) using the Green-Laplacian formula for functions and their

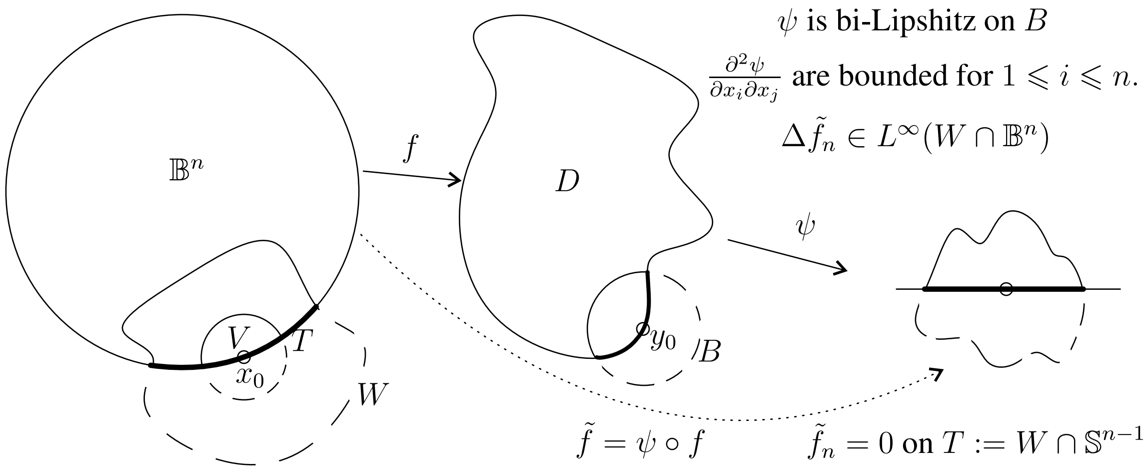

FIGURE 1. Flattening the boundary

derivatives. If in addition the considered mappings are quasiconformal (qc) between C^2 domains, M. Mateljević showed that they are Lipschitz. Some of the obtained results can be considered as versions of Kellogg-Warshawski type theorem for qc mappings. More precisely, developing further methods from Heinz paper [4]. See also Kalaj [5].

Theorem 5.1 ([9], Theorem D.)**.** *Hypothesis:*

(1) *Let Ω be a domain in $\mathbb{R}^n$ with C^2 boundary and $f : \mathbb{B}^n \xrightarrow{onto} \Omega$ be a C^2 mapping, which has continuous extension on $\overline{\mathbb{B}^n}$.*
(2) *Suppose that f satisfy the Laplacian-gradient inequality on $B_0 = B(z_0, r_0) \cap \mathbb{B}^n$, where $x_0 \in \mathbb{S}^{n-1}$ and $r_0 > 0$ and f maps $B_0 \cap \mathbb{S}^{n-1}$ into $\partial\Omega$.*

Conclusion (VIII) (a) *There is $0 < r_1 < r_0$, $c > 0$ and a unit vector fields X on $V_1 = B(x_0, r_1) \cap \mathbb{B}^n$ (i.e., to each x we associate a unit vector $h = h(x)$ with initial point at x) such that $|\,\mathrm{d}f(x)h(x)| \leqslant c$ for every $x \in V_1$.*

(b) *If in addition f is qc in B_0, then f is Lipschitz continuous on V_1.*

It is also important to note following theorem, proved in [9]. The proof of this theorem is based on, so called "bootstrap" argument, which simplifies method used in [4]. For more about this, see [7–9].

Theorem 5.2 ([9] Theorem 3.2.)**.** *Suppose the hypothesis* (1) *of the previous theorem holds, f is open and*

(3) *Δf is in $L^p(B_0)$ for some $p > n$ and*
(4) *f maps $B_0 \cap \mathbb{S}^{n-1}$ into $\partial\Omega$.*

Then Conclusion (VIII) (a) *holds.*

(c) *If in addition f is qr in B_0, then f is Lipschitz continuous on V_1.*

In [9] the author stated the following conjecture.

Proposition 5.1 ([9], Conjecture A.). *Let D and D_0 be $C^{1,\alpha}$, $0<\alpha<1$, domains (bounded) in $\mathbb{R}^n$ and $f : D \xrightarrow{onto} D_0$ a C^2 $K-qc$. If f satisfies the Laplacian-gradient inequality on D (in particular Δf is bounded), then f is Lip on D.*

It seems that local special C^2-coordinate method which works for C^2 domains needs to be modified. Namely if we work with $C^{1,\alpha}$, $0<\alpha<1$, domains the local coordinate ψ is $C^{1,\alpha}$, where $0<\alpha<1$, and therefore in general $\tilde{f}$ does not satisfy Laplacian-gradient inequality.

