

## WHEN IS A BI-JORDAN HOMOMORPHISM BI-HOMOMORPHISM?

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**ABSTRACT.** For Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$ , we show that if  $\mathcal{U} = \mathcal{A} \times \mathcal{B}$  is commutative (weakly commutative), then each bi-Jordan homomorphism from  $\mathcal{U}$  into a semisimple commutative Banach algebra  $\mathcal{D}$  is a bi-homomorphism. We also prove the same result for 3-bi-Jordan homomorphism with the additional hypothesis that the Banach algebra  $\mathcal{U}$  is unital.

### 1. INTRODUCTION

Let  $\mathcal{A}$  and  $\mathcal{B}$  be complex Banach algebras and  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  be a linear map. Then  $\varphi$  is called an  $n$ -homomorphism if for all  $a_1, a_2, \dots, a_n \in \mathcal{A}$ ,

$$\varphi(a_1 a_2 \dots a_n) = \varphi(a_1) \varphi(a_2) \dots \varphi(a_n).$$

The concept of an  $n$ -homomorphism was studied for complex algebras by Hejazian et al. in [5]. A 2-homomorphism is then just a homomorphism, in the usual sense. One may refer to [1] for certain properties of 3-homomorphisms.

The notion of  $n$ -Jordan homomorphisms was dealt with firstly by Herstein in [6]. A linear map  $\varphi$  between Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$  is called an  $n$ -Jordan homomorphism if

$$\varphi(a^n) = \varphi(a)^n, \quad a \in \mathcal{A}.$$

A 2-Jordan homomorphism is called simply a Jordan homomorphism.

It is obvious that each  $n$ -homomorphism is an  $n$ -Jordan homomorphism, but in general the converse is false. The converse statement may be true under certain conditions. For example, it is shown in [2] that every  $n$ -Jordan homomorphism between two commutative Banach algebras is an  $n$ -homomorphism for  $n \in \{2, 3, 4\}$ ,

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and this result extended to the case  $n = 5$  in [3]. Lee in [7] generalized this result and proved it for all  $n \in \mathbb{N}$ . See also [4] for another proof of Lee's theorem.

Zelazko in [9] has given a characterization of Jordan homomorphism, that we mention in the following (see also [8]). We refer to [10] for another approach to the same result.

**Theorem 1.1.** *Suppose that  $\mathcal{A}$  is a Banach algebra, which need not be commutative, and suppose that  $\mathcal{B}$  is a semisimple commutative Banach algebra. Then each Jordan homomorphism  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is a homomorphism.*

Also it is shown in [11] that Theorem 1.1 is valid for 3-Jordan homomorphism with the extra condition that the Banach algebra  $\mathcal{A}$  is unital. Some significant results concerning Jordan homomorphisms and their generalizations on Banach algebras obtained by the author in [12].

Throughout the paper, let  $\mathcal{U} = \mathcal{A} \times \mathcal{B}$ . Then  $\mathcal{U}$  is a Banach algebra for the multiplication

$$(a, b)(x, y) = (ax, by), \quad (a, b), (x, y) \in \mathcal{U},$$

and with norm

$$\|(a, b)\| = \|a\| + \|b\|.$$

Let  $\mathcal{D}$  be a complex Banach algebra. A bilinear map is a function  $\varphi : \mathcal{U} \rightarrow \mathcal{D}$  such that for any  $a \in \mathcal{A}$  the map  $b \mapsto \varphi(a, b)$  is linear map from  $\mathcal{B}$  to  $\mathcal{D}$ , and for any  $b \in \mathcal{B}$  the map  $a \mapsto \varphi(a, b)$  is linear map from  $\mathcal{A}$  to  $\mathcal{D}$ .

A bilinear map  $\varphi$  is called an  $n$ -bi-homomorphism if for all  $(a_i, b_i) \in \mathcal{U}$ ,

$$\varphi(a_1 a_2 \dots a_n, b_1 b_2 \dots b_n) = \varphi(a_1, b_1) \varphi(a_2, b_2) \dots \varphi(a_n, b_n),$$

and it is called an  $n$ -bi-Jordan homomorphism if

$$\varphi(a^n, b^n) = \varphi(a, b)^n, \quad (a, b) \in \mathcal{U}.$$

The concept of an  $n$ -bi-Jordan homomorphism introduced by the author in [13]. A (2-bi-Jordan) 2-bi-homomorphism is called simply a (bi-Jordan) bi-homomorphism.

