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CREATING AND COMPUTING GRAPHS FROM HYPERGRAPHS

M. HAMIDI[1] AND A. BORUMAND SAIED[2]

Abstract. This paper defines concept of complete hypergraph, enumerates the number of hypergraphs constructed on any nonempty set and determine a lower bound for the number of all complete hypergraphs. We define an equivalence relation on hypergraphs, this relation establishes a connection between hypergraphs and graphs. Moreover, the concept of hypergraph is redefined via the concept of hypergroupoid and a connection between of hypergraphs and hypergroupoids was considered. Finally, we consider a relation between fundamental group and fundamental graph.

1. Introduction

In 1736, Euler first introduced the concept of graph theory. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operation research, optimization, economics, networking routing, transportation and computer science.

The theory of hyperstructures was first introduced by Marty in 1934 during the 8^{th} congress of the Scandinavian Mathematicians [10]. In classical algebraic structures, the synthetic result of two elements is an element while, in the hyper algebraic system, the synthetic result of two elements is a set of elements. Hence, from this point of view the notion of hyperstructures is a generalization of classical algebraic structures. Marty introduced the concept of hypergroups and used it in different contexts like algebraic functions, rational fractions and non commutative groups.

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Hyperstructures have many applications in several sectors of both pure and applied sciences such as geometry [13, 16], automata [3, 11], cryptography, codes, artificial intelligence, probabilities and chemistry [4, 7, 9].

The notion of hypergraph has been introduced by Berge as a generalization of graph around 1960 and one of the initial concerns was to extend some classical results of graph theory and the notion of hypergraph has been considered as a useful tool to analyze the structure of a system. Further materials regarding graph and hypergraph are available in the literature too [1, 2, 8].

Regarding these points, the aim of this paper is to compute the number of hypergraphs constructed on any arbitrary set via the inclusion-exclusion principle. We construct a hypergraph on any nonempty set and define an equivalence relation on any hypergraph. With regard to this relation, we connect graphs and hypergraphs and define a new concept as the fundamental graph. Indeed, we show that any graph is a fundamental graph. Moreover, it was shown that the concepts of hypergroupoids and hypergraphs in a away are related. We define the concept of complete hypergraphs and find a lower bound for the number of all complete hypergraphs.

2. Preliminaries

In this section, we recall some definitions and results that are indispensable to our research paper.

Definition 2.1. [5] A graph G is a finite nonempty set V of objects called vertices (the singular is vertex) together with a set E of 2-element subsets of V called edges and is shown with $G = (V, E)$. Two graphs G and H are isomorphic (they have the same structure) if there exists a bijective function $\varphi : V(G) \to V(H)$ so that two vertices u and v are adjacent in G if and only if $\varphi(u)$ and $\varphi(v)$ are adjacent in H. The function φ is then called an isomorphism. If G and H are isomorphic, we write $G \cong H$.

Definition 2.2. [18] Suppose that G is a nonempty set and $P^*(G)$ is the family of all nonempty subsets of G, every function $\circ_i : G \times G \to P^*(G)$, where $i \in \{1, 2, \ldots, n\}$ and $n \in \mathbb{N}$ is called *hyperoperation*. For all x, y of G, $\circ_i(x, y)$ is called the *hyperproduct* of x and y. An algebraic system $(G, \circ_1, \circ_2, \ldots, \circ_n)$ is called a *hyperstructure* and a binary structure $(G, \circ)$ endowed with only one hyperoperation is called a *hypergroupoid*. For any two nonempty subsets A and B of $G, A \circ B$ means $\bigcup\limits_{a \in A, b \in B} a \circ b$. A hypergroupoid $(G, \circ)$ is called a *quasihypergroup* if for all $x \in G$ we have $x \circ G = G \circ x = G$ and is called a *semihypergroup* if for any $x, y, z \in G$ we have $x \circ (y \circ z) = (x \circ y) \circ z$.

A hypergroupoid $(G, \circ)$ is called a *hypergroup* if is a semihypergroup and a quasihypergroup and is called a *total* hypergroup provided that for any $x, y \in G$, $x \circ y = G$.

Definition 2.3. [6] Let $(G, \circ)$ be a hypergroup and $\mathcal{U}$ be the set of all finite products of elements of G. The relation β on G is defined by $a\beta b$ if and only if exists $u \in \mathcal{U}$: $\{a, b\} \in u$ and the relation β^* on G is defined as follows: $a\beta^* b$ if and only if exists

$z_1 = a, z_2, \dots, z_n = b \in G, u_1, u_2, \dots, u_n \in \mathfrak{U}$ s.t. $\{z_i, z_{i+1}\} \in u_i$, for all $1 \leq i \leq n$. The relation β^* is called fundamental relation.

Theorem 2.1. [6] *Let G be a group. Then β^* is the transitive closure of β, $(G/\beta^*, *)$ is a group and it is called the fundamental group.*

Definition 2.4. [1] Let $G = \{x_1, x_2, \dots, x_n\}$ be a finite set. A hypergraph on G is a family $H = (E_1, E_2, \dots, E_m) = (G, \{E_i\}_{i=1}^m)$ of subsets of G such that

(i) for all $1 \leq i \leq m$, $E_i \neq \emptyset$;

(ii) $\bigcup_{i=1}^{m} E_i = G$.

A *simple* hypergraph (*Sperner family*) is a hypergraph $H = (E_1, E_2, \dots, E_m)$ such that

(iii) $E_i \subset E_j \Rightarrow i = j$.

The elements $x_1, x_2, \dots, x_n$ of G are called *vertices*, and the sets $E_1, E_2, \dots, E_m$ are the *edges* (hyperedges) of the hypergraph. For any $1 \leq k \leq m$ if $|E_k| \geq 2$, then E_k is represented by a solid line surrounding its vertices, if $|E_k| = 1$ by a cycle on the element (loop). If for all $1 \leq k \leq m$, $|E_k| = 2$, the hypergraph becomes an ordinary (undirected) graph.

In any hypergraph, two vertices x and y are said to be adjacent if there exists a hyperedge E_i which contains the two vertices ($x, y \in E_i$). The *degree* of a vertex is the number of hyperedges which contains the vertex and is showed by $\deg(x)$ ($\deg(x) = |\{E_i \mid x \in E_i\}|$). The incidence matrix of a hypergraph is a matrix $M_G = (a_{ij})_{n\times m}$, with m columns representing the hyperedges $E_1, E_2, \dots, E_m$ and n rows representing the vertices $x_1, x_2, \dots, x_n$, where $a_{ij} = 0$ if $x_i \notin E_j$, $a_{ij} = 1$ if $x_i \in E_j$.

Theorem 2.2. [17] *Let G be a set and $|G| = 2$. Then the number of quasihypergroup, hypergroup and isomorphic hypergroup on G are* 35*,* 14*,* 8*, respectively.*

Theorem 2.3. [14, 15, 17] *Let G be a set and $|G| = 3$. Then the number of quasihypergroup, hypergroup and isomorphic hypergroup on G are* 10323979*,* 23192*,* 3999*, respectively.*

From now on, we will denote $|E| = 1$ by 1-hyperedge (loop), $|E| = 2$ by 2-hyperedge (simple hyperedge) and for any $n \in \mathbb{N}$, $|E| = n$ by n-hyperedge.

3. Construction of Hypergraphs

In this section, we will compute the number of hypergraphs that are constructed on a set with n elements. We define the concept of hyperdiagram and apply the inclusion-exclusion principle to comput the hyperdiagrams and hypergraphs.

Definition 3.1. Let $G = \{x_1, x_2, \dots, x_n\}$ be a finite set. A *hyperdiagram* on G is a family $H = (G, E_1, E_2, \dots, E_m) = (G, \{E_k\}_{k=1}^m)$ of subsets of G such that for all $1 \leq k \leq m$, $|E_k| \geq 1$.

From now on, we will denote $\mathcal{H}d(G) = \{G_k \mid \quad_k \text{ is a hyperdiagram on } G\}$ and $\mathcal{H}g(G) = \{G_k \in \mathcal{H}d(G) \mid G_k \text{ is a hypergraph on } G\}$.

Example 3.1. Let $G = \{a\}$. Then we have the following hyperdiagram on G such that is a hypergraph (Figure 1).

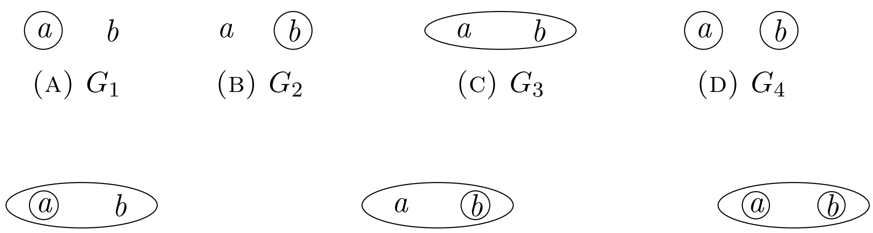
FIGURE 1. Hypergraph (G_0)

We have $\mathcal{H}d(G) = \{G_0\} = \mathcal{H}g(G) = \{G_0\}$.

Lemma 3.1. *Let G be a set and $|G| = n$. Then $|\mathcal{H}d(G)| = 2^{2^n-1} - 1$.*

Proof. Let $G = \{x_1, x_2, \ldots, x_n\}$ and E_i be a hyperedge on G. If $|E_i| = 1$, then E_i is a loop and the number of loops is equal to $\binom{n}{1}$. If $|E_i| = 2$, then E_i is a edge and the number of edges is equal to $\binom{n}{2}$. In a similar way, if $|E_i| = k$, then E_i is a k-hyperedge and the number of k-hyperedges is equal to $\binom{n}{k}$. Hence the number of all hyperedges is equal to $\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$. It follows that $|\mathcal{H}d(G)| = 2^{2^n-1} - 1$. $\square$

Example 3.2. Let $G = \{a, b\}$. Then we have $\mathcal{H}d(G) = \{G_1, G_2, G_3, G_4, G_5, G_6, G_7\}$ and $\mathcal{H}g(G) = \{G_3, G_4, G_5, G_6, G_7\}$ as follows (Figure 2).