In the recent article [2] D. Kalaj and A. Gjokaj proved that
(i) if there is a $C^{1,\alpha}$ diffeomorphism $\phi : \overline{\mathbb{B}^n} \to \overline{D}$ and
(ii) f is a harmonic quasiconformal mapping between the unit ball in $\mathbb{R}^n$ and D,
then f is Lipschitz continuous in $\mathbb{B}^n$.

This generalizes some known results for $n=2$ and improves some others in high dimensional case. Here we note that condition (i) is stronger than $C^{1,\alpha}$ condition on the domain D.

In the article in preparation [10] the author (of [10]) proposed proves of Theorem 5.3. This shows that conjecture proposed in Proposition 5.1 is true, under additional condition on the domain D.

Definition 5.1.

1. A function $g_D(x,\xi)$ defined on $\overline{D}\times D$ with the following properties:
   (1) g_D is harmonic in x in D except for $x=\xi$,
   (2) g_D is continuous in $\overline{D}$ except for $x=\xi$ and $g=0$ on ∂D,
   (3) $g_D - |x-\xi|^{2-n}$ is harmonic for $x=\xi$,
   is called Green's function for D.
2. In mathematics, a function between topological spaces is called proper if inverse images of compact subsets are compact.
3. We say that D is good Green-ian domain if $\left|\frac{\partial}{\partial x_k} g_D(x,y)\right| \leqslant c\frac{1}{|x-y|^{n-1}}$, $x,y\in D$, $k=1,\ldots,n$, for some $c>0$ and locally good Green-ian domain at $x_0\in\partial D$ if for every $\delta>0$ there is a C^{1+} domain $W=W_{x_0}\subset D\cap B(x_0,\delta)$ such that $x_0\in\partial W$ and ∂W is an open set in ∂D.
4. Domain D has a C^{1+} boundary if there exists $\alpha\in(0,1)$ such that D has $C^{1,\alpha}$ boundary.
5. D is a locally good Green-ian domain, if it is a locally good Green-ian domain at every $x_0\in\partial D$.
6. Let $SC^1(G)$ be the class of functions $f\in C^1(G)$ such that $|f'(x)|\leqslant ar^{-1}\omega_f(x,r)$ for all $B(x,r)\subset G$, where $\omega_f(x,r)=\sup\{|f(y)-f(x)| : y\in B(x,r)\}$.

It seems that K. Widman proved that $C^{1,\alpha}$ domains are examples of good Green-ian domains. For more details see [22].

Let $d(x)=d_D(x)$, $x\in D$, be the distance of point x to the boundary of D.

Question 1. If D and G are domains with C^{1+} boundary, $f : D \xrightarrow{\text{onto}} G$ a C^2 and $K - qc$ and f_i, $i = 1, 2, \ldots, n$, satisfes Laplacian-gradient inequality (in particular f is harmonic) on D. Whether f is Lip on D?

Here, we state the following theorem, which proof will be omitted at this point.

Set $\gamma = 1 - \alpha$, $A = A_\gamma := d(z)^{-\gamma}$, $B = B_\gamma := |f'(z)|^{-\gamma}$ and $M = M_\gamma := AB$.

Theorem 5.3 ([10]). *Suppose that:*

(1) *D and G are domains with C^{1+} boundary, D is locally good Green-ian domain, $f : D \xrightarrow{onto} G$ proper and there is p such that $|\nabla f| \in L^p$, $p > n$,*
(2) *$f \in SC^1(D)$.*
(3) *Suppose in addition that G is $C^{1,\alpha}$ domain, f is a C^2 vector valued function, f_i, satisfy Laplacian-gradient inequality on D for $i = 1, 2, \ldots, n$.*
(4) *f is $K - qc$ on D.*

Conclusions:

(a) *If (1) holds, then $M_\gamma \in L^l$, for $l < l_0 = \frac{p}{2-\gamma+p\gamma}$.*
(b) *If (1)–(4) hold, then f is Lipschitz continuous on D.*

In the paper [17] is proved that harmonic quasi-regularity of function f implies condition (2) of the previous theorem. It is important to note that condition (2) is equivalent with Lipshitz continuty of function f wrt to quasi-hyperbolic metric.

