Kragujevac Journal of Mathematics
Volume 41(2) (2017), Pages 251–258.

SOLUTION OF A PARTIAL DIFFERENTIAL EQUATION RELATED TO THE OPERATOR $\oplus_B^k$

S. BUPASIRI[1]

Abstract. In this paper, we consider the equation

$$\oplus_B^k u(x) = \sum_{r=o}^{m} c_r \oplus_B^r \delta,$$

where $\oplus_B^k$ is the operator iterated k-time and is defined by

$$\oplus_B^k = \left[\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^4 - \left(B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}}\right)^4\right]^k,$$

where $p+q=n, x=(x_1,\ldots,x_n) \in \mathbb{R}_n^+$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \ldots, n$, c_r is a constant, k is a nonnegative integer, δ is the Dirac-delta distribution, $\oplus_B^0 \delta = \delta$ and n is the dimension of $\mathbb{R}_n^+$. It is shown that, depending on the relationship between k and m, the solution to this equation can be ordinary functions, tempered distributions, or singular distributions.

1. Introduction

Bupasiri [5] has first introduced the elementary solution of the n-dimensional $\oplus_B^k$ operator and showed that the solution of the convolution form $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ is a unique elementary solution of the $\oplus_B^k u(x) = \delta$.

Key words and phrases. Bessel diamond operator, O-plus operator, Dirac-delta distribution.
2010 *Mathematics Subject Classification.* Primary: 46F10. Secondary: 46F20.
Received: August 7, 2016.
Accepted: October 27, 2016.

Yildirim, Sarikaya and Ozturk [3] studied the Bessel diamond operator, iterated k-times,

$$\diamondsuit_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right] \tag{1.1}$$

$$= \left[\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right]^k \left[\sum_{i=1}^{p} B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j}\right]^k.$$

Yildirim, Sarikaya and Ozturk [3] showed that the function $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution for the operator $\diamondsuit_B^k$, where $*$ indicates convolution, and $R_{2k}(x), S_{2k}(x)$ are defined by (1.4) and (1.5) respectively, that is,

$$\diamondsuit_B^k \left((-1)^k S_{2k}(x) * R_{2k}(x)\right) = \delta(x).$$

We consider the equation

$$\oplus_B^k u(x) = \sum_{r=o}^{m} c_r \oplus_B^r \delta,$$

where $\oplus_B^k$ is the operator iterated k-time and is defined by

$$\oplus_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^4 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^4\right]^k \tag{1.2}$$

$$= \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k$$

$$= \diamondsuit_B^k \circledcirc_B^k,$$

where

$$\circledcirc_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k$$

$$= \left[\left(\frac{\triangle_B + \Box_B}{2}\right)^2 + \left(\frac{\triangle_B - \Box_B}{2}\right)^2\right]^k$$

$$= \left(\frac{\triangle_B^2 + \Box_B^2}{2}\right)^k.$$

The purpose of this article, is finding the solution to the equation

$$\oplus_B^k u(x) = \sum_{r=0}^{m} c_r \oplus_B^r \delta \tag{1.3}$$

by using convolutions of the generalized function. It is also shown that the type of solution to (1.3) depends on the relationship between k and m, according to the following cases:

(1) If $m < k$ and $m = 0$, then (1.3) has the solution
$$u(x) = c_0 \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right),$$
which is an elementary solution of the $\oplus_B^k$ operator in Theorem 2.2, is an ordinary function when $6k \geq n$, and is a tempered distribution when $6k < n$.

(2) If $m < k$ then the solution of (1.3) is
$$u(x) = \sum_{r=1}^{m} c_r c_0 \left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right),$$
which is an ordinary function when $6k - 6r \geq n$ and is tempered distribution when $6k - 6r < n$.

(3) If $m \geq k$ and $k \leq m \leq M$, then (1.3) has the solution
$$u(x) = \sum_{r=k}^{M} c_r \oplus_B^{r-k} \delta,$$
which is only a singular distribution. Before going that point, the following definitions and some concepts are needed.