It is obvious that each  $n$ -bi-homomorphism is  $n$ -bi-Jordan homomorphism, but in general the converse is not true.

Recently, the author proved [13] that every bi-Jordan homomorphism from unital commutative Banach algebra  $\mathcal{U}$  into a semisimple commutative Banach algebra  $\mathcal{D}$  is a bi-homomorphism.

In this paper, we extended this result for nonunital Banach algebra  $\mathcal{U}$ . We also prove the same result for 3-bi-Jordan homomorphism with the additional hypothesis that the Banach algebra  $\mathcal{U}$  is unital.

## 2. CHARACTERIZATION OF BI-JORDAN HOMOMORPHISMS

The following Theorem is the generalization of Theorem 4 of [13].

**Theorem 2.1.** *Every bi-Jordan homomorphism  $\varphi$  from commutative Banach algebra  $\mathcal{U}$  into a semisimple commutative Banach algebra  $\mathcal{D}$  is a bi-homomorphism.*

*Proof.* We first assume that  $\mathcal{D} = \mathbb{C}$  and let  $\varphi : \mathcal{U} \rightarrow \mathbb{C}$  be a bi-Jordan homomorphism. Then for all  $(a, b) \in \mathcal{U}$ ,

$$(2.1) \quad \varphi(a^2, b^2) = \varphi(a, b)^2.$$

Replacing  $a$  by  $a + x$  and  $b$  by  $b + y$  in (2.1), gives

$$(2.2) \quad \varphi(a^2 + x^2 + 2ax, b^2 + y^2 + 2by) = \varphi(a + x, b + y)^2.$$

By Lemma 1 of [13], for all  $(a, b), (x, y) \in \mathcal{U}$  we have

$$(2.3) \quad \varphi(a^2, by) = \varphi(a, b)\varphi(a, y) \quad \text{and} \quad \varphi(ax, b^2) = \varphi(a, b)\varphi(x, b).$$

It follows from (2.2) and (2.3) that

$$(2.4) \quad 2\varphi(ax, by) = \varphi(a, b)\varphi(x, y) + \varphi(a, y)\varphi(x, b),$$

for all  $(a, b), (x, y) \in \mathcal{U}$ . Take  $I = \varphi(a, b)\varphi(x, y)$ ,  $J = \varphi(a, y)\varphi(x, b)$  and  $t = I - J$ . Then we get

$$(2.5) \quad t^2 = I^2 + J^2 - 2IJ, \quad 4\varphi(ax, by)^2 = I^2 + J^2 + 2IJ.$$

By (2.4) and (2.5), we deduce

$$\begin{aligned} 4\varphi(ax, by)^2 + t^2 &= 2(I^2 + J^2) \\ &= 2[\varphi(a, b)^2\varphi(x, y)^2 + \varphi(a, y)^2\varphi(x, b)^2] \\ &= 2[\varphi(a^2, b^2)\varphi(x^2, y^2) + \varphi(a^2, y^2)\varphi(x^2, b^2)] \\ &= 4\varphi(a^2x^2, b^2y^2) \\ &= 4\varphi(ax, by)^2. \end{aligned}$$

Hence,  $t = 0$ , which proves that  $I = J$ . Thus, by (2.4) we have

$$\varphi(ax, by) = \varphi(a, b)\varphi(x, y),$$

for all  $(a, b), (x, y) \in \mathcal{U}$ , so  $\varphi$  is a bi-homomorphism.