6. Appendix

6.1. **Belgrade Seminary of Complex Analysis.** For detailed presentation of the eariest hystory of analysis school in Serbia, see article [13]. At this point, it is important to mention prof. Dajović, as the initiator of today Seminary. After the retirement of Professor Dajović, the group for complex analysis (M. Mateljević, M. Pavlović, M. Jevtić, M. Obradović) considered problems related to the spaces of analytic functions and slowly achieved international reputation. After returning from the USA in 1990, prof. Mateljević started working with N. Lakić in the field of quasiconformal mappings. Lakić soon left for the USA and obtained significant results in the field of Teichmüller spaces. The Seminary for Complex Analysis gains an international reputation, and there is talk of the Belgrade School (On conferences: Reich, Krushkal, Cazacu, Stanojević and others, especially Olli Martio during a visit to Belgrade in 2009.). It seemed that complex analysis had reached its highest point in Belgrade. But the surprises continue. V. Marković and V. Božin appear at the seminar. Together with Mateljević and Lakić, they solve Teichmüller's problem of extremal dilations. Today, V. Marković (In 2014, he was elected a member of the British Royal Society.) is a world leader in qc mapping theory and 3-dimensional topology and geometry and a proffessor at Univercity of Oxford. D. Kalaj and D. Šarić become internationaly recognized. D. Vukotić and N. Šešum also started at the seminar. Currently, M. Marković, M. Knežević, M. Svetlik, N. Mutavdžić, B. Karapetrović, P. Melentijević are actively participating in the seminar. For the seminary (or for the complex analysis group) are also connected M. Jevtić, M. Pavlović, M. Arsenović, S. Stević, I. Anić, M. Laudanović, O. Mihić, V. Manojlović, N. Babačev, A. Abaob, A. Shkheam, D. Đurčić, A. Bulatović, I. Petrović, S. Nikčević, V. Grujić as well as the students who presented on optional courses: J. Gajić, I. Savković, A. Savić, D. Fatić, M. Milović, N. Lelas, M. Lazarević, V. Stojisavljević, S. Gajović, F. Živanović, D. Kosanović, D. Špadijer, Z. Golubović and S. Radović. D. Kečkić, R. Živaljević, D. Milinković, D. Jocić, Đ. Milićević, D. Damjanović, D. Ranković, V. Baltić, N. Jozić (Baranović) and M. Albianić.

We briefly mention some facts related to the beginning of work on hqc mapping in Belgrade. During the visiting position at Wayne State University, Detroit, 1988/89, the author (of the revised article [9]) started considering hqc mappings. In particular, the author of [9] observed that the following results hold (see Proposition 6.1 and 6.2 below) and, when returned to Belgrade, used to talk on the seminary permanently and asked several open questions related to the subject. Many research papers are based on these communications.

Since not all of these researches have been published, it happens that some researchers discovered them later. Here we only discuss a few results from Revue Roum. Math. Pures Appl. **51**(5–6) (2006), 711–722.

Proposition 6.1 (Proposition 5 [11]). *If h is a harmonic univalent orientation preserving K-qc mapping of domain D onto D', then*

$$d(z)\Lambda_h(z) \leqslant 16\,K\,d_h(z) \quad \text{and} \quad d(z)\lambda_h(z) \geqslant \frac{1-k}{4} d_h(z)\,. \tag{6.1}$$

Proposition 6.2 (Corollary 1, Proposition 5 [11]). *Every e-harmonic quasi-conformal mapping of the unit disc* (*more generally of a strongly hyperbolic domain*) *is a quasi-isometry with respect to hyperbolic distances.*

The next theorem concerns harmonic maps onto a convex domain. For the planar version of Theorem 6.1 cf. [11, 12], also [18, pp. 152-153]. The space version was communicated on International Conference on Complex Analysis and Related Topics (Xth Romanian-Finnish Seminar, August 14-19, 2005, Cluj-Napoca, Romania), by Mateljević and stated in [11], also [14].

Theorem 6.1 (Theorem 1.3, [11]). *Suppose that h is an Euclidean harmonic mapping from the unit ball $\mathbb{B}^n$ onto a bounded convex domain $D = h(\mathbb{B}^n)$, which contains the ball $h(0) + R_0\mathbb{B}^n$. Then for any $x \in \mathbb{B}^n$*

$$d(h(x), \partial D) \geqslant (1 - \|x\|)R_0/2^{n-1}.$$

For further results of this type, see [14, 17], and the literature cited there.

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