Kragujevac Journal of Mathematics
Volume 51(5) (2027), Pages 721–741.

PROBABILITY LOGIC WITH APPROXIMATE JACCARD SIMILARITY INDEX

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Abstract. The paper introduces a logical system whose language represents an enriched propositional calculus with three lists of binary operators applied to propositional formulas: $J_{\leqslant r}(\alpha, \beta)$, $J_{\geqslant r}(\alpha, \beta)$, and $J_{\approx q}(\alpha, \beta)$. These represent, respectively, "the Jaccard similarity index of formulas α and β is at most r", "the Jaccard similarity index of formulas α and β is at least r", and "the Jaccard similarity index of formulas α and β is approximately q". For the specified axiomatic system, an appropriate completeness theorem is proven, with respect to possible-world semantics with probability measure on sets of worlds. Completeness is proved by using two infinitary inference rules, one of which enables us to syntactically define the range of the similarity function. This range is chosen to be the unit interval of a recursive non-Archimedean field, allowing the expression of approximate similarity measures.

1. Introduction

Inferences derived from uncertain, vague, or imprecise data have been conducted since Leibniz's time, and likely even earlier. More recently, approaches to inference linked to computing and artificial intelligence, closely tied with mathematical logic, have been developed. Mathematical logic provides many answers to questions concerning inference, becoming more efficient when it interacts with probability theory or other theoretical frameworks. Recent interactions among these fields have led to the formation of formal systems that serve as powerful tools for reasoning under uncertainty. This interaction was initiated by Nils Nilsson [20], who, using a semantic

Key words and phrases. Probabilistic logic, Jaccard index, Approximate similarity, Infinitesimal, Completeness

2020 *Mathematics Subject Classification.* Primary: 03B60. Secondary: 03B48, 03B70, 68T27, 68T37.

DOI

Received: November 14, 2024.

Accepted: December 23, 2025.

approach, introduced methods for calculating bounds on the absolute probability of conclusions based on the probabilities of premises. Such methods have, over recent decades, motivated many researchers to develop semantic and proof-theoretic approaches with absolute probabilities, beginning with propositional calculus enhanced by "probability operators" that act like modal operators. The semantics is based on specialized Kripke models with finitely additive or σ-additive probability measures defined over possible worlds. A primary goal of any axiomatic system adopting this approach is to ensure the completeness of the system, as accomplished in many probabilistic logics (see [3–5, 8–10, 13, 14, 21–23, 25–28], and references therein). Besides interacting with probability functions, logical systems in recent years have also interacted with distance functions (see [7, 30], and references therein), where the objectives of such systems mirror those in probabilistic logics.

When organizing data into clusters, mathematical tools are essential for examining the qualitative attributes of each observed data point. Clustering is based on principles that often focus on data similarity properties (in terms of specific characteristics) or distance properties (see [6, 16, 18, 19, 32, 34]). Despite the focus initiated, clustering methods vary depending on the nature of the data and the mathematical concepts representing the observed data set. For instance, the similarity between two vectors is often determined by the value of their dot product. Many similarity indices are used across research studies, and the so-called Jaccard similarity index, known for its simplicity in value determination, attracts particular attention (see [15, 31]). Due to the few operations involved in calculating the Jaccard similarity index, it has numerous applications (see [11, 17, 24, 35]). The Jaccard similarity index ranges from 0 to 1, where a value of 0 indicates complete dissimilarity between the observed sets, and a value of 1 indicates complete similarity (total overlap) between the observed sets. In many studies, the Jaccard index is interpreted as the number of elements present in both sets divided by the total number of elements appearing in either set. The Jaccard index (denoted as J) can be expressed by the formula: $J(A, B) = \frac{|A \cap B|}{|A \cup B|}$, where A and B are the data sets being compared, $|A \cap B|$ denotes the number of elements in the intersection of the sets, and $|A \cup B|$ denotes the number of elements in the union of the observed sets. As is known, this is not the only form of the Jaccard index. In fact, the problem can be generalized to consider any arbitrary measure, including the probability measure, which will be the subject of our research in the following sections. The probability measure μ can be used to calculate the Jaccard similarity index for comparing sets A and B. This can be done as follows:

$$J(A, B) = \begin{cases} \dfrac{\mu(A \cap B)}{\mu(A \cup B)}, & \mu(A \cup B) > 0, \\ 1, & \mu(A \cup B) = 0. \end{cases}$$

Considering the results presented in [12], where events with infinitesimal probabilities are described-events that are mutually distinct and to which we devoted attention in Section 2 - and relying on studies focusing on probabilistic logics (see

[13,14,21,22,25,27,28]), we introduce a system that employs infinitary inference rules, i.e., rules where a conclusion is derived from a countable set of premises. Therefore, proofs can be countable, but all formulas, axioms, and theorems are finite. One of the infinitary rules enables us to syntactically define the similarity index rank that will appear in the interpretation. We force this rank to lie within the unit interval of a recursive non-Archimedean field, which includes all rational numbers as well as values infinitesimally close to or infinitely near a rational number. For example, such a set of values includes Hardy's field $\mathbb{Q}[\varepsilon]$, where ε is an infinitesimal. This rule was introduced following a rule given in [1], and our aim is for the axiomatic system introduced here to be sound and strongly complete. In this work, we enrich the language of propositional logic with operators of the Jaccard similarity index: $J_{\leqslant r}(\alpha, \beta)$ and $J_{\geqslant r}(\alpha, \beta)$, meaning "the Jaccard similarity index of formulas α and β is at most r" and "the Jaccard similarity index of formulas α and β is at least r", respectively. It is characteristic that usual probability operators of the form $P_{\geqslant r}\alpha$ appear as a special case of Jaccard similarity index operators, namely, $P_{\geqslant r}\alpha = J_{\geqslant r}(\alpha, \top)$, where $\top$ is any tautology. Since we have defined the similarity index rank as non-Archimedean, it is also possible to introduce operators $J_{\approx q}(\alpha, \beta)$, meaning "the Jaccard similarity index of formulas α and β is approximately q," thus enabling the representation of knowledge about approximately similar formulas. As a consequence, we can use a formula like $J_{\approx 1}(\alpha, \beta)$, which can be interpreted as "the formulas α and β are almost identical". Besides representing knowledge about similar formulas, this logic also allows for discussions about the similarity index of formulas with zero probabilities. Specifically, if the probability of formula $\alpha \wedge \beta$ equals ε and the probability of formula $\alpha \vee \beta$ equals 3ε, then the formulas α and β are similar with a similarity index of $\frac{1}{3}$.

The results obtained during the study of the observed axiomatic system are presented in the paper as follows. Section 3 outlines the syntax of the logic under consideration. Section 4 defines the corresponding models for the examined logic, while Section 5 presents the axiomatic system with a finite number of axioms and inference rules. For this axiomatic system, Section 6 provides a proof of the completeness theorem, as well as the properties necessary for its proof. In Section 7, we summarize the obtained results and propose directions for future research related to this axiomatic system.

2. Similarity of Events with Infinitesimal Probability

In the analysis of infinite sequences of random events, such as the idealized process of infinitely tossing a fair coin, a key question concerns the way probabilities are assigned to individual outcomes. In standard probability theory, each individual infinite sequence (e.g., a sequence consisting entirely of heads) is assigned probability zero. This is a direct consequence of the uncountability of the set of all infinite sequences over H, T (H-head, T-tail) and the regularity of the measure: if all singletons have positive measure, the total sum must diverge.

However, the principle of regularity - which holds that only impossible events may have zero probability - has motivated the development of alternative approaches. One of them involves the use of non-standard analysis, established by Abraham Robinson in the mid-20th century [29]. He demonstrated the existence of an extension of the real number line that includes infinitesimal and infinitely large numbers. Members of this extension are known as hyperreal numbers, and particularly interesting are elements of the hyperreal unit interval, which include positive infinitesimal numbers as potential probability values.

Within this framework, it is possible to assign each individual outcome - for instance, each infinite sequence of heads and tails - the same infinitesimal probability, thereby preserving the principle of regularity. This avoids the issue that almost all singletons receive zero probability, as occurs in standard measure theory. For example, the event $H(1\ldots)$, denoting that all tosses result in heads, receives a positive infinitesimal probability, while the event $H(2\ldots)$, in which heads appear from the second position onward (with no constraint on the first toss), encompasses more sequences and may also have an infinitesimal probability. In this way, the inclusion relation between events and their respective probabilities is preserved.

The principle of regularity has attracted the attention of many researchers, and Williamson, in his work [33], argued that even within the framework of hyperreal probabilities, regularity must fail. He considers the same case of infinite coin tossing and claims that at least one sequence must have probability zero. Williamson's argument sounds persuasive and is based on the assumption that individual outcomes can be represented as "isomorphic events" with respect to the standard setup. A proper analysis, however, shows that Williamson's conclusion rests on a misconception regarding the isomorphism of events and on imprecise notation, and it also contradicts the results of the study [2], which he himself cites.

A critique of the reasoning, logic, and conclusions presented in [33] was offered in [12]. Specifically, Howson also examines the events $H(1\ldots)$ and $H(2\ldots)$, but draws a series of conclusions, starting from the observation that the event $H(1\ldots)$ represents a singleton, as it refers to exactly one sequence of outcomes - namely, the sequence $(H, H, H, \ldots)$ in which a head appears at every position. In contrast, the event $H(2\ldots)$ is a composite event, i.e., a set containing two possible sequences: one where the first toss is a head followed by all heads, $(H, H, H, \ldots)$, and another where the first toss is a tail followed by all heads, $(T, H, H, \ldots)$. Thus, the event $H(2\ldots)$ represents a disjunction: "either the first toss is a head and the rest are heads, or the first toss is a tail and the rest are heads." The key difference between $H(1\ldots)$ and $H(2\ldots)$ lies in their cardinality: $H(1\ldots)$ contains one element, while $H(2\ldots)$ contains two. Williamson's error, as pointed out in [12], lies in attempting to use isomorphism between these events to argue that they have the same probability. However, a singleton cannot be isomorphic to a two-element set - cardinal equivalence is a necessary condition for isomorphism. Invoking isomorphism to equate their probabilities is logically flawed and constitutes a central weakness in Williamson's reasoning.