(A) G_1 (B) G_2 (C) G_3 (D) G_4

(E) G_5 (F) G_6 (G) G_7

FIGURE 2. Hyperdiagrams on $G = \{a, b\}$.

Lemma 3.2. *Let $G = \{x_1, x_2, \ldots, x_n\}$ and let $B_i = \{G \in \mathcal{H}g(G^i) \mid G^i = G \setminus \{x_1, x_2, \ldots, x_i\}\}$. Then for any $1 \leq i \neq j \leq n$, $B_i \cap B_j = \emptyset$.*

Proof. Let $i < j$ and $G \in B_i \cap B_j$. Since $G \in B_i$, we get that G is a hypergraph such that $x_1, x_2, \ldots, x_i \notin G$ and since $G \in B_j$, we get that G is a hypergraph which $x_1, x_2, \ldots, x_i, \ldots, x_j \notin G$, that is a contradiction. So for any $i \neq j$, $B_i \cap B_j = \emptyset$. $\square$

Example 3.3. Let $G = \{c, a, b\}$. Then $B_1 = \{G \in \mathcal{H}g(G^1) \mid G^1 = \{a, b\}\}$. By Examples 3.1 and 3.2, $B_1 = \{G_3, G_4, G_5, G_6, G_7\}$ and $B_2 = \{G \in \mathcal{H}g(G^2) \mid G^2 = \{c\}\} = \{G_0\}$. Clearly $B_1 \cap B_2 = \emptyset$.

Let G be a set, $|G| = i$ and $a_i = |\mathcal{H}g(G)|$. Then in the following theorem we will have a computation of a_i by the recursive sequences.

Theorem 3.1. *Let G be a set and $|G| = n$. Then* $a_n = 2^{2^n-1} - \left(1 + \sum_{i=0}^{n-1} \binom{n}{i} a_{n-i}\right)$.

Proof. Let $G = \{x_1, x_2, \dots, x_n\}$ and $B_i = \{G \in \mathcal{H}g(G^i) \mid G^i = G\backslash\{x_1, x_2, \dots, x_i\}\}$. Then we get that $|\mathcal{H}g(G)| = \left|\left(\bigcup_{i=1}^{n} B_i\right)^c\right|$, where $\left(\bigcup_{i=1}^{n} B_i\right)^c$ is the complement of $\left(\bigcup_{i=1}^{n} B_i\right)$ or $\left(\bigcup_{i=1}^{n} B_i\right) \cup \left(\bigcup_{i=1}^{n} B_i\right)^c = \mathcal{H}d(G)$. It is obvious that for any $1 \leq i \leq n$, $|B_i| = a_{n-i}$ and by Lemma 3.2, it is easy to see that $\left|\bigcap_{j=1}^{r} B_j\right| = 0$. Hence we get that

$$\begin{aligned}
|\mathcal{H}g(G)| =& |\mathcal{H}d(G)| - \left|\left(\bigcup_{i=1}^{n} B_i\right)\right| \\
=& (2^{2^n-1} - 1) - \left(\binom{n}{1} a_{n-1} + \binom{n}{2} a_{n-2} + \cdots + \binom{n}{n-1} a_0\right) \\
=& 2^{2^n-1} - \left(1 + \sum_{i=0}^{n-1} \binom{n}{i} a_{n-i}\right).
\end{aligned}$$

□

Lemma 3.3. *Let $G = \{x_1, x_2, \dots, x_n\}$ and $A_k = \{G \in \mathcal{H}d(G^{(k)}) \mid G^{(k)} = G\backslash\{x_k\}\}$. Then for any $r \in \mathbb{N}$, we get that* $\left|\bigcap_{j=1}^{r} A_j\right| = 2^{2(n-r)-1} - 1$.

Proof. Let $r \in \mathbb{N}$. Then $\bigcap_{j=1}^{r} A_j = \{G \in \mathcal{H}d(G^{(k)}) \mid G^{(k)} = G\backslash\{x_1, x_2, \dots, x_r\}\} = \{G \in \mathcal{H}d(G^{(k)}) \mid G^{(k)} = \{x_{r+1}, x_{r+2}, \dots, x_n\}\}$ and so by Lemma 3.1, $\left|\bigcap_{j=1}^{r} A_j\right| = 2^{2(n-r)-1} - 1$. □

Theorem 3.2. *Let G be set. If $|G| = n$, then* $|\mathcal{H}g(G)| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{2(n-k)-1}$.

Proof. Let $G = \{x_1, x_2, \dots, x_n\}$ and $A_k = \{G \in \mathcal{H}d(G^{(k)}) \mid G^{(k)} = G\backslash\{x_k\}\}$. Then we get that $|\mathcal{H}g(G)| = \left|\left(\bigcup_{k=1}^{n} A_k\right)^c\right|$, where $\left(\bigcup_{k=1}^{n} A_k\right)^c$ is complement of $\left(\bigcup_{k=1}^{n} A_k\right)$ or $\left(\bigcup_{k=1}^{n} A_k\right) \cup \left(\bigcup_{k=1}^{n} A_k\right)^c = \mathcal{H}d(G)$. By Lemma 3.3, it is easy to see that for any $1 \leq k \leq n$, $|A_k| = 2^{2(n-1)-1} - 1$ and so for some $r \in \mathbb{N}$, we get that $\left|\bigcap_{j=1}^{r} A_j\right| = 2^{2(n-r)-1} - 1$.

Hence

$$\begin{aligned}|\mathcal{H}g(G)| =&|\mathcal{H}d(G)| - \left|\left(\bigcup_{k=1}^{n} A_k\right)\right| \\ =&(2^{2^n-1}-1) - \left(\binom{n}{1}(2^{2^{(n-1)}-1}-1) - \binom{n}{2}\right)(2^{2^{(n-2)}-1}-1) + \cdots \\ &+(-1)^n\left(\binom{n}{n}2^{2^{(n-n)}-1}-1)\right) \\ =&\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{2^{(n-k)}-1} + \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k} \\ =&\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{2^{(n-k)}-1}.\end{aligned}$$

□

Example 3.4. Let $G = \{c, a, b\}$. Then

$$\begin{aligned}A_1 =&\{G \in \mathcal{H}g(G^{(1)}) \mid G^{(1)} = G\backslash\{c\}\} = \{G \in \mathcal{H}g(G^{(1)}) \mid G^{(1)} = \{a, b\}\} \\ =&\{G_3, G_4, G_5, G_6, G_7\}, \\ A_2 =&\{G \in \mathcal{H}g(G^{(2)}) \mid G^{(2)} = G\backslash\{b\}\} = \{G \in \mathcal{H}g(G^{(2)}) \mid G^{(2)} = \{a, c\}\} \\ =&\{G_3, G_4, G_5, G_6, G_7\}, \\ A_3 =&\{G \in \mathcal{H}g(G^{(3)}) \mid G^{(3)} = G\backslash\{a\}\} = \{G \in \mathcal{H}g(G^{(3)}) \mid G^{(3)} = \{b, c\}\} \\ =&\{G_3, G_4, G_5, G_6, G_7\}, \\ A_1 \cap A_2 =&\{G \in \mathcal{H}g(G^{(4)}) \mid G^{(4)} = \{a\}\} = \{G_0\}, \\ A_1 \cap A_3 =&\{G \in \mathcal{H}g(G^{(4)}) \mid G^{(4)} = \{b\}\} = \{G_0\}, \\ A_2 \cap A_3 =&\{G \in \mathcal{H}g(G^{(4)}) \mid G^{(4)} = \{c\}\} = \{G_0\} \text{ and} \\ A_1 \cap A_2 \cap A_3 =&\{G \in \mathcal{H}g(G^{(4)}) \mid G^{(4)} = \emptyset\} = \emptyset.\end{aligned}$$

Corollary 3.1. *Let $n \in \mathbb{N}$. Then*

$$1 + \sum_{i=0}^{n-1}\binom{n}{i}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}2^{2^{(n-i-k)}-1} + \sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{2^{(n-k)}-1} = 2^{2^n-1}.$$

Example 3.5. Let $\mathbb{N}_k = \{1, 2, 3, \ldots, k\}$. Then $|\mathcal{H}g(\mathbb{N}_2)| = 5$, $|\mathcal{H}g(\mathbb{N}_3)| = 109$, $|\mathcal{H}g(\mathbb{N}_4)| = 32302$ and $|\mathcal{H}g(\mathbb{N}_5)| = 2147321017$.

In the following we compute the number of the hypergraphs with n vertices by the Matlab software: function number

```
n=input ('Enter the number of elements of the set G =')
s = 0;
for k = 0 : n
s = s + (−1)ˆk ∗ n choose k (n, k) ∗ 2ˆ(2ˆ(n − k) − 1);
end
```

s

Remark 3.1. Let $\mathbb{N}_n = \{1, 2, \ldots, n\}$. Then we have $|P^*(P^*(\mathbb{N}_n))| = 2^{2^n-1} - 1 = k$. Now we set $t = \left|\left\{ A_i \in P^*(P^*(\mathbb{N}_n)) \mid \bigcup_{x \in A_i} x = G \right\}\right|$ and by the following algorithm will compute t as the number of hypergraphs with n vertices:

(a) Start
(b) Input the number of vertices $[n]$
(c) Compute the power set of $\mathbb{N}_n$ $[P^*(\mathbb{N}_n)]$
(d) Compute the power set of $P^*(\mathbb{N}_n)$ $[P^*(P^*(\mathbb{N}_n))]$
(e) To examine the condition of hypergraphs for $A_i \in P^*(P^*(\mathbb{N}_n))$:
if $A_i == \mathbb{N}_n$ then t=t+1
else i=i+1 and return to level (e)
(f) Print t
(g) End.