Lemma 1.1. *Given the equation $\square_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+ = \{x : x = (x_1, \cdots x_n), x_1 > 0, \ldots, x_n > 0\}$, where $\square_B^k$ is the Bessel-ultra hyperbolic operator iterated k-times. Then $u(x) = R_{2k}(x)$ is an elementary solution of the operator $\square_B^k$, where*
$$\square_B^k = \left[\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right]^k,$$
$$R_{2k}(x) = \frac{V^{\frac{2k-n-|v|}{2}}}{K_n(2k)} = \frac{\left(x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2\right)^{\left(\frac{2k-n-|v|}{2}\right)}}{K_n(2k)}, \tag{1.4}$$
for
$$V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2,$$
and
$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma\left(\frac{2+2k-n-2|v|}{2}\right) \Gamma\left(\frac{1-2k}{2}\right) \Gamma(2k)}{\Gamma\left(\frac{2+2k-p-2|v|}{2}\right) \Gamma\left(\frac{p+2|v|-2k}{2}\right)}.$$

Lemma 1.2. *Given the equation $\triangle_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\triangle_B^k$ is the Laplace Bessel operator iterated k-times. Then $u(x) = (-1)^k S_{2k}(x)$ is an elementary solution of the operator $\triangle_B^k$, where*

$$\triangle_B^k = \left[\sum_{i=1}^{p} B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j}\right]^k,$$

$$S_{2k}(x) = \frac{|x|^{2k-n-2|v|}}{w_n(2k)}, \quad p+q=n, \tag{1.5}$$

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}},$$

and

$$w_n(2k) = \frac{\Pi_{i=1}^n 2^{v_i-\frac{1}{2}}\Gamma\left(v_i+\frac{1}{2}\right)\Gamma(k)}{2^{n+2|v|-4k}\Gamma\left(\frac{n+2|v|-2k}{2}\right)}.$$

Lemma 1.3. *The convolution $R_{2k}(x) * (-1)^k S_{2k}(x)$ is an elementary solution for the operator $\diamondsuit_B^k$ iterated k-times and is defined by* (1.1)*.*

Lemma 1.4. *$R_{2k}(x)$ and $S_{2k}(x)$ are homogeneous distributions of order $(2k-n-2|v|)$.*

We need to show that $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$ satisfy the Euler equation; that is,

$$(2k-n-2|v|)\, R_{2k}(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_{2k}(x),$$

$$(2k-n-2|v|)\, S_{2k}(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} S_{2k}(x).$$

Lemma 1.5 (The B-convolution of tempered distribution)**.** *$R_{2k}(x) * S_{2k}(x)$ exists and is a tempered distribution.*

Proof. For the proofs of Lemmas 1.1–1.5, see [3, p. 378–383]. □

Lemma 1.6 (The B-convolution of $R_{2k}(x)$ and $S_{2k}(x)$)**.** *Let $R_{2k}(x)$ and $S_{2k}(x)$ defined by* (1.4) *and* (1.5) *respectively, then we obtain:*

(1) *$S_{2k}(x) * S_{2m}(x) = S_{2k+2m}(x)$, where k and m are nonnegative integers.*
(2) *$R_{2k}(x) * R_{2m}(x) = R_{2k+2m}(x)$, where k and m are nonnegative integers.*

Lemma 1.7. *The function $R_{-2k}(x)$ and $(-1)^k S_{-2k}(x)$ are the inverse in the convolution algebra of $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$, respectively. That is,*

$$R_{-2k}(x) * R_{2k}(x) = R_{-2k+2k}(x) = R_0(x) = \delta(x),$$

$$(-1)^k S_{-2k}(x) * (-1)^k S_{2k}(x) = S_{-2k+2k}(x) = S_0(x) = \delta(x).$$

Proof. For the proofs of Lemma 1.7 and Lemma 1.6, see [4]. □

Lemma 1.8. *Given the equation*

$$\oplus_B^k u(x) = \delta(x), \tag{1.6}$$

where $\oplus_B^k$ is the operator iterated k-times defined by (1.2), $\delta(x)$ is the Dirac-delta distribution, $x \in \mathbb{R}_n^+$ and k is a nonnegative integer. Then we obtain