Therefore, it is not justifiable to claim that $\mu(H(1\ldots)) = \mu(H(2\ldots))$. In addition to concluding that the observed events are not isomorphic, it is also concluded that they do not have equal probabilities, not even in nonstandard (hyperreal) probability models. This distinction is crucial, as it undermines Williamson's attempt to show that the regularity principle - that only impossible events can have probability zero - must fail even within the hyperreal framework.

The use of the unit interval of hyperrational numbers from Hardy's field allows probabilistic quantities, including infinitesimals, to be expressed in a manner compatible with rational approximations. At the same time, it enables the comparison of probabilities of events such as $H(1\ldots)$ and $H(2\ldots)$, and allows for the formalization of the inequality $\mu(H(1\ldots)) < \mu(H(2\ldots))$, even though both values are infinitesimal. This field is sufficiently rich to encompass all functional values that arise from the enumeration of infinite sequences, while also being structured enough to support operations with hyperrational measures over event spaces. Therefore, the choice of the unit interval of Hardy's field of hyperrational numbers is not merely a technical convenience, but a mathematically justified solution that enables the preservation of regularity, quantification of differences between infinite events, and axiomatization of probabilistic logics with infinitesimal structures.

Indeed, in the hyperrational (non-standard) probability framework, the event $H(2\ldots)$ is more probable than $H(1\ldots)$, because $H(1\ldots)$ refers to exactly one infinite sequence - the one consisting entirely of heads. As a singleton in an uncountably large space of all possible toss sequences, its probability in standard probability theory is exactly zero. In a nonstandard theory (with hyperrational numbers), it can be assigned a positive but infinitesimally small probability - an infinitesimal ε. The event $H(2\ldots)$ includes two distinct sequences previously described. Although both are singleton events individually, the set of two elements, within the hyperrational framework, is assigned a probability equal to the sum of the probabilities of its members - that is, a probability on the order of two infinitesimals, 2ε. This is still infinitesimally small, but strictly greater than the probability of $H(1\ldots)$.

In the paper [12], the events $H(1\ldots)$ and $H(2\ldots)$ are discussed as distinct, though no explicit statement is made regarding their similarity or the degree of their difference. Initially, given the assigned probabilities $\mu(H(1\ldots)) = \varepsilon$ and $\mu(H(2\ldots)) = 2\varepsilon$, we can determine the probability of the intersection of these two events. However, if heads occurred from the very first position, then they certainly occurred from the second position onward as well, so we can conclude that $H(1\ldots) \subseteq H(2\ldots)$, and thus $H(1\ldots) \cap H(2\ldots) = H(1\ldots)$. It follows that $\mu(H(1\ldots) \cap H(2\ldots)) = \varepsilon$, and we can easily compute that $\mu(H(1\ldots) \cup H(2\ldots)) = 2\varepsilon$. Now that we have determined the probabilities of the union and the intersection of the events, we can calculate the Jaccard similarity index for $H(1\ldots)$ and $H(2\ldots)$. That is,

$$J(H(1\ldots), H(2\ldots)) = \frac{\mu(H(1\ldots) \cap H(2\ldots))}{\mu(H(1\ldots) \cup H(2\ldots))} = \frac{1}{2}.$$

Thus, although the events $H(1\ldots)$ and $H(2\ldots)$ have infinitesimal probabilities, they are indeed distinct events and have a similarity index of $\frac{1}{2}$. Clearly, making such a statement within the framework of standard probability theory would involve significant difficulties. In a similar manner, we can compute the Jaccard similarity index for the events $H(1\ldots)$ and $H(3\ldots)$, as well as for $H(1\ldots)$ and $H(4\ldots)$, and observe that this index decreases, indicating that these events differ more from each other than the events $H(1\ldots)$ and $H(2\ldots)$.

3. Syntax

The set of rational numbers $\mathbb{Q}$, although powerful for data representation, has certain limitations. Specifically, it is not possible to find a positive number q in $\mathbb{Q}$ such that $q < \frac{1}{n}$ for every natural number n. To overcome this, one may extend $\mathbb{Q}$ by adding an infinitesimal element ε to obtain the smallest ordered field that contains all rationals and one infinitesimal, known as the Hardy field and denoted by $\mathbb{Q}[\varepsilon]$. The element ε belongs to the nonstandard extension $^*\mathbb{R}$ of the real numbers and satisfies $|\varepsilon| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Clearly, the set $\mathbb{Q}[\varepsilon]$ contains all rational numbers.

For the purposes of our study, we are interested only in the unit interval of the Hardy field $\mathbb{Q}[\varepsilon]$, which we denote by $\mathbb{Q}_{[0,1]}^{[\varepsilon]}$. In contrast, $\mathbb{Q}_{[0,1]}$ will denote the set of all rational numbers within the interval $[0, 1]$.

Let us recall that the Hardy field $\mathbb{Q}[\varepsilon]$ consists of all rational expressions of the form $\frac{p(\varepsilon)}{q(\varepsilon)}$, where $p(\varepsilon)$ and $q(\varepsilon)$ are polynomials with coefficients from $\mathbb{Q}$, and $q(\varepsilon) \not\equiv 0$. Two such expressions $\frac{p(\varepsilon)}{q(\varepsilon)}$ and $\frac{p_1(\varepsilon)}{q_1(\varepsilon)}$ are considered equal if the polynomials $p(\varepsilon)q_1(\varepsilon)$ and $p_1(\varepsilon)q(\varepsilon)$ have equal non-zero coefficients.

Addition and multiplication in $\mathbb{Q}[\varepsilon]$ are defined in the usual way, making it a well-defined field. Each element $\eta \in \mathbb{Q}[\varepsilon]$ can be written in a normalized form as

$$\eta = \frac{a\varepsilon^k + \sum_{i=k+1}^{n} a_i\varepsilon^i}{1 + \sum_{j=1}^{m} b_j\varepsilon^j}, \quad k < n, 0 < m,$$

with uniquely determined integer k and leading coefficient $a \neq 0$, except in the case $\eta = 0$. The order $<$ on $\mathbb{Q}[\varepsilon]$ is defined such that $\eta > 0$ if and only if $a > 0$. Hence, to compare $\alpha, \beta \in \mathbb{Q}[\varepsilon]$, we consider $\eta = \alpha - \beta$ and inspect the sign of a in the normalized form of η.

As a result, the field $\mathbb{Q}[\varepsilon]$ is a non-Archimedean ordered field because it contains an infinitesimal ε, and clearly $\mathbb{Q} \subsetneq \mathbb{Q}[\varepsilon]$. The elements of $\mathbb{Q}[\varepsilon] \setminus \mathbb{Q}$ are called non-standard rational numbers.

Two elements $x, y \in \mathbb{Q}[\varepsilon]$ are said to be infinitely close, denoted $x \approx y$, if $x - y$ is infinitesimal. The unique integer k in the normalized form of η is called the order of η, denoted $\mathrm{ord}(\eta)$, with the convention $\mathrm{ord}(0) = \infty$. If $k = 0$, then η differs infinitesimally from a nonzero rational number, and the rational number a is called the standard part of η, denoted $\mathrm{st}(\eta) = a$. If $k > 0$, then $\mathrm{st}(\eta) = 0$.

The set $\mathbb{Q}[\varepsilon]$ enjoys the following several useful properties.

- It is countable and recursive; its operations are computable and its ordering is decidable.
- It forms a dense subfield of ${}^*\mathbb{R}$.
- The monad of a rational number $q \in [0,1]$, defined as $\text{monad}(q) = \{y \in \mathbb{Q}[\varepsilon] \mid y \approx q\}$, can be characterized as

$$\text{monad}(q) = \bigcap_{n \in \mathbb{N}^+} \left[\max\left\{0, q - \frac{1}{n}\right\}, \min\left\{1, q + \frac{1}{n}\right\}\right].$$

The language of the logic $\text{LPP}^{\approx}_{\text{Jccrd}}$ contains the following: a countable set of propositional letters $P = \{p, q, r, \ldots\}$, classical propositional connectives $\neg$ and $\wedge$, and three lists of binary operators in the forms $(J_{\geqslant r})_{r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}}$, $(J_{\leqslant r})_{r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}}$, and $(J_{\approx q})_{q \in \mathbb{Q}_{[0,1]}}$. The set For_C of all classical propositional formulas over P is defined in the usual way. We will denote elements of the set For_C with lowercase Greek letters: $\alpha, \beta, \gamma, \ldots$

Assuming that α and β are classical propositional formulas and $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$, $q \in \mathbb{Q}_{[0,1]}$, then the formulas $J_{\geqslant r}(\alpha, \beta)$, $J_{\leqslant r}(\alpha, \beta)$, and $J_{\approx q}(\alpha, \beta)$ are called elementary probability formulas (it is clear that similarity operators are present in these formulas, but due to the method of determining similarity indices, we refer to such formulas as probability formulas). The set of probability formulas, denoted For_J, is the smallest set that satisfies the following properties.

- The set For_J contains all elementary probability formulas.
- If A and B are probability formulas, then $\neg A$ and $A \wedge B$ are also probability formulas.