In the Table 1, for any $n \in \mathbb{N}$ we describe the previous program how enumerate the hypergraphs on any set.

TABLE 1

n	$\mathbb{N}_n$	$\lvert P^*(\mathbb{N}_n)\rvert = 2^n - 1$	$\lvert P^*(P^*(\mathbb{N}_n))\rvert = 2^{2^n-1} - 1$	t
2	$\{1, 2\}$	3	7	5
3	$\{1, 2, 3\}$	7	127	109
4	$\{1, 2, 3, 4\}$	15	32767	32302
5	$\{1, 2, 3, 4, 5\}$	31	2147483647	2147321017

Definition 3.2. Let $(G, \{E_i\}_{i=1}^n)$ and $(G', \{E_i'\}_{i=1}^n)$ be hypergraphs. We say that $(G, \{E_i\}_{i=1}^n)$ and $(G', \{E_i'\}_{i=1}^n)$ are isomorphic if there exists a bijective function $\varphi : V(G) \to V(G')$ such that two vertices u and v are adjacent in G if and only if $\varphi(u)$ and $\varphi(v)$ are adjacent in G'.

Example 3.6. Let $G = \{a, b\}$. Then $|\mathcal{H}g(G)| = 5$, while the following hypergraphs are isomorphic (Figure 3) and the other hypergraphs on $G = \{a, b\}$ are not isomorphic

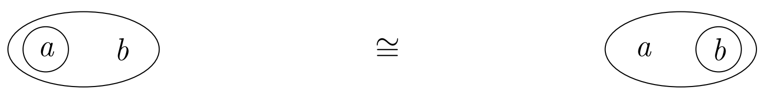


FIGURE 3. Isomorphic hypergraphs with two elements

as follows (Figure 4). It follows that, the number of hypergraphs with two elements is equal to 4 up to isomorphism.

Let G be a set such that $|G| = i$. In the following, we will show $||\mathcal{H}g(G)||$ as the number of all hypergroups on G up to isomorphism.

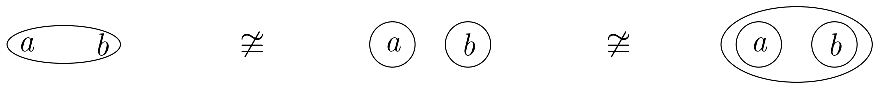


FIGURE 4. Not isomorphic hypergraphs with two elements

Example 3.7. Let $|G| = 3$. Then $||\mathcal{H}g(G)|| = 34$ as follows:

$$\binom{3}{3}\binom{3}{0}\binom{1}{0} + \sum_{i=1}^{3}\binom{3}{i}\binom{3}{1}\binom{1}{0} + \sum_{i=0}^{3}\binom{3}{i}\binom{3}{2}\binom{1}{0} + \sum_{i=0}^{3}\binom{3}{i}\binom{3}{3}\binom{1}{0}$$
$$+ \sum_{i=0}^{3}\binom{3}{i}\binom{3}{0}\binom{1}{1} + \sum_{i=0}^{3}\binom{3}{i}\binom{3}{1}\binom{1}{1} + \sum_{i=0}^{3}\binom{3}{i}\binom{3}{2}\binom{1}{1} + \sum_{i=0}^{3}\binom{3}{i}\binom{3}{3}\binom{1}{1}$$
$$=1 + 3 + 6 + 4 + 4 + 6 + 6 + 4 = 34.$$

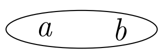
Open Problem. Let G be a set and $|G| = n$. Then $||\mathcal{H}g(G)|| =?$

3.1. **Complete Hypergraphs.** In this subsection, we define a new type of hypergraphs as complete hypergraph. Also we find a lower bound for the number of all complete hypergraps and prove some theorems for computing.

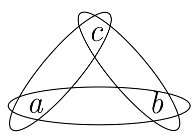
Definition 3.3. Let $(G, \{E_x\}_{x\in G})$ be a hypergraph. Then we say that $(G, \{E_x\}_{x\in G})$ is a complete hypergraph, if for any $x, y \in G$ there exits a hyperedge E such that $\{x, y\} \subseteq E$. We will show a complete hypergraph with n elements by K_n^* and the set of all hypergraphs on a set G by $\mathcal{CH}g(G)$.

Example 3.8. (i) For any $n \in \mathbb{N}$, complete graphs shown by K_n are complete hypergraphs.

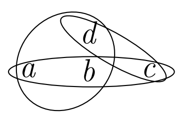
(ii) For $n = 2$, $n = 3$ and $n = 4$, we have defined complete hypergraphs K_2^*, K_3^* and K_4^* as follows (Figure 5).



(A) K_2^*



(B) K_3^*



(C) K_4^*

FIGURE 5. Complete hypergraphs with 2, 3 and 4 elements

Lemma 3.4. *Let G be a hypergraph, $|G| = n$ and $(H, \{E_i\}_{i=1}^m)$ be a complete subhypergraph of G in such a way that has not trivial hyperedge. Then*

(i) *for any $1 \leq i, j \leq m$, $E_i \cap E_j \neq \emptyset$;*
(ii) *$m \neq 2$;*
(iii) *if $m \geq 3$, then $\sum_{i=1}^{m} |E_i| \geq 2n$.*

Proof. (i) Since $(H, \{E_i\}_{i=1}^m)$ is a subhypergraph, we get that for any $x, y \in H$, there exist E_x, E_y so that $x \in E_x, y \in E_y$. Now, if there exist some $1 \leq i, j \leq m$ in such a way that $E_i \cap E_j = \emptyset$, then there exist some $x, y \in G$ for any $1 \leq i \leq n, \{x, y\} \nsubseteq E_i$. But H is a complete subhypergraph and it is a contradiction. Hence for any $1 \leq i, j \leq n$, $E_i \cap E_j \neq \emptyset$.

(ii) Let $m = 2$. Then $H = E_1 \cup E_2$ in such a way that $E_1 \neq E_2$ and by (i), $E_1 \cap E_2 \neq \emptyset$. Clearly $E_1 \backslash E_2$ and $E_2 \backslash E_1$ are nonempty and $(E_1 \backslash E_2) \cap (E_2 \backslash E_1) = \emptyset$. It follows that there exist $x \in (E_1 \backslash E_2)$ and $y \in (E_2 \backslash E_1)$ such that $\{x, y\} \nsubseteq E_1$ and $\{x, y\} \nsubseteq E_2$ that it is a contradiction, hence $m \neq 2$.

(iii) Let $G = \{a_1, a_2, \ldots, a_n\}$ and $m = 3$. Then $G = E_1 \cup E_2 \cup E_3$ and by (i), $\bigcap\limits_{1 \leq i \neq j \leq 3} (E_i \cap E_j) \neq \emptyset$. If $|E_1| = n - k$ and $|E_2| = n - k'$ where $k, k' \in \mathbb{N}$, then we must have $|E_3| \geq k + k'$ and so $\sum\limits_{i=1}^{3} |E_i| = (n - k) + (n - k') + (k + k') \geq 2n$. Similarly, we compute for $m \geq 3$. □

Example 3.9. Let $G = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $(G, \{E_i\}_{i=1}^3)$ and $(G, \{E'_i\}_{i=1}^4)$ be complete subhypergraphs. If $E_1 = \{a_1, a_2, a_3, a_4, a_5\}$, $E_2 = \{a_2, a_3, a_4, a_5, a_6\}$ and $E_3 = \{a_1, a_2, a_6\}$. Clearly $|E_1| + |E_2| + |E_3| = 13$. Now, for $m = 4$, consider $E'_1 = \{a_1, a_2, a_3, a_4\}$, $E'_2 = \{a_4, a_5, a_6\}$, $E'_3 = \{a_3, a_2, a_6\}$ and $E'_4 = \{a_1, a_5\}$. Clearly $|E'_1| + |E'_2| + |E'_3| + |E'_4| = 12$.

Theorem 3.3. *Let G be a hypergraph, $|G| = n$ and H be a subhypergraph of G and for any $1 \leq i \leq n$, $S_H(n, i)$ be the number of i-hyperedge. Then*

(i) $S_H(n, n) \geq 1$ *if and only if H is one of complete subhypergraphs;*

(ii) $S_H(n, n-1) \geq 3$ *and* $n \geq 3$ *if and only if H is one of complete subhypergraphs;*

(iii) $S_H(n, 2) = \frac{n(n-1)}{2}$ *if and only if H is one of complete subhypergraphs.*

Proof. (i) Let E be a hyperedge of H and $|E| = n$. Then for any $x, y \in H, \{x, y\} \subseteq E$. Hence H is a complete hypergraph. The converse follows immediately.

(ii) Let $G = \{a_1, a_2, \ldots, a_{n-1}, a_n\}$, E_1, E_2 and E_3 be hyperedges of H, where $|E_1| = |E_2| = |E_3| = n - 1$. By Lemma 3.4, clearly, $\bigcap\limits_{1 \leq i \neq j \leq 3} (E_i \cap E_j) \neq \emptyset$ and for $n \geq 3$ we get that

$$|E_1| + |E_2| + |E_3| = 3(n - 1) = n + (2n - 3) \geq n + 3 \geq 2n.$$

Consider $E_1 = \{a_1, a_2, \ldots, a_{n-1}\}, E_2 = \{a_1, a_n\}$ and $E_3 = \{a_2, a_3, \ldots, a_{n-1}, a_n\}$. If (H, E_1, E_2, E_3) is a subhypergraph of G, then clearly H is a complete subhypergraph. The converse follows immediately.