$$u(x) = \left(R_{6k}(x) * (-1)^{3k} S_{6k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{1.7}$$

is a Green's function or an elementary solution for the operator $\oplus_B^k$ iterated k-times where $\oplus_B^k$ is defined by (1.2), and

$$C(x) = \frac{1}{2} R_4(x) + \frac{1}{2}(-1)^2 S_4(x), \tag{1.8}$$

where $C^{*k}(x)$ denotes the convolution of C with itself k times, $\left(C^{*k}(x)\right)^{*-1}$ denotes the inverse of $C^{*k}(x)$ in the convolution algebra. Moreover $u(x)$ is a tempered distribution.

Proof. For a proof of the above lemma, see [5]. □

2. Main Results

Theorem 2.1. *For $0 < r < k$,*

$$\begin{aligned} &\oplus_B^k \left(c_0\left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right)\right) \\ &= \left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right) \end{aligned}$$

and for $k \leq m$,

$$\oplus_B^m \left(c_0\left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right)\right) = \oplus_B^{m-k} \delta.$$

Proof. For $0 < r < k$, from (1.6),

$$\oplus_B^k \left(c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right) = \delta.$$

Thus,

$$\oplus_B^{k-r} \oplus_B^r \left(c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right) = \delta$$

or

$$\oplus_B^{k-r} \delta * \oplus_B^r \left(c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right) = \delta.$$

Convolving both sides by $\left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right)$, we obtain

$$\begin{aligned} &\oplus_B^{k-r} \left(\left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right)\right) \\ &* \oplus_B^r \left(c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right) \\ &= \left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right) * \delta. \end{aligned}$$

By Lemma 1.8,

$$\begin{aligned} &\delta * \oplus_B^r \left(c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right) \\ &= \left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right) * \delta. \end{aligned}$$

It follows that

$$\begin{aligned} &\oplus_B^r \left(c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right) \\ &= \left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right), \end{aligned}$$

as required. For $k \leq m$

$$\begin{aligned}&\oplus_B^m \left(c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right) \\ &= \oplus_B^{m-k} \oplus_B^k \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right).\end{aligned}$$

It follows that

$$\oplus_B^m \left(c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right) = \oplus_B^{m-k} \delta$$

by Lemma 1.8. This completes the proof. □

Theorem 2.2. *Consider the linear differential equation*

$$\oplus_B^k u(x) = \sum_{r=0}^{m} c_r \oplus_B^r \delta, \tag{2.1}$$

where $p + q = n$, n is odd with p odd and q even, or n is even with p odd and q odd, $x \in \mathbb{R}_n^+ = \{x : x = (x_1, \ldots, x_n), x_1 > 0, \ldots, x_n > 0\}$, c_r is a constant, δ is the Dirac-delta distribution, and $\oplus_B^0 \delta = \delta$. Then the type of solution to (2.1) depends on the relationship between k and m, according to the following cases:

(1) *If $m < k$ and $m = 0$, then (2.1) has the solution*

$$u(x) = c_0 \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right),$$

which is an elementary solution of the $\oplus_B^k$ operator in Theorem 2.1, is an ordinary function when $6k \geq n$, and is a temper distribution when $6k < n$.

(2) *If $m < k$ then the solution of (2.1) is*

$$u(x) = \sum_{r=1}^{m} c_r \left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right),$$

which is an ordinary function when $6k - 6r \geq n$ and is tempered distribution when $6k - 6r < n$.

(3) *If $m \geq k$ and $k \leq m \leq M$, then (2.1) has the solution*

$$u(x) = \sum_{r=k}^{M} c_r \oplus_B^{r-k} \delta,$$

which is only a singular distribution.

Proof. (1) For $m = 0$, we have $\oplus_B^k u(x) = c_0 \delta$, and by Theorem 2.1 we obtain

$$u(x) = \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right).$$

Now, $(-1)^{3k} S_{6k}(x)$ and $R_{6k}(x)$ are the analytic function for $6k \geq n$ and also $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ exits and is an analytic function by (1.7). It follows that $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ is an ordinary function for $6k \geq n$. By Lemma 1.5, $(-1)^{3k} S_{6k}(x)$, $R_{6k}(x)$ are tempered distributions with $6k < n$, we obtain $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ exits and is a tempered distribution.