We will denote formulas from For_J with uppercase Latin letters: $A, B, C, \ldots$. Let $\text{For}(\text{LPP}^{\approx}_{\text{Jccrd}}) = \text{For}_C \cup \text{For}_J$. We will denote formulas from the set $\text{For}(\text{LPP}^{\approx}_{\text{Jccrd}})$ with $\varphi, \psi, \ldots$

Based on the previously stated facts, note that mixing classical propositional and probability formulas in constructing new formulas is not allowed, nor is the iteration of the aforementioned operators permitted. Thus, we say that $J_{\approx 0.2}(\alpha, \gamma) \wedge \neg J_{\geqslant 1-\varepsilon}(\beta, \alpha)$ and $\alpha \to (\beta \vee \gamma)$ are formulas, while the sequence of symbols $J_{\geqslant \frac{1}{2}}(\beta \wedge \neg J_{\approx \frac{9}{17}}(\neg\alpha, \gamma), \gamma) \wedge \alpha$ does not represent a formula, as it includes the iteration of probability formulas and the mixing of classical and probability formulas, which is not permitted.

The remaining propositional connectives $\vee$, $\to$, and $\leftrightarrow$ are introduced in the usual way: $\alpha \vee \beta =_{def} \neg(\neg\alpha \wedge \neg\beta)$, $\alpha \to \beta =_{def} \neg\alpha \vee \beta$, and $\alpha \leftrightarrow \beta =_{def} (\alpha \to \beta) \wedge (\beta \to \alpha)$. We denote $\bot$ for the formulas $\alpha \wedge \neg\alpha$ and $A \wedge \neg A$, assuming the meaning is clear from the context, while $\top$ denotes $\neg\bot$. In the following text, some other binary and unary operators will also appear, so we will denote:

- $J_{<r}(\alpha, \beta)$ for $\neg J_{\geqslant r}(\alpha, \beta)$, $\alpha, \beta \in \text{For}_C$, $r \in \mathbb{Q}^{[\varepsilon]}_{(0,1]}$,
- $J_{>r}(\alpha, \beta)$ for $\neg J_{\leqslant r}(\alpha, \beta)$, $\alpha, \beta \in \text{For}_C$, $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1)}$,
- $J_{=r}(\alpha, \beta)$ for $J_{\geqslant r}(\alpha, \beta) \wedge J_{\leqslant r}(\alpha, \beta)$, $\alpha, \beta \in \text{For}_C$, $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$,
- $P_{xr}\alpha$ for $J_{xr}(\alpha, \top)$, $\alpha \in \text{For}_C$, $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$, and $x \in \{\leqslant, \geqslant, <, >, =\}$,

- $P_{\approx q}\alpha$ for $J_{\approx q}(\alpha, \top)$, $\alpha \in \mathrm{For}_C$, $q \in \mathbb{Q}_{[0,1]}$.

We need three lists of binary operators because the operators $J_{\leqslant}$ and $J_{\geqslant}$ are not interdefinable, i.e., they cannot be expressed in terms of each other.

4. Semantics

To assign values to the formulas in the set $\mathrm{For}(\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}})$ we use models where probabilities are defined over possible worlds.

Definition 4.1. An $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}}$-model is a structure $M = \langle W, H, \mu, v\rangle$, where:

- W is a nonempty set of elements called worlds,
- H is an algebra of subsets of W,
- $\mu : H \to \mathbb{Q}^{[\varepsilon]}_{[0,1]}$ is a finitely additive probability,
- $v : W \times P \to \{true, false\}$ is a valuation assigning $true$ or $false$ to each world and each propositional letter.

To clarify the previous definition, let's consider an $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}}$-model in the following example.

Example 4.1. Let us consider a set of three propositional letters, i.e., $P = \{p, q, r\}$, and let model $M = \langle W, H, \mu, v\rangle$ be given as follows:

- $W = \{w_1, w_2, w_3\}$,
- let H be the set $\mathbb{P}(W)$ (the power set of W),
- $\mu(\{w_1\}) = \mu(\{w_3\}) = \frac{1}{3} + \varepsilon$ and $\mu(\{w_2\}) = \frac{1}{3} - 2\varepsilon$, with other values easily determined, giving us: $\mu(\emptyset) = 0$, $\mu(\{w_1, w_3\}) = \frac{2}{3} + 2\varepsilon$, $\mu(\{w_2, w_3\}) = \mu(\{w_1, w_2\}) = \frac{2}{3} - \varepsilon$, $\mu(W) = 1$,
- $v(w_1, p) = v(w_1, q) = v(w_1, \neg r) = true$, $v(w_2, \neg p) = v(w_2, q) = v(w_2, \neg r) = true$ and $v(w_3, \neg p) = v(w_3, \neg q) = v(w_3, r) = true$.

When observing an $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}}$-model, say M, and a formula $\alpha \in \mathrm{For}_C$, we identify the set of all worlds in which the formula α holds. We denote this set by $[\alpha]_M$, and we can write

$$[\alpha]_M = \{w \in W \mid v(w, \alpha) = true\}.$$

If it is clear from the context which structure M is being referenced, we will write simply $[\alpha]$. As in all similar logical systems, the satisfiability relation is defined naturally.

Definition 4.2. An $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}}$-model M is measurable if $[\alpha]_M \in H$ for every $\alpha \in \mathrm{For}_C$. Let $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$ denotes the class of all measurable $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}}$-models.

Definition 4.3. Let M be any $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$-model. The satisfiability relation $\models$ is defined as follows:

- if $\alpha \in \mathrm{For}_C$, $M \models \alpha$ if and only if for every world $w \in W$, $v(w, \alpha) = true$,
- for $\alpha, \beta \in \mathrm{For}_C$, $M \models J_{\geqslant r}(\alpha, \beta)$ if and only if $\mu([\alpha] \cup [\beta]) = 0$ or $\mu([\alpha] \cup [\beta]) > 0$ and $\frac{\mu([\alpha]\cap[\beta])}{\mu([\alpha]\cup[\beta])} \geqslant r$,

- for $\alpha, \beta \in \mathrm{For}_C$, $M \models J_{\leqslant r}(\alpha, \beta)$ if and only if $\mu([\alpha] \cup [\beta]) = 0$ and $r = 1$ or $\mu([\alpha] \cup [\beta]) > 0$ and $\frac{\mu([\alpha] \cap [\beta])}{\mu([\alpha] \cup [\beta])} \leqslant r$,
- for $\alpha, \beta \in \mathrm{For}_C$, $M \models J_{\approx q}(\alpha, \beta)$ if and only if $\mu([\alpha] \cup [\beta]) = 0$ and $q = 1$ or $\mu([\alpha] \cup [\beta]) > 0$ and
$$\frac{\mu([\alpha] \cap [\beta])}{\mu([\alpha] \cup [\beta])} \in \bigcap_{n \in \mathbb{N}^+} \left[\max\left\{0, q - \frac{1}{n}\right\}, \min\left\{1, q + \frac{1}{n}\right\}\right] = \mathrm{monad}(q),$$
- for $A \in \mathrm{For}_J$, $M \models \neg A$ if and only if $M \not\models A$,
- for $A, B \in \mathrm{For}_J$, $M \models A \wedge B$ if and only if $M \models A$ and $M \models B$.

A formula $\varphi \in \mathrm{For}(\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}})$ is satisfiable if there exists an $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$-model M such that $M \models \varphi$. A formula φ is valid if for every $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$-model M, $M \models \varphi$. A set of formulas is satisfiable if there is an $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$-model in which every formula in the set is satisfiable.

To clarify the previously defined concepts, consider the following example.

Example 4.2. For $\alpha = (p \wedge q) \to r$ and $\beta = p \wedge \neg r$, let us examine the satisfiability of the formulas
$$J_{\leqslant 0.3}(\alpha, \beta) \quad \text{and} \quad J_{\approx 0.2}(\neg\alpha, \beta)$$
in the model M from the previous example.

We know that $M \models J_{\leqslant 0.3}(\alpha, \beta)$ if $\mu([\alpha] \cup [\beta]) > 0$ and $\frac{\mu([\alpha] \cap [\beta])}{\mu([\alpha] \cup [\beta])} \leqslant 0.3$.

First, from the formula $\alpha = \neg(p \wedge q \wedge \neg r)$, we find that $[\alpha] = \{w_2, w_3\}$ and $[\neg\alpha] = \{w_1\}$, while for the formula $\beta = p \wedge \neg r$ we find that $[\beta] = \{w_1\}$.

Since $\mu([\alpha] \cap [\beta]) = \mu(\emptyset) = 0$, and $\mu([\alpha] \cup [\beta]) = \mu(\{w_1, w_2, w_3\}) = 1 > 0$, it follows that $\frac{\mu([\alpha] \cap [\beta])}{\mu([\alpha] \cup [\beta])} = 0 \leqslant 0.3$. Thus, $M \models J_{\leqslant 0.3}(\alpha, \beta)$.

Similarly, since $\mu([\neg\alpha] \cap [\beta]) = \mu(\{w_1\}) = \frac{1}{3} + \varepsilon$, and $\mu([\neg\alpha] \cup [\beta]) = \mu(\{w_1\}) = \frac{1}{3} + \varepsilon > 0$, it follows that $\frac{\mu([\neg\alpha] \cap [\beta])}{\mu([\neg\alpha] \cup [\beta])} = 1$, so we conclude that $M \not\models J_{\approx 0.2}(\neg\alpha, \beta)$.

5. Axiomatization

The axiomatic system of $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}}$ logic, denoted by $Ax(\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}})$, contains the following set of axiom schemes.