(iii) The proof is clear. □

Corollary 3.2. *Let G be a hypergraph, $|G| = n$, H be a subhypergraph of G and $1 \leq t < n$. If $n \geq 3t$, then $S_H(n, n-t) \geq t+2$ and so H is a complete subhypergraph.*

Proof. By Theorem 3.3, we must have $3(n - t) \geq 2n$, that implies $n \geq 3t$. □

Theorem 3.4. *Let G be a set, $|G| = n$ and $0 \leq j \leq n$. If $N(n, i_j)$ is the number of complete subhypergraphs with j loops and for any $1 \leq i \leq n$, $\alpha_i = \binom{n}{i}$, then*

(i) *if $k_1 = 0$ and $k_n = 1$, then*
$$\sum_{k_n=1}^{1} \sum_{k_{n-1}=0}^{\alpha_{n-1}} \cdots \sum_{k_2=0}^{\alpha_2} \sum_{k_1=0}^{0} \binom{\alpha_1}{k_1}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\cdots\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n} = 2^{2^n-(n+2)};$$

(ii) *for any $1 \leq j \leq n$ we have $N(n, i_j) = \binom{n}{j} N(n, i_0)$;*

(iii) $|\mathcal{CH}g(G)| = 2^n N(n, i_0)$.

Proof. (i) Let H be a subhypergraph of the hypergraph G. Then for any $1 \leq i \leq n$ and $\alpha_i = \binom{n}{i}$, there exist $0 \leq k_i \leq \alpha_i$ in such a way that H has $\binom{\alpha_i}{k_i}$ choices of types i-hyperedges. Hence,
$$\sum_{k_n=0}^{\alpha_n} \sum_{k_{n-1}=0}^{\alpha_{n-1}} \cdots \sum_{k_2=0}^{\alpha_2} \sum_{k_1=0}^{\alpha_1} \binom{\alpha_1}{k_1}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\cdots\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n}$$
is the number of all subhyperhraphs of G. It is easy to see that the number of all i-hyperedges is equal to 2^{α_i} and so the number of all hyperedges is equal to $2^{(\sum_{i=0}^{n}\alpha_i)} = 2^{2^n}$. If $k_1 = 0$ and $k_n = 1$, then
$$\begin{aligned}&\sum_{k_n=1}^{\alpha_n} \sum_{k_{n-1}=0}^{\alpha_{n-1}} \cdots \sum_{k_2=0}^{\alpha_2} \sum_{k_1=0}^{\alpha_1} \binom{\alpha_1}{k_1}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\cdots\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n}\\ =&2^{2^n-(\binom{n}{0}+\binom{n}{1}+\binom{n}{n})} = 2^{2^n-(n+2)}.\end{aligned}$$

(ii) Since $N(n, i_j)$ is the number of complete subhypergraphs with j loops, we get that $k_1 = j$. There is $\binom{n}{j}$ cases for choosing of j loops, so by (i), $N(n, i_j) = \binom{n}{j} N(n, i_0)$.

(iii) Let G be a set and $|G| = n$. Then
$$\begin{aligned}|\mathcal{CH}g(G)| =&\sum_{j=1}^{n} N(n, i_j) = \sum_{j=1}^{n} \binom{n}{j} N(n, i_0)\\ =&N(n, i_0)\sum_{j=1}^{n}\binom{n}{j} = 2^n N(n, i_0).\end{aligned}$$
□

Theorem 3.5. *Let G be a set and $|G| = n$. If $n < 4$, $2k_2+3k_3+\cdots+(n-1)k_{n-1} \geq 2n$, where $k_{n-1} = \cdots = k_3 = 0$ implies that $k_2 = \binom{n}{2}$, then*
$$N(n, i_0) = 2^{2^n-(n+2)} + \sum_{1\neq k_n=0}^{\alpha_n} \sum_{k_{n-1}=0}^{\alpha_{n-1}} \cdots \sum_{k_2=0}^{\alpha_2} \sum_{k_1=0}^{0} \binom{\alpha_1}{0}\binom{\alpha_2}{k_1}\binom{\alpha_3}{k_2}\cdots\binom{\alpha_n}{k_n}.$$

Proof. Let H be a complete subhypergraphs. If $k_n = 1$, then $S_H(n, n) = 1$ and so by Theorem 3.3, H is complete. But if $k_n = 0$, by Lemma 3.4, we must have

$\sum_{i=0}^{n} S_H(n,i) \geq 2n$. It follows that the number of complete subhypergraphs of G without n-hyperedge is equal to $\sum_{1\neq k_n=0}^{\alpha_n} \sum_{k_{n-1}=0}^{\alpha_{n-1}} \cdots \sum_{k_2=0}^{\alpha_2} \sum_{k_1=0}^{0} \binom{\alpha_1}{0}\binom{\alpha_2}{k_1}\binom{\alpha_3}{k_2}\cdots\binom{\alpha_n}{k_n}$ where $2k_2 + 3k_3 + \cdots + (n-1)k_{n-1} \geq 2n$ and $k_{n-1} = \cdots = k_3 = 0$ implies that $k_2 = \binom{n}{2}$. Therefore, by Theorem 3.4, we have

$$N(n,i_0) = 2^{2^n-(n+2)} + \sum_{1\neq k_n=0}^{\alpha_n} \sum_{k_{n-1}=0}^{\alpha_{n-1}} \cdots \sum_{k_2=0}^{\alpha_2} \sum_{k_1=0}^{0} \binom{\alpha_1}{0}\binom{\alpha_2}{k_1}\binom{\alpha_3}{k_2}\cdots\binom{\alpha_n}{k_n},$$

where $2k_2 + 3k_3 + \cdots + (n-1)k_{n-1} \geq 2n$ and $k_{n-1} = \cdots = k_3 = 0$ implies that $k_2 = \binom{n}{2}$. $\square$

Example 3.10. Let $G = \{a,b,c\}$. Then we enumerate $|\mathcal{CH}g(G)|$ as follows:

$$\sum_{j=0}^{3} N(n,i_j) = N(n,i_0) + N(n,i_1) + N(n,i_2) + N(n,i_3),$$

where

$$\begin{aligned} N(n,i_0) =& \binom{3}{0}\binom{3}{0}\binom{1}{1} + \binom{3}{0}\binom{3}{1}\binom{1}{1} + \binom{3}{0}\binom{3}{2}\binom{1}{1} + \binom{3}{0}\binom{3}{3}\binom{1}{1} \\ &+ \binom{3}{0}\binom{3}{3}\binom{1}{0} = 9, \\ N(n,i_1) =& \binom{3}{1}\binom{3}{0}\binom{1}{1} + \binom{3}{1}\binom{3}{1}\binom{1}{1} + \binom{3}{1}\binom{3}{2}\binom{1}{1} + \binom{3}{1}\binom{3}{3}\binom{1}{1} \\ &+ \binom{3}{1}\binom{3}{3}\binom{1}{0} = \binom{3}{1} \times 9, \\ N(n,i_2) =& \binom{3}{2}\binom{3}{0}\binom{1}{1} + \binom{3}{2}\binom{3}{1}\binom{1}{1} + \binom{3}{2}\binom{3}{2}\binom{1}{1} + \binom{3}{2}\binom{3}{3}\binom{1}{1} \\ &+ \binom{3}{2}\binom{3}{3}\binom{1}{1} = \binom{3}{2} \times 9 \end{aligned}$$

and

$$\begin{aligned} N(n,i_3) =& \binom{3}{3}\binom{3}{0}\binom{1}{1} + \binom{3}{3}\binom{3}{1}\binom{1}{1} + \binom{3}{3}\binom{3}{2}\binom{1}{1} + \binom{3}{3}\binom{3}{3}\binom{1}{1} \\ &+ \binom{3}{0}\binom{3}{3}\binom{1}{0} = \binom{3}{0} \times 9. \end{aligned}$$

Hence, $|\mathcal{CH}g(G)| = \sum_{j=0}^{3} N(n, i_j) = 9 + 27 + 27 + 9 = 72$. Moreover,

$$\begin{aligned} S_H(3,3) =&\left(\binom{3}{0}\binom{3}{0}\binom{1}{1} + \binom{3}{0}\binom{3}{1}\binom{1}{1} + \binom{3}{0}\binom{3}{2}\binom{1}{1} + \binom{3}{0}\binom{3}{3}\binom{1}{1}\right) \\ &+ \left(\binom{3}{1}\binom{3}{0}\binom{1}{1} + \binom{3}{1}\binom{3}{1}\binom{1}{1} + \binom{3}{1}\binom{3}{2}\binom{1}{1} + \binom{3}{1}\binom{3}{3}\binom{1}{1}\right) \\ &+ \left(\binom{3}{3}\binom{3}{0}\binom{1}{1} + \binom{3}{2}\binom{3}{1}\binom{1}{1} + \binom{3}{2}\binom{3}{2}\binom{1}{1} + \binom{3}{2}\binom{3}{3}\binom{1}{1}\right) \\ &+ \left(\binom{3}{3}\binom{3}{0}\binom{1}{1} + \binom{3}{3}\binom{3}{1}\binom{1}{1} + \binom{3}{3}\binom{3}{2}\binom{1}{1} + \binom{3}{3}\binom{3}{3}\binom{1}{1}\right) \\ =&56. \end{aligned}$$

Corollary 3.3. $|\mathcal{CH}g(\mathbb{N}_2)| = 4$ *and* $|\mathcal{CH}g(\mathbb{N}_3)| = 72$.