Now suppose that  $\mathcal{D}$  is semisimple and commutative. Let  $\mathfrak{M}(\mathcal{D})$  be the maximal ideal space of  $\mathcal{D}$ . We associate with each  $f \in \mathfrak{M}(\mathcal{D})$  a function  $\varphi_f : \mathcal{U} \rightarrow \mathbb{C}$  defined by

$$\varphi_f(a, b) := f(\varphi(a, b)), \quad (a, b) \in \mathcal{U}.$$

Pick  $f \in \mathfrak{M}(\mathcal{D})$  arbitrary. It is easy to see that  $\varphi_f$  is a bi-Jordan homomorphism, so by the above argument it is a bi-homomorphism. Thus, by the definition of  $\varphi_f$  we have

$$f(\varphi(ax, by)) = f(\varphi(a, b))f(\varphi(x, y)) = f(\varphi(a, b)\varphi(x, y)).$$

Since  $f \in \mathfrak{M}(\mathcal{D})$  was arbitrary and  $\mathcal{D}$  is assumed to be semisimple,

$$\varphi(ax, by) = \varphi(a, b)\varphi(x, y),$$

for all  $(a, b), (x, y) \in \mathcal{U}$ . This complete the proof.  $\square$

A bilinear map  $\varphi : \mathcal{U} \rightarrow \mathcal{D}$  is called co-bi-homomorphism if

$$\varphi(ax, by) = -\varphi(a, b)\varphi(x, y),$$

for all  $(a, b), (x, y) \in \mathcal{U}$ , and it is called co-bi-Jordan homomorphism if

$$\varphi(a^2, b^2) = -\varphi(a, b)^2, \quad (a, b) \in \mathcal{U}.$$

By a same method as Theorem 2.1, we have the following result for co-bi-Jordan homomorphisms.

**Theorem 2.2.** *Every co-bi-Jordan homomorphism from commutative Banach algebra  $\mathcal{U}$  into a semisimple commutative Banach algebra  $\mathcal{D}$  is a co-bi-homomorphism.*

We say that the Banach algebra  $\mathcal{A}$  is weakly commutative if

$$(ax)^2 = a^2x^2 \quad \text{and} \quad ax^2a = x^2a^2,$$

for all  $a, x, \in \mathcal{A}$ . Clearly, every commutative Banach algebra is weakly commutative, but in general, the converse is false. For example, let

$$\mathcal{A} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Then it is obvious to check that with the usual matrix product for all  $x, y \in \mathcal{A}$ ,

$$(xy)^2 = x^2y^2 \quad \text{and} \quad xy^2x = y^2x^2.$$

Thus,  $\mathcal{A}$  is weakly commutative, but it is neither unital nor commutative.

Note that a unital Banach algebra is weakly commutative if and only if it is commutative.

**Lemma 2.1.** *Let  $\mathcal{U}$  be a weakly commutative Banach algebra, and  $\varphi : \mathcal{U} \rightarrow \mathbb{C}$  be a bi-Jordan homomorphism. Then*

$$\varphi(ax, by) = \varphi(ax, yb) = \varphi(xa, by),$$

for all  $(a, b), (x, y) \in \mathcal{U}$ .

*Proof.* By Lemma 1 of [13],

$$(2.6) \quad \varphi(a^2, by + yb) = 2\varphi(a, b)\varphi(a, y), \quad (a, b), (a, y) \in \mathcal{U}.$$

Replacing  $a$  by  $ax$  in (2.6) we get

$$(2.7) \quad \varphi((ax)^2, by + yb) = 2\varphi(ax, b)\varphi(ax, y).$$

Replacing  $b$  by  $by$  and  $y$  by  $yb$  in (2.7), gives

$$(2.8) \quad \varphi((ax)^2, by^2b + yb^2y) = 2\varphi(ax, by)\varphi(ax, yb).$$

Since  $\mathcal{U}$  is weakly commutative, by (2.8) we have

$$\begin{aligned} 2\varphi(ax, by)\varphi(ax, yb) &= \varphi((ax)^2, by^2b + yb^2y) \\ &= \varphi((ax)^2, y^2b^2 + b^2y^2) \\ &= \varphi((ax)^2, b^2y^2) + \varphi((ax)^2, y^2b^2) \\ &= \varphi(ax, by)^2 + \varphi(ax, yb)^2. \end{aligned}$$

Thus,

$$\left(\varphi(ax, by) - \varphi(ax, yb)\right)^2 = 0,$$

which proves that

$$\varphi(ax, by) = \varphi(ax, yb),$$

for all  $(a, b), (x, y) \in \mathcal{U}$ . In a similar way, we can prove that  $\varphi(ax, by) = \varphi(xa, by)$ . This complete the proof.  $\square$

The next result is the generalization of Theorem 2.1.