($A1$) All For_C-instances of classical propositional tautologies.
($A2$) All For_J-instances of classical propositional tautologies.
($A3$) $J_{\geqslant 0}(\alpha, \beta)$.
($A4$) $J_{\leqslant r}(\alpha, \beta) \to J_{<s}(\alpha, \beta)$, $r < s$.
($A5$) $J_{<r}(\alpha, \beta) \to J_{\leqslant r}(\alpha, \beta)$.
($A6$) $P_{\geqslant 1}(\alpha \leftrightarrow \beta) \to (P_{=r}\alpha \to P_{=r}\beta)$.
($A7$) $P_{\leqslant r}\alpha \leftrightarrow P_{\geqslant 1-r}\neg\alpha$.
($A8$) $(P_{=r}\alpha \wedge P_{=s}\beta \wedge P_{\geqslant 1}\neg(\alpha \wedge \beta)) \to P_{=\min(1, r+s)}(\alpha \vee \beta)$.
($A9$) $P_{=0}(\alpha \vee \beta) \to J_{=1}(\alpha, \beta)$.
($A10$) $(P_{=r}(\alpha \vee \beta) \wedge P_{=s}(\alpha \wedge \beta)) \to J_{=\frac{s}{r}}(\alpha, \beta)$, $r \neq 0$.
($A11$) $J_{\approx q}(\alpha, \beta) \to J_{\geqslant q_1}(\alpha, \beta)$, for every rational number $q_1 \in [0, q)$.

(A12) $J_{\approx q}(\alpha,\beta) \to J_{\leqslant q_1}(\alpha,\beta)$, for every rational number $q_1 \in (q,1]$.

And the following inference rules.

(R1) From α and $\alpha \to \beta$, infer β.

(R2) If $\alpha \in \mathrm{For}_C$, from α infer $P_{\geqslant 1}\alpha$.

(R3) From $A \to P_{\neq r}\alpha$, for every $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$, infer $A \to \bot$.

(R4) For every $q \in \mathbb{Q}_{[0,1]} \setminus \{0,1\}$, from $A \to J_{\geqslant q-\frac{1}{n}}(\alpha,\beta)$, for every natural number $n \geqslant \frac{1}{q}$, and $A \to J_{\leqslant q+\frac{1}{n}}(\alpha,\beta)$ for every natural number $n \geqslant \frac{1}{1-q}$, infer $A \to J_{\approx q}(\alpha,\beta)$.
From $A \to J_{\geqslant 1-\frac{1}{n}}(\alpha,\beta)$, for all $n \geqslant 1$, infer $A \to J_{\approx 1}(\alpha,\beta)$.
From $A \to J_{\leqslant \frac{1}{n}}(\alpha,\beta)$, for all $n \geqslant 1$, infer $A \to J_{\approx 0}(\alpha,\beta)$.

As mentioned at the beginning of this section, this axiomatic system will be denoted by $Ax(\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}})$. The motivation for introducing these axioms and inference rules follows from the discussion below. Axioms $A1$ and $A2$, as well as inference rule $R1$, ensure that classical propositional logic is a sublogic of this observed logic. If, in axiom $A3$, we replace the formula β with $\top$, then axiom $A3$ states that each formula is satisfied in a set of worlds with probability at least 0. The meanings of axioms $A4$ and $A5$ are trivial, while axiom $A6$ means that equivalent formulas have equal probabilities. The meaning of axiom $A7$ is straightforward, while axiom $A8$ ensures finite additivity of probability. To ensure a well-defined system, and to ensure that if the probability of the union is 0, then the similarity index of the observed formulas is 1, we introduced axiom $A9$. Axiom $A10$ expresses the definition of the Jaccard index of similarity in terms of probability measures. Axioms $A11$ and $A12$, together with rule $R4$, describe the relationship between the value of the standard Jaccard index and the value of the Jaccard index that is infinitely close to a rational number $q \in \mathbb{Q}_{[0,1]}$. It is easy to see that by using axioms $A3$ and $A7$, as well as inference rule $R2$, we ensure that from α we infer $P_{=1}\alpha$. As we can see, rules $R3$ and $R4$ are infinite, with rule $R3$ guaranteeing that the probabilities of formulas belong to the set $\mathbb{Q}^{[\varepsilon]}_{[0,1]}$, i.e., that the rank of the observed logic is the set $\mathbb{Q}^{[\varepsilon]}_{[0,1]}$.

After presenting the set of axioms and inference rules, it is important to emphasize that our aim was not to introduce new proof techniques or to innovate within the Henkin-style framework with infinitary rules. Our formal system adopts the foundational structural principles from [28], with the key distinction lying in the semantic approach. Specifically, we introduce the Hardy field as a nonstandard domain for interpreting probabilities and connect it with notions of similarity between events, thereby providing a theoretical foundation for modeling infinitesimal probabilities within the context of probabilistic logics.

We define the concepts of proof theory necessary for further work. All concepts are defined similarly to other axiomatic systems.

Definition 5.1. A formula φ is deducible from a set T of formulas, denoted $T \vdash \varphi$, $T \subseteq \mathrm{For}(\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}})$, if there is an at most denumerable sequence of formulas

$\varphi_0, \varphi_1, \dots, \varphi$, such that every formula φ_i is either an axiom, a member of the set T, or can be derived from the previous formulas in the sequence by applying one of the inference rules. A formula φ is a theorem (denoted $\vdash \varphi$) if it is deducible from the empty set of formulas.

Definition 5.2. A set T of formulas is consistent if there are at least a formula from For_C and at least a formula from For_J that are not deducible from T; otherwise, T is an inconsistent set.

Definition 5.3. A consistent set of formulas T is a maximal consistent set of formulas if it satisfies:

- for every formula $\alpha \in \mathrm{For}_C$, if $T \vdash \alpha$, then $\alpha \in T$ and $P_{\geqslant 1}\alpha \in T$; and
- for every formula $A \in \mathrm{For}_J$, either $A \in T$ or $\neg A \in T$.

Definition 5.4. A set T is deductively closed if for every formula $\varphi \in \mathrm{For}(\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}})$, it holds that if $T \vdash \varphi$, then $\varphi \in T$.

6. Soundness and Completeness

At the beginning of this section, we prove that the axiomatic system $Ax(\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}})$ is sound, i.e., we prove the following theorem.

Theorem 6.1 (Soundness theorem)**.** *The axiomatic system* $Ax(\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}})$ *is sound with respect to the class of models* $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$.

Proof. We need to show that every instance of an axiom schema is true in every $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$-model and that the inference rules preserve validity in the sense that if all premises of a rule hold in a model, then the conclusion derived from these premises also holds in that model. The proof is fairly straightforward and trivial in many parts. For illustration, let's show that rule R4 preserves validity.

Let $q \in \mathbb{Q}_{[0,1]} \setminus \{0,1\}$ and let M be an $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$-model. From the fact that

$$M \models \left\{A \to J_{\geqslant q-\frac{1}{n}}(\alpha,\beta) \mid n \geqslant \frac{1}{q}\right\} \cup \left\{A \to J_{\leqslant q+\frac{1}{n}}(\alpha,\beta) \mid n \geqslant \frac{1}{1-q}\right\},$$

we need to show that $M \models A \to J_{\approx q}(\alpha,\beta)$. Assume that $M \models A$. Then, we have $M \models J_{\geqslant q-\frac{1}{n}}(\alpha,\beta)$ for every $n \geqslant \frac{1}{q}$ and $M \models J_{\leqslant q+\frac{1}{n}}(\alpha,\beta)$ for every $n \geqslant \frac{1}{1-q}$. It is not possible that $\mu([\alpha]\cup[\beta]) = 0$, because $M \models J_{\leqslant q+\frac{1}{n}}(\alpha,\beta)$ and $q+\frac{1}{n} < 1$ for $n > \frac{1}{1-q}$ (see Definition 4.3). Therefore, we have $\mu([\alpha]\cup[\beta]) > 0$. Then $\frac{\mu([\alpha]\cap[\beta])}{\mu([\alpha]\cup[\beta])} \geqslant q - \frac{1}{n}$ for every $n \geqslant \frac{1}{q}$, and $\frac{\mu([\alpha]\cap[\beta])}{\mu([\alpha]\cup[\beta])} \leqslant q + \frac{1}{n}$ for every $n \geqslant \frac{1}{1-q}$. We thus get

$$\frac{\mu([\alpha]\cap[\beta])}{\mu([\alpha]\cup[\beta])} \in \bigcap_{n\in\mathbb{N}^+}\left[\max\left\{0, q-\frac{1}{n}\right\}, \min\left\{1, q+\frac{1}{n}\right\}\right] = \mathrm{monad}(q),$$

i.e., $\frac{\mu([\alpha]\cap[\beta])}{\mu([\alpha]\cup[\beta])} \approx q$, and so $M \models J_{\approx q}(\alpha,\beta)$. $\square$

The completeness theorem and its proof are structured in several clear steps. We begin by establishing the deduction theorem and proving several auxiliary results that will be used later. Then, we demonstrate how a given consistent set of formulas can be systematically extended to a maximally consistent set, denoted T^*. Finally, we construct the canonical model M for the logic $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$ based on the formulas in T^*. This model is defined in such a way that a formula φ is satisfied in M (i.e., $M \models \varphi$) if and only if $\varphi \in T^*$.

Theorem 6.2 (Deduction theorem). *Let φ and ψ be formulas of the same type. If T is a set of formulas and $T \cup \{\varphi\} \vdash \psi$, then $T \vdash \varphi \to \psi$.*

Proof. We proceed by transfinite induction on the length of the proof of the formula ψ from the set $T \cup \{\varphi\}$. If the proof has length 1, the existing cases are easy to show, as well as the case where the formula ψ is obtained by applying inference rule R1.