Theorem 3.6. *Let G be a set, $|G| = n$ and*

$$T_i^* = \sum_{k_n=0}^{\alpha_n} \sum_{k_{n-1}=0}^{\alpha_{n-1}} \cdots \sum_{k_i=0}^{\alpha_i} \cdots \sum_{k_2=0}^{\alpha_2} \binom{\alpha_2}{k_1}\binom{\alpha_3}{k_2} \cdots \binom{\alpha_i}{k_i} \cdots \binom{\alpha_n}{k_n},$$

be the number of hypergraphs so that have all i-hyperedges and are complete

(i) *for $n > 3$, $T_{n-1}^* = \left(2^{n-1} - \left(1 + \frac{n(n-1)}{2}\right)\right) 2^{2^n - 2n}$;*

(ii) $T_2^* = 2^{2^n - \frac{n^2+n+2}{2}}$;

(iii) $T_n^* = \frac{1}{2}\left(T_{n-1}^* + \sum_{k_n=0}^{\alpha_n} \sum_{k_{n-1}=0}^{\alpha_{n-1}} \cdots \sum_{k_i=0}^{\alpha_i} \cdots \sum_{k_3=0}^{2} \sum_{k_2=0}^{\alpha_2} \binom{\alpha_2}{k_1}\binom{\alpha_3}{k_2} \cdots \binom{\alpha_i}{k_i} \cdots \binom{\alpha_n}{k_n}\right)$.

Proof. (i) By Theorem 3.3,

$$\begin{aligned} T_{n-1}^* =& \sum_{k_n=0}^{\alpha_n} \sum_{k_{n-1}=3}^{\alpha_{n-1}} \sum_{k_{n-2}=0}^{\alpha_{n-2}} \cdots \sum_{k_3=0}^{\alpha_3} \sum_{k_2=0}^{\alpha_2} \binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3} \cdots \binom{\alpha_{n-2}}{k_{n-2}}\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n} \\ =& 2^{\alpha_n}\left(2^{n-1} - \left(1 + \frac{n(n-1)}{2}\right)\right) \sum_{k_{n-2}=0}^{\alpha_{n-2}} \cdots \sum_{k_3=0}^{\alpha_3} \sum_{k_2=0}^{\alpha_2} \binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3} \cdots \binom{\alpha_{n-2}}{k_{n-2}} \\ =& 2^{\alpha_{n-2}}\left(2^{n-1} - \left(1 + \frac{n(n-1)}{2}\right)\right) \sum_{k_{n-3}=0}^{\alpha_{n-3}} \cdots \sum_{k_3=0}^{\alpha_3} \sum_{k_2=0}^{\alpha_2} \binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3} \cdots \binom{\alpha_{n-2}}{k_{n-2}} \\ &\vdots \\ =& 2^{\alpha_n} \cdot 2^{\alpha_{n-2}} \cdot 2^{\alpha_{n-3}} \cdots 2^{\alpha_3} 2^{\alpha_2}\left(2^{n-1} - \left(1 + \frac{n(n-1)}{2}\right)\right) \\ =& 2^{\alpha_n + \alpha_{n-2} + \alpha_{n-3} + \cdots + \alpha_3 + \alpha_2}\left(2^{n-1} - \left(1 + \frac{n(n-1)}{2}\right)\right) \end{aligned}$$

$$=2^{2^n-2n}\left(2^{n-1}-\left(1+\frac{n(n-1)}{2}\right)\right).$$

(ii) By Theorem 3.3,

$$\begin{aligned}
T_2^* &= \sum_{k_n=0}^{\alpha_n}\sum_{k_{n-1}=0}^{\alpha_{n-1}}\sum_{k_{n-2}=0}^{\alpha_{n-2}}\cdots\sum_{k_3=0}^{\alpha_3}\sum_{k_2=\alpha_2}^{\alpha_2}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\cdots\binom{\alpha_{n-2}}{k_{n-2}}\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n}\\
&= \sum_{k_n=0}^{\alpha_n}\sum_{k_{n-1}=0}^{\alpha_{n-1}}\sum_{k_{n-2}=0}^{\alpha_{n-2}}\cdots\sum_{k_4=0}^{\alpha_4}\sum_{k_3=0}^{\alpha_3}\binom{\alpha_3}{k_3}\binom{\alpha_4}{k_4}\cdots\binom{\alpha_{n-2}}{k_{n-2}}\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n}\\
&=2^{\alpha_3}\sum_{k_n=0}^{\alpha_n}\sum_{k_{n-1}=0}^{\alpha_{n-1}}\sum_{k_{n-2}=0}^{\alpha_{n-2}}\cdots\sum_{k_5=0}^{\alpha_5}\sum_{k_4=0}^{\alpha_4}\binom{\alpha_4}{k_4}\binom{\alpha_5}{k_5}\cdots\binom{\alpha_{n-2}}{k_{n-2}}\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n}\\
&\vdots\\
&=2^{\alpha_3}2^{\alpha_4}\cdots2^{\alpha_{n-2}}2^{\alpha_{n-1}}2^{\alpha_n}\\
&=2^{\alpha_n+\alpha_{n-1}+\alpha_{n-2}+\cdots+\alpha_3}\\
&=2^{2^n-\left(1+\frac{n(n+1)}{2}\right)}.
\end{aligned}$$

(iii) By Theorem 3.3,

$$\begin{aligned}
T_n^* &= \sum_{k_n=1}^{\alpha_n}\sum_{k_{n-1}=3}^{\alpha_{n-1}}\sum_{k_{n-2}=0}^{\alpha_{n-2}}\cdots\sum_{k_3=3}^{\alpha_3}\sum_{k_2=0}^{\alpha_2}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\cdots\binom{\alpha_{n-2}}{k_{n-2}}\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n}\\
&=\frac{1}{2}\Bigg(\sum_{k_n=0}^{\alpha_n}\sum_{k_{n-1}=3}^{\alpha_{n-1}}\sum_{k_{n-2}=0}^{\alpha_{n-2}}\cdots\sum_{k_3=3}^{\alpha_3}\sum_{k_2=0}^{\alpha_2}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\cdots\binom{\alpha_{n-2}}{k_{n-2}}\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n}\\
&\quad+\sum_{k_n=0}^{\alpha_n}\sum_{k_{n-1}=3}^{2}\sum_{k_{n-2}=0}^{\alpha_{n-2}}\cdots\sum_{k_3=0}^{2}\sum_{k_2=0}^{\alpha_2}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\cdots\binom{\alpha_{n-2}}{k_{n-2}}\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n}\Bigg)\\
&=\frac{1}{2}\left(T_{n-1}^*+\sum_{k_n=0}^{\alpha_n}\sum_{k_{n-1}=3}^{\alpha_{n-1}}\sum_{k_{n-2}=0}^{\alpha_{n-2}}\cdots\sum_{k_3=0}^{2}\sum_{k_2=0}^{\alpha_2}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\cdots\binom{\alpha_{n-2}}{k_{n-2}}\binom{\alpha_{n-1}}{k_{n-1}}\binom{\alpha_n}{k_n}\right)\square
\end{aligned}$$

In the following example, for any $n\in\mathbb{N}$, we enumerate T_i^*'s.

Example 3.11. Let G be a set and $|G|=9$. We have

$$\begin{aligned}
T_9^* &= \sum_{k_8=0}^{9}\sum_{k_7=0}^{36}\sum_{k_6=0}^{84}\sum_{k_5=0}^{126}\sum_{k_4=0}^{126}\sum_{k_3=0}^{84}\sum_{k_2=0}^{36}\binom{36}{k_2}\binom{84}{k_3}\binom{126}{k_4}\binom{126}{k_5}\binom{84}{k_6}\binom{36}{k_7}\binom{9}{k_8}\\
&=2^9\sum_{k_7=0}^{36}\sum_{k_6=0}^{84}\sum_{k_5=0}^{126}\sum_{k_4=0}^{126}\sum_{k_3=0}^{84}\sum_{k_2=0}^{36}\binom{36}{k_2}\binom{84}{k_3}\binom{126}{k_4}\binom{126}{k_5}\binom{84}{k_6}\binom{36}{k_7}\\
&=2^92^{36}\sum_{k_6=0}^{84}\sum_{k_5=0}^{126}\sum_{k_4=0}^{126}\sum_{k_3=0}^{84}\sum_{k_2=0}^{36}\binom{36}{k_2}\binom{84}{k_3}\binom{126}{k_4}\binom{126}{k_5}\binom{84}{k_6}
\end{aligned}$$

$$
\begin{aligned}
&=2^9 2^{36} 2^{84} \sum_{k_5=0}^{126}\sum_{k_4=0}^{126}\sum_{k_3=0}^{84}\sum_{k_2=0}^{36}\binom{36}{k_2}\binom{84}{k_3}\binom{126}{k_4}\binom{126}{k_5}\\
&=2^9 2^{36} 2^{84} 2^{126} \sum_{k_4=0}^{126}\sum_{k_3=0}^{84}\sum_{k_2=0}^{36}\binom{36}{k_2}\binom{84}{k_3}\binom{126}{k_4}\\
&=2^9 2^{36} 2^{84} 2^{126} 2^{126} \sum_{k_3=0}^{84}\sum_{k_2=0}^{36}\binom{36}{k_2}\binom{84}{k_3}\\
&=2^9 2^{36} 2^{84} 2^{126} 2^{126} 2^{84} \sum_{k_2=0}^{36}\binom{36}{k_2}\\
&=2^9 2^{36} 2^{84} 2^{126} 2^{126} 2^{84} 2^{36} = 2^{501}.
\end{aligned}
$$