**Theorem 2.3.** *Suppose that  $\varphi$  is a bi-Jordan homomorphism from weakly commutative Banach algebra  $\mathcal{U}$  into a semisimple commutative Banach algebra  $\mathcal{D}$ . Then  $\varphi$  is a bi-homomorphism.*

*Proof.* We first assume that  $\mathcal{D} = \mathbb{C}$  and let  $\varphi : \mathcal{U} \rightarrow \mathbb{C}$  be a bi-Jordan homomorphism. Then for all  $(a, b) \in \mathcal{U}$ ,

$$(2.9) \quad \varphi(a^2, b^2) = \varphi(a, b)^2.$$

Replacing  $a$  by  $a + x$  and  $b$  by  $b + y$  in (2.9), gives

$$(2.10) \quad \varphi(ax + xa, by + yb) = 2\varphi(a, b)\varphi(x, y) + 2\varphi(a, y)\varphi(x, b),$$

for all  $(a, b), (x, y) \in \mathcal{U}$ . It follows from (2.10) and Lemma 2.1 that

$$\begin{aligned} 4\varphi(ax, by) &= \varphi(ax + xa, by + yb) \\ &= 2\varphi(a, b)\varphi(x, y) + 2\varphi(a, y)\varphi(x, b). \end{aligned}$$

Hence,

$$2\varphi(ax, by) = \varphi(a, b)\varphi(x, y) + \varphi(a, y)\varphi(x, b),$$

for all  $(a, b), (x, y) \in \mathcal{U}$ . Thus, the relation (2.4) in Theorem 2.1 holds. Now the rest of proof is similar to the proof of Theorem 2.1.  $\square$

As a consequence of Theorem 2.3 we have the next result.

**Corollary 2.1.** *Suppose that  $\mathcal{U}$  is weakly commutative and  $\varphi : \mathcal{U} \rightarrow \mathbb{C}$  satisfies*

$$(2.11) \quad |\varphi(ax, by) - \varphi(a, b)\varphi(x, y)| \leq \delta(\|(a, b)\| + \|(x, y)\|),$$

for all  $(a, b), (x, y) \in \mathcal{U}$  and for some  $\delta \geq 0$ . Then  $\varphi$  is a bi-homomorphism.

*Proof.* Replacing  $(x, y)$  by  $(a, b)$  in (2.11), gives

$$(2.12) \quad |\varphi(a^2, b^2) - \varphi(a, b)^2| \leq 2\delta(\|a\| + \|b\|),$$

for all  $(a, b) \in \mathcal{U}$ . Take  $a = 2^n x$  and  $b = 2^n y$  in (2.12), then

$$|\varphi(x^2, y^2) - \varphi(x, y)^2| \leq \frac{2^{n+1}\delta(\|x\| + \|y\|)}{2^{4n}} \rightarrow 0,$$

as  $n \rightarrow \infty$ . Hence,

$$\varphi(x^2, y^2) = \varphi(x, y)^2, \quad (x, y) \in \mathcal{U}.$$

Therefore,  $\varphi$  is a bi-Jordan and so it is a bi-homomorphism by Theorem 2.3. □

*Example 2.1.* Let

$$\mathcal{U} = \left\{ \left( \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \right) : a, b, x, y \in \mathbb{R} \right\}.$$

Then  $\mathcal{U}$  is a weakly commutative Banach algebra, but it is not commutative. Hence by Theorem 2.3, each bi-Jordan homomorphism from  $\mathcal{U}$  into a semisimple commutative Banach algebra  $\mathcal{D}$  is a bi-homomorphism and via versa.