Let the formula $\psi = P_{\geqslant 1}\alpha$ be obtained from the set $T \cup \{\varphi\}$ applying inference rule R2, and let $\varphi \in \mathrm{For}_J$. However, since $\alpha \in \mathrm{For}_C$ and $\varphi \in \mathrm{For}_J$, φ is not used in the proof of α from $T \cup \{\varphi\}$, and we have:

$T \vdash \alpha$,
$T \vdash P_{\geqslant 1}\alpha$ by Rule R2,
$T \vdash P_{\geqslant 1}\alpha \to (\varphi \to P_{\geqslant 1}\alpha)$,
$T \vdash \varphi \to P_{\geqslant 1}\alpha$ by Rule R1.

Now, assume that the formula $\psi = A \to \bot$ is obtained by applying rule R3 to the set $T \cup \{\varphi\}$, and $\varphi \in \mathrm{For}_J$. Then:

$T, \varphi \vdash A \to P_{\neq r}\alpha$, for every $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$,
$T \vdash \varphi \to (A \to P_{\neq r}\alpha)$, for every $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$, by the Induction hypothesis,
$T \vdash (\varphi \wedge A) \to P_{\neq r}\alpha$, for every $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$,
$T \vdash (\varphi \wedge A) \to \bot$, by Rule R3,
$T \vdash \varphi \to \psi$.

Finally, assume that the formula $\psi = A \to J_{\approx q}(\alpha, \beta)$ is derived from T, φ by R4 with premises $A \to J_{\geqslant q - \frac{1}{n}}(\alpha, \beta)$, $n \geqslant \frac{1}{q}$, $A \to J_{\leqslant q + \frac{1}{n}}(\alpha, \beta)$, $n \geqslant \frac{1}{1-q}$, $q \in \mathbb{Q}_{[0,1]} \setminus \{0, 1\}$.

$T, \varphi \vdash A \to J_{\geqslant q - \frac{1}{n}}(\alpha, \beta)$, for every $n \geqslant \frac{1}{q}$,
$T, \varphi \vdash A \to J_{\leqslant q + \frac{1}{n}}(\alpha, \beta)$, for every $n \geqslant \frac{1}{1-q}$,
$T \vdash \varphi \to (A \to J_{\geqslant q - \frac{1}{n}}(\alpha, \beta))$, for every $n \geqslant \frac{1}{q}$, by the Induction hypothesis,
$T \vdash \varphi \to (A \to J_{\leqslant q + \frac{1}{n}}(\alpha, \beta))$, for every $n \geqslant \frac{1}{1-q}$, by the Induction hypothesis,
$T \vdash (\varphi \wedge A) \to J_{\geqslant q - \frac{1}{n}}(\alpha, \beta)$, for every $n \geqslant \frac{1}{q}$,
$T \vdash (\varphi \wedge A) \to J_{\leqslant q + \frac{1}{n}}(\alpha, \beta)$, for every $n \geqslant \frac{1}{1-q}$,
$T \vdash (\varphi \wedge A) \to J_{\approx q}(\alpha, \beta)$, by Rule R4,
$T \vdash \varphi \to (A \to J_{\approx q}(\alpha, \beta))$.
$T \vdash \varphi \to \psi$. □

The following theorem provides some auxiliary statements needed for the proof of the completeness theorem.

Theorem 6.3. *Let α and β be classical formulas. Then, the following holds.*

(1) $J_{\geqslant s}(\alpha,\beta) \to J_{\geqslant r}(\alpha,\beta)$, $s > r$.
(2) $J_{\leqslant s}(\alpha,\beta) \to J_{\leqslant r}(\alpha,\beta)$, $s < r$.
(3) $J_{=s}(\alpha,\beta) \to \neg J_{=r}(\alpha,\beta)$, $s \neq r$.
(4) $J_{=s}(\alpha,\beta) \to \neg J_{\geqslant r}(\alpha,\beta)$, $s < r$.
(5) $J_{=s}(\alpha,\beta) \to \neg J_{\leqslant r}(\alpha,\beta)$, $s > r$.
(6) $J_{=q}(\alpha,\beta) \to J_{\approx q}(\alpha,\beta)$, $q \in \mathbb{Q}_{[0,1]}$.
(7) $J_{\approx q_1}(\alpha,\beta) \to \neg J_{\approx q_2}(\alpha,\beta)$, *for* $q_1, q_2 \in \mathbb{Q}_{[0,1]}$, $q_1 \neq q_2$.
(8) $P_{=0}(\alpha \vee \beta) \to \neg J_{\leqslant r}(\alpha,\beta)$, *for* $r < 1$.
(9) $P_{\leqslant 1}\alpha$.

Proof. The proofs for (1) and (2) follow directly from axioms A4 and A5. The properties expressed by these statements can be referred to as the monotonicity of the similarity operator. As an illustration, we prove statements (3) and (6). For brevity, we omit details related to obvious arguments.

Indeed, from

$\vdash J_{\leqslant s}(\alpha,\beta) \to J_{<r}(\alpha,\beta)$, $s < r$ by axiom A4,
$\vdash J_{\geqslant s}(\alpha,\beta) \to J_{>r}(\alpha,\beta)$, $s > r$ by axiom A4,
$\vdash J_{\leqslant s}(\alpha,\beta) \to \neg J_{\geqslant r}(\alpha,\beta)$, $s < r$
$\vdash J_{\geqslant s}(\alpha,\beta) \to \neg J_{\leqslant r}(\alpha,\beta)$, $s > r$
$\vdash (J_{\leqslant s}(\alpha,\beta) \wedge J_{\geqslant s}(\alpha,\beta)) \to (\neg J_{\geqslant r}(\alpha,\beta) \vee \neg J_{\leqslant r}(\alpha,\beta))$, $s \neq r$
$\vdash J_{=s}(\alpha,\beta) \to \neg J_{=r}(\alpha,\beta)$, $s \neq r$

we conclude that the considered statement holds.

Finally, we verify statement (6) for $q \in \mathbb{Q}_{[0,1]} \setminus \{0,1\}$.

$\vdash J_{=q}(\alpha,\beta) \leftrightarrow J_{\geqslant q}(\alpha,\beta) \wedge J_{\leqslant q}(\alpha,\beta)$,
$\vdash J_{\geqslant q}(\alpha,\beta) \to J_{\geqslant q-\frac{1}{n}}(\alpha,\beta)$, $n \geqslant \frac{1}{q}$
$\vdash J_{\leqslant q}(\alpha,\beta) \to J_{\leqslant q+\frac{1}{n}}(\alpha,\beta)$, $n \geqslant \frac{1}{1-q}$
$\vdash J_{=q}(\alpha,\beta) \to J_{\geqslant q-\frac{1}{n}}(\alpha,\beta)$, $n \geqslant \frac{1}{q}$
$\vdash J_{=q}(\alpha,\beta) \to J_{\leqslant q+\frac{1}{n}}(\alpha,\beta)$, $n \geqslant \frac{1}{1-q}$
$\vdash J_{=q}(\alpha,\beta) \to J_{\approx q}(\alpha,\beta)$.

□

Theorem 6.4. *Every consistent set of formulas T can be extended to a maximal consistent set.*

Proof. Let T be a consistent set of formulas. Denote by $Con_C(T)$ the set of all classical consequences of T. Let the sequence of formulas $A_0, A_1, A_2, \dots$ be a sequence of all formulas from the set For_J, and let the sequence $\alpha_0, \alpha_1, \alpha_2, \dots$ be a sequence of all formulas from the set For_C. We define a sequence of sets $T_i, i = 0, 1, 2, \dots$, as shown below.

(1) $T_0 = T \cup Con_C(T) \cup \{P_{\geqslant 1}\alpha \mid \alpha \in Con_C(T)\}$,
(2) for every $i \geqslant 0$, we have the following:

(a) if the set $T_{2i} \cup \{A_i\}$ is consistent, then $T_{2i+1} = T_{2i} \cup \{A_i\}$;
(b) otherwise, if the set $T_{2i} \cup \{A_i\}$ is not consistent, we have the following:
(i) if the formula A_i is of the form $A \to J_{\approx q}(\alpha, \beta)$, then $T_{2i+1} = T_{2i} \cup \{\neg A_i, A \to \neg J_{\geqslant q-\frac{1}{n}}(\alpha, \beta)\}$ or $T_{2i+1} = T_{2i} \cup \{\neg A_i, A \to \neg J_{\leqslant q+\frac{1}{n}}(\alpha, \beta)\}$, for some non-negative integer n, chosen such that T_{2i+1} is consistent (we will show that this is possible);
(ii) otherwise, $T_{2i+1} = T_{2i} \cup \{\neg A_i\}$,

(3) for every $i \geqslant 0$, $T_{2i+2} = T_{2i+1} \cup \{P_{=r}\alpha_i\}$, where r is chosen to be any arbitrary element of the set $\mathbb{Q}^{[\varepsilon]}_{[0,1]}$ such that T_{2i+2} is consistent (we will show that this is possible),

(4) for every $i \geqslant 0$, if the set T_i is enriched with a formula of the form $P_{=0}\alpha$, we add the formula $\neg\alpha$ to the set $T_i \cup \{P_{=0}\alpha\}$.