In a similar way

$$
\begin{aligned}
T_2^* &= \sum_{k_8=0}^{9}\sum_{k_7=0}^{36}\sum_{k_6=0}^{84}\sum_{k_5=0}^{126}\sum_{k_4=0}^{126}\sum_{k_3=0}^{84}\binom{84}{k_3}\binom{126}{k_4}\binom{126}{k_5}\binom{84}{k_6}\binom{36}{k_7}\binom{9}{k_8} = 2^{465},\\
T_8^* &= \sum_{k_8=3}^{9}\sum_{k_7=0}^{36}\sum_{k_6=0}^{84}\sum_{k_5=0}^{126}\sum_{k_4=0}^{126}\sum_{k_3=0}^{84}\sum_{k_2=0}^{36}\binom{36}{k_2}\binom{84}{k_3}\binom{126}{k_4}\binom{126}{k_5}\binom{84}{k_6}\binom{36}{k_7}\binom{9}{k_8}\\
&= 466 \times 2^{492},\\
T_7^* &= \sum_{k_7=4}^{36}\sum_{k_8=0}^{9}\sum_{k_6=0}^{84}\sum_{k_5=0}^{126}\sum_{k_4=0}^{126}\sum_{k_3=0}^{84}\sum_{k_2=0}^{36}\binom{36}{k_2}\binom{84}{k_3}\binom{126}{k_4}\binom{126}{k_5}\binom{84}{k_6}\binom{36}{k_7}\binom{9}{k_8}\binom{36}{k_7}\\
&= 2^{465} \times (2^{36} - 8227)
\end{aligned}
$$

and

$$
\begin{aligned}
T_6^* =& \sum_{k_6=5}^{84}\sum_{k_7=0}^{36}\sum_{k_8=0}^{9}\sum_{k_5=0}^{126}\sum_{k_4=0}^{126}\sum_{k_3=0}^{84}\sum_{k_2=0}^{36}\binom{36}{k_2}\binom{84}{k_3}\binom{126}{k_4}\binom{126}{k_5}\binom{9}{k_8}\binom{36}{k_7}\binom{84}{k_6}\\
&=2^{417} \times (2^{84} - 2028365).
\end{aligned}
$$

Theorem 3.7. *Let G be a set, $|G| = n$ and $T_i = \sum_{k_2=0}^{\alpha_1}\sum_{k_3=0}^{\alpha_3}\cdots\sum_{k_i=0}^{\alpha_i}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\cdots\binom{\alpha_i}{k_i}$, be the number of hypergraphs so that have i-hyperedges and are complete*

(i) *$T_n = T_n^*$;*

(ii) *For $n \geq 4$, $T_{n-1} = \left(2^{n-1} - \left(1 + \frac{n(n-1)}{2}\right)\right) 2^{2^n-(2n+1)}$;*

(iii) *$T_2 = 1$;*

(iv) *If $n \geq 3$, then $k_{n-1} \geq 3$ and so*

$$
|\mathcal{CH}g(G)| \geq 2^n\left(1 + 2^{2^n-(n+2)} + \left(2^{n-1} - \left(1 + \frac{n(n-1)}{2}\right)\right) 2^{2^n-(2n+1)}\right).
$$

Proof. (i) Is obtained by Theorem 3.6, and (iii) is clear.

(ii) Since $n \geq 3$ by Theorem 3.3, we get $S_H(n, n-1) \geq 3$ and so we have

$$T_{n-1} = \sum_{k_2=0}^{\alpha_2} \sum_{k_3=0}^{\alpha_3} \sum_{k_4=0}^{\alpha_4} \cdots \sum_{k_{n-2}=0}^{\alpha_{n-2}} \sum_{k_{n-1}=3}^{\alpha_{n-1}} \binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\binom{\alpha_4}{k_4} \cdots \binom{\alpha_{n-2}}{k_{n-2}}\binom{\alpha_{n-1}}{k_{n-1}}$$
$$= \left(2^{n-1} - \left(1 + \frac{n(n-1)}{2}\right)\right) 2^{2^n-(2n+1)}.$$

(iii) By Theorem 3.3, if $k_n = 1$, clearly H is a complete subhypergraph and so $T_n^* = 2^{2^n-(n+2)}$. If $n \geq 3$ by Theorem 3.3, we get $S_H(n, n-1) \geq 3$ and H is a complete subhypergraph. Moreover, for $S_H(n, 2) = \frac{n(n-1)}{2}$ always H is a complete subhypergraph. Therefore, $|\mathcal{CH}g(G)| \geq 2^n[T_n + T_{n-1} + T_2]$. □

Corollary 3.4. *Let G be a hypergraph, $|G| = n$ and $1 \leq t \leq n$. If $n \geq 3t$, then*

$$|\mathcal{CH}g(G)| \geq 2^n[T_2 + T_n + T_{n-1} + T_{n-2} + \cdots + T_{n-t}].$$

Proof. Let H be a subhypergraph of G. Since $n \geq 3t$, by Corollary 3.2, we get that $S_H(n, n-t) \geq t+2$ and so H is a complete subhypergraph. □

Example 3.12. Let G be a set and $|G| = 3$. Then

$$|\mathcal{CH}g(G)| \geq 2^3 \left(1 + 2^{2^3-(3+2)} + \left(2^{3-1} - \left(1 + \frac{3(3-1)}{2}\right)\right) 2^{2^3-(2\times 3+1)}\right) = 8(1+8+0) = 72.$$

Open Problem. Let G be a set and $|G| = n$. Then $|N(n, i_0)| =?$
Open Problem. Let G be a set and $|G| = n$. Then $||\mathcal{CH}g(G)|| =?$

4. Redefining of Hypergraphs with Regarding to Hypergroupoid

In this section, we define the concept of hypergraphs via the hyperoperations on any set and prove that on any nonempty set, we can construct at least a hypergraph. It has also proved that a hypergraph was obtained from any quasihypergroup. Moreover, it has shown that a hypergroupoid was obtained from any hypergraph. Finally we define the concept of complete hypergraph and show that some complete hypergraph yields a hypergroup.

Definition 4.1. Let $(G, \circ)$ be a hypergroupoid. Then for any $x_i, y_j \in G$, we set $E_{i,j} = x_i \circ y_j$. If $(G, \{E_{i,j}\}_{\{i,j\}})$ is a hypergraph, then we will call $(G, \{E_{i,j}\}_{\{i,j\}})$ is an *associated* hypergraph and we will denote it by $G^{\downarrow}$.

Example 4.1. Let $G = \{a, b, c\}$. Then $(G, \circ)$ is a hypergroupoid as follows:

$\circ$	a	b	c
a	$\{a\}$	$\{a,b\}$	$\{a\}$
b	$\{a,b\}$	$\{b\}$	$\{b\}$
c	$\{b\}$	$\{a\}$	$\{a,b\}$

then yields a hyperdiagram while is not a hypergraph as follows (Figure 6).

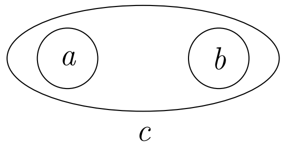


FIGURE 6. Hyperdiagram obtained of above table

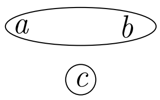
Theorem 4.1. *Let G be a nonempty set. Then there exists at least a hyperoperation $\circ$ on G so that $(G,\circ)$ is an its associated hypergraph.*

Proof. For any $x,y \in G$, we apply the B-hyperoperation [11, 12], on G as $x \circ y = \{x,y\}$. Clearly $(G,\circ)$ is a hypergroupoid and for any $x,y \in G, E_{x,y} = \{x,y\}$. Now, $\bigcup\limits_{x,y\in G}(x \circ y) = G$ and so $G^{\downarrow} = (G,\{E_{x,y}\})$ is an associated hypergraph. □

Theorem 4.2. *Let $(G,\circ)$ be a quasihypergroup. Then $(G,\circ)$ is an associated hypergraph.*

Proof. Let $x \in G$. Since $(G,\circ)$ is a quasihypergroup, we get that $x \circ G = G$ and so $(G,\circ)$ is an associated hypergraph. □

Example 4.2. Let $G = \{a,b,c\}$. Then $(G,\circ)$ is an its associated hypergraph as Figure 7. Thus the converse of Theorem 4.2, necessarily is not true.



$\circ$	a	b	c
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$

FIGURE 7. Associated hypergraph of hypergroupoid $(G,\circ)$

Theorem 4.3. *Let G be a total hypergroup. Then any its associated hypergraph is complete.*

Proof. Since $(G,\circ)$ is a total hypergroup, for any $x,y \in G$, $x \circ y = G$. Thus $(G,\circ)$ is an its associated hypergraph and has a hyperedge as $E = G$, in such a way that for any $x,y \in G$, $\{x,y\} \subseteq E$, so it is complete. □

Definition 4.2. Let $(G,\{E_x\}_{x\in G})$ be a hypergraph. Then for any $x,y \in G$ we set $x \circ y = E_{x,y}$, where $E_{x,y}$ is a hyperedge in such a way that has the smallest cardinal that contains the vertices x,y. If $(G,\circ)$ is a hypergroup, then we will call $(G,\circ)$ is an its *associated* hypergroup and we will denote it by $G^{\uparrow}$.

Lemma 4.1. *Let $(G,\{E_x\}_{x\in G})$ be a hypergraph. Then there exists at least a hyperoperation $\circ$ on G such that $(G,\circ)$ a quasihypergroup.*

Proof. Define a hyperoperation $\circ$ on G as follows:

$$x \circ y = \begin{cases} E, & \text{if there exists } E \text{ such that } x, y \in E, \\ G, & \text{otherwise.} \end{cases}$$

Clearly, $(G, \circ)$ is a quasihypergroup. $\square$

Example 4.3. Let $G = \{a, b, c\}$ be a hypergraph as follows (Figure 8).

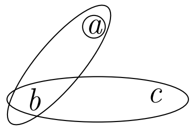


FIGURE 8. Hypergraph

then $(G, \circ_1)$ and $(G, \circ_2)$ are hypergroups as follows:

$\circ_1$	a	b	c
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$
b	$\{a,b\}$	$\{b,c\}$	$\{b,c\}$
c	$\{a,b,c\}$	$\{b,c\}$	$\{b,c\}$

and

$\circ_2$	a	b	c
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$
b	$\{a,b\}$	$\{b,a\}$	$\{b,c\}$
c	$\{a,b,c\}$	$\{b,c\}$	$\{b,c\}$

.