The commutativity of Banach algebra  $\mathcal{D}$  in Theorem 2.3 is essential. For example, let

$$\mathcal{A} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\},$$

as above and let  $\mathcal{A}^\sharp$  be the unitization of  $\mathcal{A}$  with the identity matrix as a unit. Set  $\mathcal{U} = \mathcal{A} \times \mathcal{A}^\sharp$  and define  $\varphi : \mathcal{U} \rightarrow \mathcal{A}$  by  $\varphi(x, y) = xy$ . Then for all  $(x, y) \in \mathcal{U}$ ,

$$\varphi(x^2, y^2) = \varphi(x, y)^2.$$

Hence  $\varphi$  is bi-Jordan homomorphism, but it is not bi-homomorphism. Because, let

$$x = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}, \quad m = \begin{bmatrix} s & t \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad n = I,$$

where  $I$  is the identity matrix. Then  $(x, y), (m, n) \in \mathcal{U}$ , but

$$\varphi(xm, yn) \neq \varphi(x, y)\varphi(m, n).$$

### 3. CHARACTERIZATION OF 3-BI-JORDAN HOMOMORPHISMS

Clearly, the Banach algebra  $\mathcal{U}$  is unital if and only if both  $\mathcal{A}$  and  $\mathcal{B}$  are unital. Without any confusion we denote by  $e$ , the unit element of both  $\mathcal{A}$  and  $\mathcal{B}$ .

**Lemma 3.1.** *Let  $\mathcal{U}$  be a unital commutative Banach algebra, and  $\varphi : \mathcal{U} \rightarrow \mathbb{C}$  be a 3-bi-Jordan homomorphism. Then for all  $(a, b) \in \mathcal{U}$ ,*

- (a)  $\varphi(a^3, b^2 + b) = \varphi(a, b)^2\varphi(a, e) + \varphi(a, b)\varphi(a, e)^2,$
- (b)  $\varphi(a^2 + a, b^3) = \varphi(a, b)^2\varphi(e, b) + \varphi(a, b)\varphi(e, b)^2.$

*Proof.* The proof is straightforward. □

**Lemma 3.2.** *By the hypotheses of above Lemma, for all  $(a, b), (x, y) \in \mathcal{U}$ ,*

- (a)  $3\varphi(ax^2, b) = \varphi(a, b)\varphi(x, e)^2 + 2\varphi(a, e)\varphi(x, e)\varphi(x, b)$ ,  
 (b)  $3\varphi(a, by^2) = \varphi(a, b)\varphi(e, y)^2 + 2\varphi(e, b)\varphi(e, y)\varphi(a, y)$ .

*Proof.* We prove (a), that the assertion (b) can be proved similarly. Let  $\varphi : \mathcal{U} \rightarrow \mathbb{C}$  be a 3-bi-Jordan homomorphism. Then for all  $(a, b) \in \mathcal{U}$ ,

$$(3.1) \quad \varphi(a^3, b^3) = \varphi(a, b)^3.$$

Replacing  $b$  by  $b + y$  in (3.1), gives

$$(3.2) \quad \varphi(a^3, b^2y + by^2) = \varphi(a, b)^2\varphi(a, y) + \varphi(a, b)\varphi(a, y)^2,$$

for all  $(a, b), (a, y) \in \mathcal{U}$ . Replacing  $y$  by  $-y$  in (3.2), we get

$$(3.3) \quad \varphi(a^3, -b^2y + by^2) = -\varphi(a, b)^2\varphi(a, y) + \varphi(a, b)\varphi(a, y)^2.$$

By (3.2) and (3.3) we have

$$(3.4) \quad \varphi(a^3, by^2) = \varphi(a, b)\varphi(a, y)^2.$$

Replacing  $y$  by  $e$  in (3.4), gives

$$(3.5) \quad \varphi(a^3, b) = \varphi(a, b)\varphi(a, e)^2.$$

Replacing  $a$  by  $a + x$  in (3.5), to obtain

$$(3.6) \quad 3\varphi(ax^2 + a^2x, b) = I + J,$$

where

$$I = \varphi(x, b)\varphi(a, e)^2 + 2\varphi(a, b)\varphi(a, e)\varphi(x, e)$$

and

$$J = \varphi(a, b)\varphi(x, e)^2 + 2\varphi(a, e)\varphi(x, b)\varphi(x, e).$$

Replacing  $x$  by  $-x$  in (3.6), we get

$$(3.7) \quad 3\varphi(ax^2 - a^2x, b) = -I + J.$$

By (3.6) and (3.7) we have

$$3\varphi(ax^2, b) = \varphi(a, b)\varphi(x, e)^2 + 2\varphi(a, e)\varphi(x, b)\varphi(x, e),$$

for all  $(a, b), (x, e) \in \mathcal{U}$ , as required.  $\square$

Now we state and prove the main Theorem of this section.