(2) For the case $0 < m < k$, we have

$$\oplus_B^k u(x) = c_1 \oplus_B \delta + c_2 \oplus_B^2 \delta + \cdots + c_m \oplus_B^m \delta.$$

We convolved both sides of the above equation by $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ to obtain

$$\begin{aligned}
&\oplus_B^k \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right) * u(x)\\
=& c_1 \oplus_B \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right)\\
&+ c_2 \oplus_B^2 \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right)\\
&+ \cdots + c_m \oplus_B^m \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right).
\end{aligned}$$

By Theorem 2.1, we obtain

$$\begin{aligned}
u(x) =& c_1 \left((-1)^{3(k-1)} S_{6(k-1)}(x) * R_{6(k-1)}(x) * (C^{*(k-1)}(x))^{*-1}\right)\\
&+ c_2 \left((-1)^{4(k-2)} S_{6(k-2)}(x) * R_{6(k-2)}(x) * (C^{*(k-2)}(x))^{*-1}\right)\\
&+ \cdots + c_m \left((-1)^{3(k-m)} S_{6(k-m)}(x) * R_{6(k-m)}(x) * (C^{*(k-m)}(x))^{*-1}\right),
\end{aligned}$$

or

$$u(x) = \sum_{r=1}^{m} c_r \left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right).$$

Similarly, as in case (1), $u(x)$ is an ordinary function for $6k - 6r \geq n$ and is a tempered distribution for and $6k - 6r < n$.

(3) For the case $m \geq k$ and $k \leq m \leq M$, we have

$$\oplus_B^k u(x) = c_k \oplus_B^k \delta + c_{k+1} \oplus_B^{k+1} \delta + \cdots + c_M \oplus_B^M \delta.$$

Convolved both sides of the above equation by

$$(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$$

to obtain

$$\begin{aligned}
&\oplus_B^k \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right) * u(x)\\
=& c_k \oplus_B^k \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right)\\
&+ c_{k+1} \oplus_B^{k+1} \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right)\\
&+ \cdots + c_M \oplus_B^M \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right).
\end{aligned}$$

By Theorem 2.1 again, we obtain

$$u(x) = c_k \delta + c_{k+1} \oplus_B \delta + c_{k+2} \oplus_B^2 \delta + \cdots + c_M \oplus_B^{M-k} \delta = \sum_{r=k}^{M} c_r \oplus_B^{r-k} \delta.$$

Since $\oplus_B^{r-k} \delta$ is a singular distribution, hence $u(x)$ is only the singular distribution. This completes the proofs. $\square$

Acknowledgements. The author would like to thank the referees for their suggestions which enhanced the presentation of the paper. The author was supported by Sakon Nakhon Rajabhat University.

References

[1] A. H. Zemanian, *Distribution and Transform Analysis,* McGraw-Hill, New York, 1965.
[2] B. M. Levitan, *Expansion in Fourier series and integrals with Bessel functions*, Uspekhi Mat. Nauk **6**(42), (1951), 102–143.
[3] H. Yildirim, M. Z. Sarikaya and S. Ozturk, *The solutions of the n-dimensional Bessel diamond operator and the Fourier-Bessel transform of their convolution*, Proc. Indian Acad. Sci. Math. Sci. **114**(4) (2004), 375–387.
[4] M. Z. Sarikaya and H. Yildirim, *On the B-convolutions of the Bessel diamond kernel of Riesz*, Appl. Math. Comput. **208** (2009), 18–22.
[5] S. Bupasiri, *On the solution of the n-dimensional $\oplus_B^k$ operator*, Applied Mathematical Sciences **9**(10), (2015), 469–479.
[6] W. Satsanit, *Green function and Fourier transform for o-plus operator*, Electron. J. Differential Equations **2010** (2010), Article ID: 48.

[1]Faculty of Education
Sakon Nakhon Rajabhat University
Thailand 47000
E-mail address: sudprathai@gmail.com