Since $b \circ_1 b \neq b \circ_2 b$, so $(G, \circ_1) \ncong (G, \circ_2)$.

Theorem 4.4. *Let $|G| = n$, $(G, \{E_x\}_{x\in G})$ be a complete hypergraph without n loops and has n-hyperedge. Then $G^{\uparrow}$ is a commutative hypergroup.*

Proof. Since $(G, \{E_x\}_{x\in G})$ is a complete hypergraph without n loops and has n-hyperedge, for any $x, y \in G$ we obtain a hyperoperation $\circ$ on G as follows:

$$x \circ y = \begin{cases} \{x, y\}, & \text{if } x \neq y, \\ G, & \text{otherwise.} \end{cases}$$

Clearly, $(G, \circ)$ is a semihypergroup, so by Lemma 4.1, $(G, \circ)$ is a hypergroup. $\square$

Example 4.4. Let $G = \{a, b, c\}$ be a hypergraph as Figure 9 (K_3^*), then $(G, \circ_1)$ is not

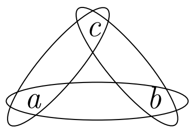


FIGURE 9. Complete hypergraph(K_3^*)

a hypergroup, $(G, \circ_2)$ is a hypergroup and $(G, \circ_3)$ is one of associated hypergroups as follows:

$\circ_1$	a	b	c
a	$\{a,b\}$	$\{a,b\}$	$\{a,c\}$
b	$\{a,b\}$	$\{b,c\}$	$\{b,c\}$
c	$\{a,c\}$	$\{b,c\}$	$\{b,c\}$

,

$\circ_2$	a	b	c
a	G	$\{a,b\}$	$\{a,c\}$
b	$\{a,b\}$	G	$\{b,c\}$
c	$\{a,c\}$	$\{b,c\}$	G

,

$\circ_3$	a	b	c
a	$\{a,b\}$	$\{a,b\}$	$\{a,c\}$
b	$\{a,b\}$	$\{a,b\}$	$\{b,c\}$
c	$\{a,c\}$	$\{b,c\}$	$\{b,c\}$

.

Theorem 4.5. *Let $|G| = n$, G be a complete hypergraph with n loops. Then $G^{\uparrow}$ is a commutative hypergroup.*

Proof. Since G is a complete hypergraph in such a way has n loops, for any $x, y \in G$ we obtain the B-hyperoperation *"$\circ$"* [11, 12] on G as follows:

$$x \circ y = \{x, y\}.$$

Clearly $(G, \circ)$ is a quasihypergroup. Since for any $x, y, z \in G, (x \circ y) \circ z = \{x, y\} \circ z = \{x, y, z\} = x \circ (y \circ z)$, we get that $(G, \circ)$ is also a semihypergroup. Moreover, for any $x, y \in G, x \circ y = \{x, y\} = y \circ x$, it follows that $(G, \circ)$ is a commutative transposition hypergroup. $\square$

Corollary 4.1. *With any finite set, we can construct at least two commutative hypergroups.*

5. Connection Between Hypergraphs and Graphs via Equivalence Relation ρ

In this section, we will define an equivalence relation as ρ on hypergraph G, in such a way, we can construct a quotient G/ρ and show that it is a graph. Moreover, the concept of fundamental graph is defined and it is shown that any graph is a fundamental graph.

Definition 5.1. Let $(G, \{E_x\}_{x \in G})$ be a hypergraph, where for any $x \in G, E_x$ is one of hyperedges such that $x \in E_x$. Then define a binary relation ρ on G as follows: for every integer $n \geq 1$, ρ_n is defined as follows:

$$x \rho_n y \Leftrightarrow |E_x^m| = |E_y^m|,$$

where $|E_x^m| = \min\{|E_t|;\ x \in E_t\}$ or $|E_x^m| \leq |E_x|$ and $n = \min\{\deg(x), \deg(y)\}$.

Obviously the relation $\rho = \bigcup\limits_{n \geq 1} \rho_n$ is an equivalence relation on G. We denote the set of all equivalence classes of ρ by G/ρ. Hence $G/\rho = \{\rho(x) \mid x \in G\}$.

Theorem 5.1. *Let $(G, \{E_x\}_{x \in G})$ be a hypergraph. Then there exists an operation $*$ on G/ρ such that $(G/\rho, *)$ is a graph.*

Proof. For any $\rho(x), \rho(y) \in G/\rho$, define an operation "$*$" on G/ρ by

$$\rho(x) * \rho(y) = \begin{cases} \widehat{\rho(x), \rho(y)}, & \text{if } E_x \cap E_y \neq \emptyset, \\ \widehat{\emptyset}, & \text{otherwise,} \end{cases}$$

where for any $x, y \in G$, $(\widehat{\rho(x), \rho(y)})$ is represented as an ordinary (simple) edge and $\widehat{\emptyset} = \widehat{\rho(x)}$ means that there is not edge. It is easy to see that $(G/\rho, *)$ is a graph. $\square$

Example 5.1. Let $G = \{a, b, c\}$, $E_1 = \{a\}$, $E_2 = \{a, b\}$ and $E_3 = \{a, b, c\}$. Then (G, E_1, E_2, E_3) is a hypergraph as Figure 10. Since $E_a^m = \{a\}, E_b^m = \{a, b\}$ and

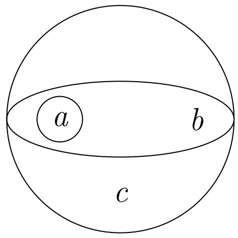


FIGURE 10. Hypergraph

$E_c^m = \{a, b, c\}$, we get that $\rho(a) = \{a\}$, $\rho(b) = \{b\}$ and $\rho(c) = \{c\}$. So we obtained the graph in Figure 11.

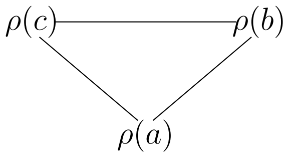


FIGURE 11. Graph obtained by hypergraph Figure 10

Definition 5.2. A graph $G = (V, E)$ is said to be a *fundamental graph* if there exists a nontrivial hypergraph say, $(G, \{E_k\}_{k=1}^n)$ such that $(G, \{E_k\}_{k=1}^n)/\rho \cong G = (V, E)$. In other words, it is equal to the fundamental of nontrivial hypergraph up to isomorphic.

Theorem 5.2. *Any graph is a fundamental graph.*

Proof. Let $G = (V, E, *')$ be a graph, in such a way that $V = \{a_1, a_2, \ldots, a_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, where $m \leq n$. Suppose that for any $1 \leq i \leq m$, $e_i = \{a_i, a_{i'}\} = a_i *' a_{i'}$. Define a hypergraph $\overline{G} = (\overline{V}, \{\overline{E_i}\}_{i=1}^n)$ as follows:

$$\overline{E_i} = \{a_i\} \cup A_i,$$

such that for any $1 \leq k \leq n$ we have $|A_k| = k$ and for any $1 \leq i, i' \leq n$, $|\overline{E_i}| < |\overline{E_{i+1}}|$ and $\overline{E_i} \cap \overline{E_{i'}} \neq \emptyset$. It is easy to see that $\overline{V} = \bigcup\limits_{i=1}^{n} A_i \cup V$ and $(\overline{V}, \{\overline{E_i}\}_{i=1}^n)$ is a hypergraph. Clearly for any $1 \leq i \leq n$, $\rho(a_i) = \overline{E_i}$ and since $\overline{E_i} \cap \overline{E_{i'}} \neq \emptyset$, we get that $G/\rho = \{\rho(a_i) \mid 1 \leq i \leq n\}$ and so obtain

$$\rho(a_i) * \rho(a_j) = \begin{cases} \widehat{\rho(a_i), \rho(a_{i'})}, & \text{if } j = i', \\ \widehat{\emptyset} & \text{if } j = i. \end{cases}$$

Now define a map $\varphi : (G/\rho, *) \to G = (V, E)$ by $\varphi(\rho(a_i)) = a_i$ and $\varphi((\widehat{\rho(a_i), \rho(a_{i'})})) = e_i$. Let $x, y \in \overline{V}$. If $\rho(x) = \rho(y)$, then $|\overline{E_x}| = |\overline{E_y}|$ and so $\overline{E_x} = \overline{E_y}$. Thus $\varphi(\rho(x)) = \varphi(\rho(y))$. Since for any $1 \leq i, i' \leq n$,

$$\varphi(\rho(a_i) * \rho(a_{i'})) = \varphi((\widehat{\rho(a_i), \rho(a_{i'})})) = e_i = a_i *' a_{i'} = \varphi(\rho(a_i)) *' \varphi(\rho(a_{i'})),$$

in other words, if $\rho(a_i)$ and $\rho(a'_i)$ in G/ρ are adjacent, then $\varphi(\rho(a_i))$ and $\varphi(\rho(a'_i))$ in G are adjacent. So φ is a homomorphism. It is easy to see that φ is bijection and so is an isomorphism. It follows that any graph is a fundamental graph. $\square$

Example 5.2. Let $V = \{a_1, a_2, a_3, a_4, a_5\}$, $E = \{e_1, e_2\}$. Consider the unconnected graph $G = (V, E)$ as Figure 12.