**Theorem 3.1.** *Suppose that  $\varphi$  is a 3-bi-Jordan homomorphism from unital commutative Banach algebra  $\mathcal{U}$  into  $\mathbb{C}$ . Then  $\varphi$  is a 3-bi-homomorphism.*

*Proof.* Let  $\varphi : \mathcal{U} \rightarrow \mathbb{C}$  be a 3-bi-Jordan homomorphism. Then

$$(3.8) \quad \varphi(a^3, b^3) = \varphi(a, b)^3, \quad (a, b) \in \mathcal{U}.$$

Replacing both of  $a$  and  $b$  by  $e$ , gives  $\varphi(e, e) = \varphi(e, e)^3$ . Since  $\varphi(e, e) \neq 0$ , so  $\varphi(e, e) = 1$  or  $\varphi(e, e) = -1$ . We first assume that  $\varphi(e, e) = 1$ . Replacing  $a$  by  $a + e$  and  $b$  by  $b + e$  in (3.8), and simplifies the result by Lemma 3.1, we get

$$(3.9) \quad 9\varphi(a^2 + a, b^2 + b) = 3\{\varphi(a, b)^2 + \varphi(a, b) + P + Q + R + S\},$$

where

$$\begin{aligned} P &= 2\varphi(a, b)\varphi(a, e) + \varphi(e, b)\varphi(a, e)^2, & Q &= 2\varphi(a, b)\varphi(e, b) + \varphi(a, e)\varphi(e, b)^2, \\ R &= 2\varphi(a, e)\varphi(e, b), & S &= 2\varphi(a, e)\varphi(a, b)\varphi(e, b). \end{aligned}$$

It follows from preceding Lemma that for all  $(a, b) \in \mathcal{U}$ ,

$$(3.10) \quad P = 3\varphi(a^2, b), \quad Q = 3\varphi(a, b^2), \quad R = 2\varphi(a, b) \quad \text{and} \quad S = 2\varphi(a, b)^2.$$

By (3.9) and (3.10) we obtain

$$\varphi(a^2, b^2) = \varphi(a, b)^2,$$

for all  $(a, b) \in \mathcal{U}$ . Hence,  $\varphi$  is bi-Jordan homomorphism and so it is bi-homomorphism by Theorem 2.1. Thus,  $\varphi$  is 3-bi-homomorphism.

Now suppose that  $\varphi(e, e) = -1$ . Then by a similar argument we have

$$\varphi(a^2, b^2) = -\varphi(a, b)^2, \quad (a, b) \in \mathcal{U}.$$

Therefore by Theorem 2.2,  $\varphi$  is co-bi-homomorphism. That is,

$$\varphi(ax, by) = -\varphi(a, b)\varphi(x, y),$$

for all  $(a, b), (x, y) \in \mathcal{U}$ . Thus,

$$\begin{aligned} \varphi(axu, byv) &= -\varphi(a, b)[\varphi(xu, yv)] \\ &= -\varphi(a, b)[- \varphi(x, y)\varphi(u, v)] \\ &= \varphi(a, b)\varphi(x, y)\varphi(u, v), \end{aligned}$$

for all  $(a, b), (x, y), (u, v) \in \mathcal{U}$ . So  $\varphi$  is 3-bi-homomorphism, as claimed.  $\square$

As a consequence of Theorem 3.1 we have the next result.

**Corollary 3.1.** *Suppose that  $\varphi$  is a 3-bi-Jordan homomorphism from unital commutative Banach algebra  $\mathcal{U}$  into a semisimple commutative Banach algebra  $\mathcal{D}$ . Then  $\varphi$  is a 3-bi-homomorphism.*

In view of Theorem 1.1 and Theorem 2.1, the following question suggests itself: does Theorem 2.1 hold without commutativity of  $\mathcal{U}$ ?

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