We need to show that every set T_i is consistent. The set T_0 is consistent because it is the set of consequences of a consistent set. The sets obtained in steps 2.a and 2.b.ii are clearly consistent. Now, let us consider step 2.b.i. Suppose that the formula A_i is of the form $A \to J_{\approx q}(\alpha, \beta)$ and that the set $T_{2i} \cup \{A \to J_{\approx q}(\alpha, \beta)\}$ is not consistent. Also, assume that all sets $T_{2i} \cup \{\neg(A \to J_{\approx q}(\alpha, \beta)), A \to \neg J_{\geqslant q-\frac{1}{n}}(\alpha, \beta)\}$, for every non-negative integer n such that $q - \frac{1}{n} \geqslant 0$, and $T_{2i} \cup \{\neg(A \to J_{\approx q}(\alpha, \beta)), A \to \neg J_{\leqslant q+\frac{1}{n}}(\alpha, \beta)\}$, for every n such that $q + \frac{1}{n} \leqslant 1$, are inconsistent. Then, the following contradicts the consistency of T_{2i}:

$T_{2i}, \neg(A \to J_{\approx q}(\alpha, \beta)), A \to \neg J_{\geqslant q-\frac{1}{n}}(\alpha, \beta) \vdash \bot$, for every n,
$T_{2i}, \neg(A \to J_{\approx q}(\alpha, \beta)), A \to \neg J_{\leqslant q+\frac{1}{n}}(\alpha, \beta) \vdash \bot$, for every n,
$T_{2i}, \neg(A \to J_{\approx q}(\alpha, \beta)) \vdash A \to J_{\geqslant q-\frac{1}{n}}(\alpha, \beta)$, for every n,
$T_{2i}, \neg(A \to J_{\approx q}(\alpha, \beta)) \vdash A \to J_{\leqslant q+\frac{1}{n}}(\alpha, \beta)$, for every n,
$T_{2i}, \neg(A \to J_{\approx q}(\alpha, \beta)) \vdash A \to J_{\approx q}(\alpha, \beta)$, by Rule R4.

Now consider step 3 of our construction. Suppose that for every $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$, the set $T_{2i+1} \cup \{P_{=r}\alpha_i\}$ is not consistent. Let $T_{2i+1} = T_0 \cup T'_{2i+1}$, where T'_{2i+1} denotes the set of all formulas $B \in \mathrm{For}_J$ that were added to T_0 in the previous steps of the construction. Then, the following contradicts the consistency of T_{2i+1}:

$T_0, T'_{2i+1}, P_{=r}\alpha_i \vdash \bot$, for every $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$, by the Hypothesis,
$T_0, T'_{2i+1} \vdash \neg P_{=r}\alpha_i$, for every $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$, by Deduction theorem,
$T_0 \vdash (\bigwedge_{B \in T'_{2i+1}} B) \to \neg P_{=r}\alpha_i$, for every $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$, by Deduction theorem,
$T_0 \vdash (\bigwedge_{B \in T'_{2i+1}} B) \to \bot$, by Rule R3,
$T_{2i+1} \vdash \bot$.

Finally, consider step 4 of our construction. Suppose that for some $\alpha \in \mathrm{For}_C$, $T_i \cup \{P_{=0}\alpha, \neg\alpha\} \vdash \bot$. By the Deduction theorem, we have $T_i \cup \{P_{=0}\alpha\} \vdash \alpha$. Since $\alpha \in \mathrm{For}_C$, α a is a consequence of $Con_C(T)$, the set of all classical formulas that are consequences of T, and $\alpha \in Con_C(T)$. Now, by the construction, we have $P_{\geqslant 1}\alpha \in T_0$ which contradicts $T_i \cup \{P_{=0}\alpha\}$, as follows:

$T_i, P_{=0}\alpha \vdash P_{\leqslant 1}\alpha$, by Theorem 6.3.(9),
$T_i, P_{=0}\alpha \vdash P_{\geqslant 1}\alpha$, since $P_{\geqslant 1}\alpha \in T_0 \subseteq T_i$,
$T_i, P_{=0}\alpha \vdash P_{=1}\alpha$, by the definition of the operator $P_{=1}$,
$T_i \vdash P_{=0}\alpha \to P_{=1}\alpha$, by the Deduction theorem,
$T_i \vdash P_{=0}\alpha \to \neg P_{=1}\alpha$, by Theorem 6.3.(3),
$T_i, P_{=0}\alpha \vdash \bot$.

Now, let $T^* = \cup_i T_i$. The key point is to show that the set T^* is a maximal consistent set.

Initially, note that if $P_{=r}\alpha \in T^*$, then for every formula B from the set For_J, it holds that $B \to P_{=r}\alpha \in T^*$. If we assumed that this is not the case, based on the given construction, for some formula B from For_J and some j, we would have that both $P_{=r}\alpha$ and $\neg(B \to P_{=r}\alpha)$ belong to the set T_j. This statement would lead us to the conclusion that $T_j \vdash P_{=r}\alpha \wedge \neg P_{=r}\alpha$, which is a contradiction. In the continuation of the proof, it is necessary to show that the set T^* is a deductively closed set which does not contain all formulas. If the formula α belongs to the set For_C, based on the construction of the set T_0, we have that α and $\neg\alpha$ cannot both belong to T_0 simultaneously. For a formula A from the set For_J, the set T^* does not contain both formulas $A = A_i$ and $\neg A = A_j$, because the set $T_{\max\{2i,2j\}+1}$ is consistent.

If the formula α is from the set For_C and $T^* \vdash \alpha$, then based on the construction of the set T_0, we have that $\alpha \in T^*$ and $P_{\geqslant 1}\alpha \in T^*$. Now, let us consider the formula A from the set For_J. By induction on the length of the inference, it can be shown that if $T^* \vdash A$, then $A \in T^*$. Note that if $A = A_j$ and $T_i \vdash A$, it must hold that $A \in T^*$, because the set $T_{\max\{i,2j\}+1}$ is consistent. Suppose that the sequence of formulas $\varphi_1, \varphi_2, \dots, A$ is a proof for the formula A from T^*. If the assumed sequence is finite, then there must exist a set T_i such that $T_i \vdash A$ and $A \in T^*$. Therefore, suppose that the observed sequence is countably infinite. We can show that for each i, if the formula φ_i is obtained by applying an inference rule and all premises belong to the set T^*, then $\varphi_i \in T^*$ must also hold. If the inference rule is finite, then by reasoning as above, we conclude that $\varphi_i \in T^*$. In the following steps, we are interested in infinite inference rules.

Let $\varphi_i = A \to \bot$ be a formula obtained by applying inference rule R3 to the set of premises of the form $A \to \neg P_{=r}\alpha$ for all $r \in \mathbb{Q}_{[0,1]}^{[\varepsilon]}$. Assume that $\varphi_i \notin T^*$. Based on step (3) of the construction described above, there exists some $r_l \in \mathbb{Q}_{[0,1]}^{[\varepsilon]}$ and an index l such that $P_{=r_l}\alpha \in T_l$. Reasoning as above, we conclude that $A \to P_{=r_l}\alpha \in T^*$. Thus, there must be some j such that $A \to \neg P_{=r_l}\alpha, A \to P_{=r_l}\alpha \in T_j$, $T_j \vdash A \to \neg P_{=r_l}\alpha$, $T_j \vdash A \to P_{=r_l}\alpha$, and $T_j \vdash A \to \bot$, which means that $\varphi_i = A \to \bot \in T^*$, which is a contradiction.

Let $\varphi_i = A \to J_{\approx q}(\alpha, \beta)$ be a formula obtained by applying inference rule R4 to the set of premises of the form $A \to J_{\geqslant q-\frac{1}{n}}(\alpha, \beta)$ for every non-negative integer n such that $q - \frac{1}{n} \geqslant 0$ and $A \to J_{\leqslant q+\frac{1}{n}}(\alpha, \beta)$, for every n such that $q + \frac{1}{n} \leqslant 1$. Assume that $\varphi_i \notin T^*$. Based on step 2.b.i. of the construction described above, there exist some n

and some j such that the formulas $A \to \neg J_{\geqslant q-\frac{1}{n}}(\alpha,\beta)$ or $A \to \neg J_{\leqslant q+\frac{1}{n}}(\alpha,\beta)$ belong to the set T_j. Assume that the first formula belongs to it, while the case with the second formula will be similar. The previously mentioned statement means that there exists some l such that $A \to J_{\geqslant q-\frac{1}{n}}(\alpha,\beta), A \to \neg J_{\geqslant q-\frac{1}{n}}(\alpha,\beta) \in T_l$. Thus, we obtain that $T_l \vdash A \to \bot$ and $T_l \vdash A \to J_{\approx q}(\alpha,\beta)$. Therefore, we obtain that $\varphi_i \in T^*$, which is a contradiction. □

Theorem 6.5. *T^* has the following properties.*

(1) *T^* contains all theorems.*
(2) *If $\varphi \in T^*$, then $\neg\varphi \notin T^*$.*
(3) *$\varphi \wedge \psi \in T^*$ if and only if $\varphi \in T^*$ and $\psi \in T^*$.*
(4) *If $\varphi, \varphi \to \psi \in T^*$, then $\psi \in T^*$.*
(5) *There is exactly one $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$ such that $P_{=r}\alpha \in T^*$.*
(6) *There is exactly one $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$ such that $J_{=r}(\alpha,\beta) \in T^*$.*
(7) *If $J_{\geqslant r}(\alpha,\beta) \in T^*$, there exists $s \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$ such that $s \geqslant r$ and $J_{=s}(\alpha,\beta) \in T^*$.*
(8) *If $J_{\leqslant r}(\alpha,\beta) \in T^*$, there exists $s \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$ such that $s \leqslant r$ and $J_{=s}(\alpha,\beta) \in T^*$.*
(9) *If $J_{\approx q_1}(\alpha,\beta) \in T^*$ and $q_2 \in \mathbb{Q}_{[0,1]} \setminus \{q_1\}$, then $J_{\approx q_2}(\alpha,\beta) \notin T^*$.*

Proof. We will prove properties (5), (6), and (7). The proofs for (1)-(4) are standard, so we omit them. The proof for (8) is carried out in the same way as for (7), and the proof for (9) directly follows from Theorem 6.3.(7).

(5) Based on Theorem 6.3.(3), if it holds that $P_{=r}\alpha \in T^*$, then for every value $s \neq r$, we have $P_{=s}\alpha \notin T^*$. If it were true that for every $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$, $\neg P_{=r}\alpha \in T^*$, then $T^* \vdash \neg P_{=r}\alpha$ for every $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$. By the application of inference rule R3, we get $T^* \vdash \bot$ which is a contradiction with the consistency of the set T^*.

(6) According to (5), for any classical propositional formulas α and β, there exists exactly one $s \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$ and exactly one $t \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$ such that $P_{=s}(\alpha \vee \beta) \in T^*$ and $P_{=t}(\alpha \wedge \beta) \in T^*$. If $s = 0$, then for $r = 1$, we have $J_{=1}(\alpha,\beta) \in T^*$ by axiom A9. For $s \neq 0$, there exists a unique $r = \frac{t}{s}$ such that $J_{=r}(\alpha,\beta) \in T^*$ by axiom A10.

(7) Now suppose $J_{\geqslant r}(\alpha,\beta) \in T^*$. From item (6) of this statement, there exists exactly one $s \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$ such that $J_{=s}(\alpha,\beta) \in T^*$. It follows from Theorem 6.3.(4) that s cannot be less than r. Thus, it must be $s \geqslant r$. □

Using T^*, we can define the structure $M = \langle W, H, \mu, v\rangle$ as follows:

- $W = \{w \models Con_C(T)\}$ contains all the classical propositional interpretations that satisfy the set $Con_C(T)$ of all classical consequences of the set T,
- for each formula $\alpha \in For_C$, let $[\alpha] = \{w \in W \mid w \models \alpha\}$ and $H = \{[\alpha] \mid \alpha \in For_C\}$,
- for every world w and every propositional letter $p \in Var$, $v(w,p) = true$ if and only if $w \models p$, and
- μ is defined on H by $\mu([\alpha]) = r$ if and only if $P_{=r}\alpha \in T^*$.

The next theorem states that M is an $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$-model.

Theorem 6.6. *Let $M = \langle W, H, \mu, v\rangle$ be defined as above. Then, the following hold.*

(1) *μ is well-defined.*
(2) *H is an algebra of subsets of W.*
(3) *μ is a finitely additive probability measure.*
(4) *For every classical formula α, $\mu([\alpha]_M) = 0$ if and only if $[\alpha]_M = \emptyset$.*

Proof. (1) It follows from Theorem 6.5. property (5), that for every $\alpha \in \mathrm{For}_C$ there is exactly one $r \in \mathbb{Q}^{[\varepsilon]}_{[0,1]}$ such that $\mu([\alpha]) = r$. On the other hand, let $[\alpha] = [\beta]$ for some $\alpha, \beta \in \mathrm{For}_C$. Essentially, this means that for every $w \in W$, $w \models \alpha \leftrightarrow \beta$. By the completeness of classical propositional logic, we have $\alpha \leftrightarrow \beta \in Con_C(T)$. And using the previously described construction, we find that $P_{\geqslant 1}(\alpha \leftrightarrow \beta) \in T^*$. Now, A6 guarantees that $P_{=r}\alpha \in T^*$ if and only if $P_{=r}\beta \in T^*$. Thus, $[\alpha] = [\beta]$ implies that $\mu([\alpha]) = \mu([\beta])$.

(2) H is an algebra of subsets of W, because for every classical formula α, the following holds:

(*i*) $W = [\alpha \vee \neg\alpha] \in H$;
(*ii*) if $[\alpha] \in H$, then the complement of $[\alpha]$, which is $[\neg\alpha]$, belongs to H;
(*iii*) if $[\alpha_1], \dots, [\alpha_n] \in H$, then $\bigcup_{i=1}^n [\alpha_i] \in H$, because $[\alpha_1] \cup \cdots \cup [\alpha_n] = [\alpha_1 \vee \cdots \vee \alpha_n]$.

(3) Based on axiom $A3$ and the fact that if we set $\beta = \top$, we find that $\mu : H \to [0, 1]$. Let $\alpha \in \mathrm{For}_C$. Then $W = [\alpha \vee \neg\alpha]$, $\alpha \vee \neg\alpha \in T^*$, and $P_{\geqslant 1}(\alpha \vee \neg\alpha) \in T^*$. From the previous, we have that $\mu(W) = 1$. Let $\alpha, \beta \in \mathrm{For}_C$, $[\alpha] \cap [\beta] = \emptyset$, and let $\mu([\alpha]) = r$ and $\mu([\beta]) = s$. Since, $[\alpha] \cap [\beta] = \emptyset$, we have that $[\neg(\alpha \wedge \beta)] = W$, and $\mu([\neg(\alpha \wedge \beta)]) = 1$. From the assumptions, we have that $P_{=r}\alpha$, $P_{=s}\beta$, $P_{\geqslant 1}\neg(\alpha \wedge \beta) \in T^*$. Using $A8$ we get $P_{=r+s}(\alpha \vee \beta) \in T^*$, i.e., we find that $\mu([\alpha \vee \beta]) = r + s = \mu([\alpha]) + \mu([\beta])$.

(4) The statement follows directly from the construction of the maximal consistent set, i.e., from the construction given in the proof of Theorem 6.4. □

Theorem 6.7 (Extended completeness theorem)**.** *A set T of formulas is consistent if and only if T has an $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$-model.*

Proof. If T has an $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$-model, then the set T is consistent, which follows from Theorem 6.1. Let us prove the reverse. Assuming that T is consistent, we construct an $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd,Meas}}$-model as described above and prove that for every $\varphi \in \mathrm{For}(\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}})$, $M \models \varphi$ if and only if $\varphi \in T^*$.

Let $\varphi \in \mathrm{For}_C$. If $\varphi \in T^*$, then certainly $\varphi \in Con_C(T)$, and for every $w \in W$, $w \models \varphi$, so we conclude that $M \models \varphi$. Conversely, if $M \models \varphi$, then by the completeness of classical propositional logic, we have $\varphi \in Con_C(T)$, and $\varphi \in T^*$.

Let $\varphi = \neg A$, where $A \in \mathrm{For}_J$. Indeed, it holds that $M \models \neg A$ if and only if $M \models A$ does not hold, if and only if $A \notin T^*$, if and only if $\neg A \in T^*$.

Let $\varphi = A \wedge B$, where $A, B \in \mathrm{For}_J$. Indeed, it holds that $M \models A \wedge B$ if and only if $M \models A$ and $M \models B$ if and only if $A \in T^*$ and $B \in T^*$ if and only if $A \wedge B \in T^*$.

Let $\varphi = J_{\geqslant r}(\alpha,\beta)$. First, assume that $J_{\geqslant r}(\alpha,\beta) \in T^*$. By Theorem 6.5. property (5), there are unique values $s, t \in \mathbb{Q}_{[0,1]}^{[\varepsilon]}$ such that $\mu([\alpha \wedge \beta]) = s$ and $\mu([\alpha \vee \beta]) = t$, i.e., $P_{=s}(\alpha \wedge \beta), P_{=t}(\alpha \vee \beta) \in T^*$. If $t = 0$, by Definition 4.3. we have $M \models J_{\geqslant r}(\alpha,\beta)$. If $t \neq 0$, based on A10, we have $J_{=\frac{s}{t}}(\alpha,\beta) \in T^*$. It follows from Theorem 6.3. property (4), that $\frac{s}{t} \geqslant r$, so $M \models J_{\geqslant r}(\alpha,\beta)$. Conversely, assume that $M \models J_{\geqslant r}(\alpha,\beta)$. In the first case, let $\mu([\alpha \vee \beta]) = 0$. We have that $P_{=0}(\alpha \vee \beta) \in T^*$, and by A9, $J_{=1}(\alpha,\beta) \in T^*$. Now, by Theorem 6.3. property (1), we have $J_{\geqslant r}(\alpha,\beta) \in T^*$. In the second case, let $\mu([\alpha \wedge \beta]) = s$ and $\mu([\alpha \vee \beta]) = t$, where $t > 0$ and $\frac{s}{t} \geqslant r$. Based on A10, we must have $J_{=\frac{s}{t}}(\alpha,\beta) \in T^*$, so it easily follows that $J_{\geqslant r}(\alpha,\beta) \in T^*$.

The case for $\varphi = J_{\leqslant r}(\alpha,\beta)$ is shown in a similar manner.

Finally, let $\varphi = J_{\approx q}(\alpha,\beta)$. To show the equivalence, first assume that $J_{\approx q}(\alpha,\beta) \in T^*$. In the case that $\mu([\alpha \vee \beta]) = 0$, by A9 and properties (6) and (7) from Theorem 6.3, it must be that $q = 1$, and $M \models J_{\approx 1}(\alpha,\beta)$. If $\mu([\alpha \vee \beta]) > 0$, by A11 and A12, we have $J_{\geqslant q_1}(\alpha,\beta), J_{\leqslant q_2}(\alpha,\beta) \in T^*$, for all rational numbers q_1, q_2 such that $0 \leqslant q_1 < q < q_2 \leqslant 1$. Applying the reasoning described previously, we have $M \models J_{\geqslant q_1}(\alpha,\beta)$ and $M \models J_{\leqslant q_2}(\alpha,\beta)$, which actually means that $M \models J_{\approx q}(\alpha,\beta)$. To prove the second implication, assume that $M \models J_{\approx q}(\alpha,\beta)$. Then, for all rational numbers q_1, q_2 such that $0 \leqslant q_1 < q < q_2 \leqslant 1$, we have $M \models J_{\geqslant q_1}(\alpha,\beta)$ and $M \models J_{\leqslant q_2}(\alpha,\beta)$. Based on the facts obtained, we have $J_{\geqslant q_1}(\alpha,\beta), J_{\leqslant q_2}(\alpha,\beta) \in T^*$. If we assumed that $J_{\approx q}(\alpha,\beta) \notin T^*$, by the construction of the set T^* we would obtain that for some rational number q_0, either $q_0 < q$ and $\neg J_{\geqslant q_0}(\alpha,\beta) \in T^*$ or $q_0 > q$ and $\neg J_{\leqslant q_0}(\alpha,\beta) \in T^*$, which in either case is a contradiction, because the set T^* is consistent. Therefore, $J_{\approx q}(\alpha,\beta) \in T^*$, which is what we aimed to prove. □

7. Conclusion

In this paper, we present a new logical system denoted as $\mathrm{LPP}^{\approx}_{\mathrm{Jccrd}}$, in which the language of probabilistic logic is enriched with binary operators $J_{\leqslant r}(\alpha,\beta)$, $J_{\geqslant r}(\alpha,\beta)$, and $J_{\approx q}(\alpha,\beta)$, whose intended meaning is motivated by the Jaccard similarity index. The main result of the paper lies in the proof of soundness and strong completeness of the axiomatic system, thereby establishing a theoretical foundation for further exploration of logics that combine probability and similarity.

One possible direction for future research concerns the role of infinitesimal probabilities in fine-grained differentiation among events, particularly those that would all be assigned zero probability in the standard framework. Although this paper does not introduce conditional probabilities, it remains an open question whether combining infinitesimal values with similarity-based indices could contribute to a better understanding or alternative formalizations of certain forms of uncertain reasoning.

Additionally, the study of operator iteration and its combination with classical propositional formulas requires special care, as such systems often entail deeper changes in the semantic framework. For that purpose, it is necessary to develop an independent line of investigation, which would require a new semantics and a thorough analysis of meta-theoretical properties. This opens the possibility for research that connects similarity-based structures with richer linguistic and semantic layers.

Acknowledgements. The work of all authors is supported by the Serbian Ministry of Science, Technological Development and Innovation (Agreement No. 451-03-65/2024-03/200122).

References

[1] N. Alechina, *Logic with probabilistic operators*, Proceedings of the ACCOLADE'94 (1995), 121–138.

[2] A. R. Bernstein and F. Wattenberg, *Nonstandard measure theory*, in: W. A. J. Luxemburg (Ed.), *Applications of Model Theory to Algebra, Analysis and Probability*, Holt, Rinehart and Winston, New York, 1969, 171–186.

[3] M. Boričić, *Suppes-style sequent calculus for probability logic*, J. Logic Comput. **27**(4) (2017), 1157–1168. https://doi.org/10.1093/logcom/exv068

[4] M. Boričić, *Inference rules for probability logic*, Publications de l'Institut Mathématique **100**(114) (2016), 77–86. https://doi.org/10.2298/PIM1614077B

[5] M. Boričić, N. Ikodinović and N. Stojanović, *Probability and natural deduction*, J. Logic Comput. **35**(2) (2025), exae007. https://doi.org/10.1093/logcom/exae007

[6] M. Brusco, J. D. Cradit and D. Steinley, *A comparison of* 71 *binary similarity coefficients: The effect of base rates*, PLOS One **16**(4) (2021), Article ID e0247751. https://doi.org/10.1371/journal.pone.0247751

[7] R. Djordjević, N. Ikodinović and N. Stojanović, *A propositional metric logic with fixed finite ranges*, Fund. Inform. **174**(2) (2020), 185–199. https://doi.org/10.3233/FI-2020-1938

[8] R. Djordjević, M. Rašković and Z. Ognjanović, *Completeness theorem for propositional probabilistic models whose measures have only finite ranges*, Arch. Math. Logic **43** (2004), 557–563. https://doi.org/10.1007/s00153-004-0217-3

[9] R. Fagin and J. Halpern, *Reasoning about knowledge and probability*, Journal of the ACM **41**(2) (1994), 340–367. https://doi.org/10.1145/174652.174658

[10] R. Fagin, J. Halpern and N. Megiddo, *A logic for reasoning about probabilities*, Inform. and Comput. **87**(1–2) (1990), 78–128. https://doi.org/10.1016/0890-5401(90)90060-U

[11] L. Hamers, Y. Hemeryck, G. Herweyers, M. Janssen, H. Ketters, H. Rousseau and A. Vanhoutte, *Similarity measures in scientometric research: The Jaccard index versus Salton's cosine formula*, Information Processing and Management **25**(3) (1989), 315–318. https://doi.org/10.1016/0306-4573(89)90048-4

[12] C. Howson, *Regularity and infinitely tossed coins*, Eur. J. Philos. Sci. **7** (2017), 97–102. https://doi.org/10.1007/s13194-016-0147-z

[13] N. Ikodinović and Z. Ognjanović, *A logic with coherent conditional probabilities*, Lecture Notes in Comput. Sci. **3571** (2005), 726–736. https://doi.org/10.1007/11518655_61

[14] N. Ikodinović, M. Rašković, Z. Marković and Z. Ognjanović, *A first-order probabilistic logic with approximate conditional probabilities*, Logic Journal of the IGPL **22**(4) (2014), 539–564. https://doi.org/10.1093/jigpal/jzt048

[15] P. Jaccard, *Distribution de la flore alpine dans le bassin des Dranses*, Bulletin de la Société Vaudoise des Sciences Naturelles **37** (1901), 241–272. https://doi.org/10.5169/seals-266440

[16] R. A. Jarvis and E. A. Patrick, *Clustering using a similarity measure based on shared near neighbors*, IEEE Transactions on Computers **C-22** (1973), 1025–1034. https://doi.org/10.1109/T-C.1973.223640

[17] L. Leydesdorff, *On the normalization and visualization of author co-citation data: Salton's cosine versus the Jaccard index*, Journal of the American Society for Information Science and Technology **59**(1) (2008), 77–85. https://doi.org/10.1002/asi.20732

[18] R. A. Miranda-Quintana, D. Bajusz, A. Rácz and K. Héberger, *Extended similarity indices: The benefits of comparing more than two objects simultaneously. Part 1: Theory and characteristics*, Journal of Cheminformatics **13** (2021), Article No. 17. https://doi.org/10.1186/s13321-021-00505-3

[19] B. Mirkin, *Mathematical Classification and Clustering*, Kluwer Academic Publishers, Dordrecht, 1996. https://doi.org/10.1007/978-1-4613-0457-9

[20] N. Nilsson, *Probabilistic logic*, Artificial Intelligence **28** (1986), 71–87. https://doi.org/10.1016/0004-3702(86)90031-7

[21] Z. Ognjanović and N. Ikodinović, *A logic with higher order conditional probabilities*, Publications de l'Institut Mathématique **82**(96) (2007), 141–154. https://doi.org/10.2298/PIM0796141O

[22] Z. Ognjanović and M. Rašković, *Some probability logics with new types of probability operators*, J. Logic Comput. **9**(2) (1999), 181–195. https://doi.org/10.1093/logcom/9.2.181

[23] Z. Ognjanović, M. Rašković and Z. Marković, *Probability Logics: Probability-Based Formalization of Uncertain Reasoning*, Springer, Cham, 2016. https://doi.org/10.1007/978-3-319-47012-2

[24] S. Park and D.-Y. Kim, *Assessing language discrepancies between travelers and online travel recommendation systems: Application of the Jaccard distance score to web data mining*, Technological Forecasting and Social Change **123** (2017), 381–388. https://doi.org/10.1016/j.techfore.2017.03.031

[25] M. Rašković, *Classical logic with some probability operators*, Publications de l'Institut Mathématique **53**(67) (1993), 1–3.

[26] M. Rašković, R. Djordjević and N. Stojanović, *Completeness theorem for probability models with finitely many valued measure*, Open Math. **17** (2019), 168–171. https://doi.org/10.1515/math-2019-0016

[27] M. Rašković, Z. Ognjanović and Z. Marković, *A logic with conditional probabilities*, in: *Logics in Artificial Intelligence (JELIA 2004)*, Lecture Notes in Artificial Intelligence **3229** (2004), Springer, Berlin, 226–238. https://doi.org/10.1007/978-3-540-30227-8_21

[28] M. Rašković, Z. Ognjanović and Z. Marković, *A logic with approximate conditional probabilities that can model default reasoning*, Internat. J. Approx. Reason. **49**(1) (2008), 52–66. https://doi.org/10.1016/j.ijar.2007.08.006

[29] A. Robinson, *Non-standard Analysis*, North-Holland, Amsterdam, 1966.

[30] N. Stojanović, N. Ikodinović and R. Djordjević, *A propositional logic with binary metric operators*, J. Appl. Logics **5**(8) (2018), 1605–1622.

[31] T. T. Tanimoto, *An elementary mathematical theory of classification and prediction*, IBM Internal Technical Report, 1958.

[32] M. K. Vijaymeena and K. Kavitha, *A survey on similarity measures in text mining*, Machine Learning and Applications **3**(1) (2016), 19–28. https://doi.org/10.5121/mlaij.2016.3103

[33] T. Williamson, *How probable is an infinite sequence of heads?*, Analysis **67**(3) (2007), 173–180. https://doi.org/10.1111/j.1467-8284.2007.00671.x

[34] H. Wolda, *Similarity indices, sample size and diversity*, Oecologia **50** (1981), 296–302. https://doi.org/10.1007/BF00344966

[35] Y. Yuan, M. Chao and Y.-C. Lo, *Automatic skin lesion segmentation using deep fully convolutional networks with Jaccard distance*, IEEE Transactions on Medical Imaging **36**(9) (2017), 1876–1886. https://doi.org/10.1109/TMI.2017.2695227

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