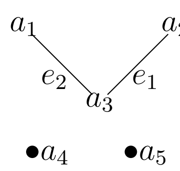
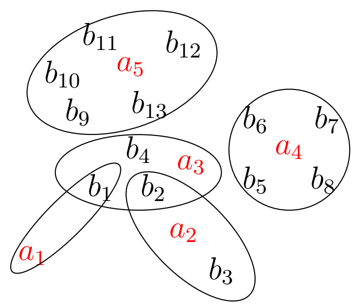


FIGURE 12. Graph (G)

Now for any arbitrary set $B = \bigcup_{i=1}^{13} b_i$ where for any i, j have $a_i \neq b_j$, we define a hypergraph $\overline{G} = (\overline{V}, \{\overline{E_i}\})$ in a way that $\overline{E_1} = \{a_1\} \cup \{b_1\}$, $\overline{E_2} = \{a_2\} \cup \{b_2, b_3\}$, $\overline{E_3} = \{a_3\} \cup \{b_1, b_2, b_4\}$, $\overline{E_4} = \{a_4\} \cup \{b_5, b_6, b_7, b_8\}$, $\overline{E_5} = \{a_5\} \cup \{b_9, b_{10}, b_{11}, b_{12}, b_{13}\}$. Hence consider the hypergraph $\overline{G} = (\overline{V}, \{\overline{E_i}\})$ in Figure 13 $(A), \overline{G}$. Clearly for any



(A) $\overline{G}$

FIGURE 13. Associated hypergraph with graph Figure 12

$1 \leq i \leq 5, \rho(a_i) = \overline{E_i}$, so

$$\overline{G}/\rho = \Big(\Big\{\rho(a_1), \rho(a_2), \rho(a_3), \rho(a_4), \rho(a_5)\Big\}, \Big\{\{\rho(a_1), \rho(a_3)\}, \{\rho(a_2), \rho(a_3)\}\Big\}\Big)$$

is a graph as Figure 14 (G).

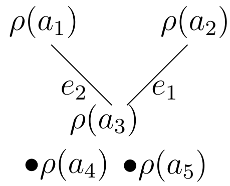


Figure 14. Graph obtained of hypergraph Figure 13

It is easy to see that $G \cong \overline{G}/\rho$ and so $G = (V, E)$ is a fundamental graph.

Example 5.3. Let $G = \{a_1, a_2, \ldots, a_n\}$. Then it is easy to see that $K_n^* = (G, \{E_k\}_{k=1}^n)$ is a complete hypergraph, where $E_k = \{a_1, a_2, \ldots, a_k\}$. Clearly $E_{a_1}^m = \{a_1\}$, $E_{a_2}^m = \{a_1, a_2\}$ and for any $1 \leq i \leq n, E_{a_i}^m = \{a_1, a_2, \ldots, a_i\}$. Hence $K_n^*/\rho = \{\rho(a_i) \mid 1 \leq i \leq n\}$ and so $K_n^*/\rho \cong K_n$. Therefore for any $n \in \mathbb{N}, K_n$ is a fundamental graph.

Theorem 5.3. *Let $(G, \{E_x\}_{x \in G})$ be a complete hypergraph and $|G| = n$. Then its fundamental graph is isomorphic to one of the complete graphs as $K_n, K_{n-1}, \ldots, K_1$.*

Proof. Let $(G, \{E_x\}_{x \in G})$ be a complete hypergraph. Since for any $x, y \in G$ there exists a hyperedge $E_{x,y}$ in such a way that $x, y \in E_{x,y}$, we get G/ρ is a complete graph. Moreover, let $G = \{a_1, a_2, \ldots, a_n\}$ and $m < n$. Then for any $2 \leq i \leq m$, consider $E_1 = \{a_1\}, E_2 = \{a_2\}, E_3 = \{a_3\}, \ldots, E_{n-m} = \{a_{n-m}\}, E_{n-m+1} = \{a_{n-m+1}\}$ and $E_{n-m+i} = \bigcup\limits_{j=1}^{n-m+i-1} E_j \cup \{a_{n-m+i}\}$. Clearly $(G, \{E_i\}_{i=1}^m)$ is a hypergraph and it is easy to see that for all $1 \leq m \leq n$, $G/\rho \cong K_m$. □

5.1. **Fundamental group and fundamental graph.**

In this subsection, we define concept of discrete and joint complete hypergraph and study the relation between fundamental graphs and fundamental groups.

Theorem 5.4. *Let G be a total hypergroup. Then*

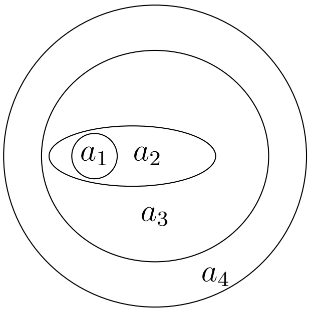
(i) *its associated fundamental graph is isomorphic to complete graph K_1;*
(ii) $|G^{\downarrow}/\rho| = |G/\beta^*|$.

Proof. (i) Let $(G, \circ)$ be a total hypergroup and $G^{\downarrow} = (G, \{E_{i,j}\}_{\{i,j\}})$ be its associated hypergraph. Then by Theorem 4.3, $G^{\downarrow}$ is a complete hypergraph, where for any $1 \leq i, j \leq n$, $\{E_{i,j}\}_{\{i,j\}} = G$ and by Theorem 5.3, $G/\rho \cong K_1$.
(ii) It is immediate. □

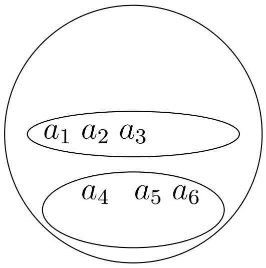
Definition 5.3. Let $(G, \{E_i\}_{i=1}^{n+1})$ be a complete hypergraph. We say that

(i) $(G, \{E_i\}_{i=1}^{n+1})$ is a joint complete hypergraph, if for any $1 \leq i \leq n$, $|E_i| = i, E_i \subseteq E_{i+1}$ and $|E_{n+1}| = n$;
(ii) $(G, \{E_i\}_{i=1}^{n+1})$ is a discrete complete hypergraph, if for any $1 \leq i, j \leq n, |E_i| = |E_j|$ and $|E_{n+1}| = n$.

Example 5.4. Let $G = \{a_1, a_2, a_3, a_4\}$ and $G' = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Then we have that $(G, \{E_x\}_{x\in G})$ and $(G', \{E_x\}_{x\in G'})$ are joint complete hypergraph and discrete complete hypergraph respectively. Figure 15 ((A) and (B)).



(A) G



(B) G'

FIGURE 15. Joint and discrete hypergaphs

Theorem 5.5. *Let $(G, \{E_x\}_{x\in G})$ be a complete hypergraph and $|G| = n$. Then*

(i) *if $(G, \{E_x\}_{x\in G})$ is a joint complete hypergraph, then in its associated hypergroup there exists a hyperproduct so that is equal to G;*
(ii) *if $(G, \{E_x\}_{x\in G})$ is a discrete complete hypergraph, then for any $x \in G$ we have* $\deg(x) \in \{1, 2, 3, \ldots, n\}$.

Proof. (i) Since $(G, \{E_x\}_{x\in G})$ is a joint complete hypergraph, then there exists a hyperedge as E such that $|E| = n$ and so $G = E$. Now for some $x, y \in G$ we obtain a hyperoperation $\circ$ on G by $x \circ y = E_{x,y}$. Clearly $(G, \circ)$ is a hypergroup and there exists a hyperproduct is equal to G.
(ii) Since $(G, \{E_x\}_{x\in G})$ is a discrete complete hypergraph, then for any $1 \leq i, j \leq n, |E_i| = |E_j|$ and $|E_{n+1}| = n$. Now for any $1 \leq i \leq n$, $|E_i| \in \{1, 2, \ldots, n\}$, so for any $x \in G, \deg(x) \in \{1, 2, \ldots, n\}$. $\square$

Theorem 5.6. *Let $G = \{a_1, a_2, \ldots, a_n\}$. Then*

(i) *$(G, \{E_x\}_{x\in G})$ is a joint complete hypergraph if and only if $|G/\rho| = n$;*
(ii) *if $(G, \{E_x\}_{x\in G})$ is a discrete complete hypergraph, then $|G/\rho| = 1$;*
(iii) *if $|G/\rho| = 1$ and $(G, \{E_x\}_{x\in G})$ is complete hypergraph, then $(G, \{E_x\}_{x\in G})$ is discrete;*
(iv) *if $|G/\rho| = n$, then $|G^{\uparrow}/\beta^*| = 1$;*
(v) *if $(G, \{E_x\}_{x\in G})$ is complete hypergraph and $|G/\rho| = 1$, then $|G^{\uparrow}/\beta^*| = 1$.*

Proof. (i) Since $(G, \{E_i\}_{i=1}^{n+1})$ is a joint complete hypergraph, we get that for any $1 \leq i \leq n+1$, $E^m_{a_i} = \{a_1, a_2, \dots, a_i\}$, hence $G/\rho = \{\rho(a_i) \mid 1 \leq i \leq n\}$ and so $|G/\rho| = n$. Conversely, let $|G/\rho| = n$ then $G/\rho = \{\rho(a_1), \rho(a_2), \dots, \rho(a_n)\}$. It follows that for any $1 \leq i, j \leq n$, $|\rho(a_i)| \neq |\rho(a_j)|$ and $\rho(a_i) \cap \rho(a_j) \neq \emptyset$. Hence without loss of generality for any $1 \leq i \leq n$, we can rearrange $\rho(a_i) = \{a_1, a_2, \dots, a_i\}$. Thus for any $1 \leq i \leq n$, $|\rho(a_i)| = i, \rho(a_i) \subseteq \rho(a_{i+1})$ and so $(G, \{E_i\}_{i=1}^{n+1})$ is a joint complete hypergraph.

(ii) Since $(G, \{E_i\}_{i=1}^{n+1})$ is a discrete complete hypergraph, we have for any $1 \leq i, j \leq n$, $|E^m_{a_i}| = |E^m_{a_j}|$, $E^m_{a_{n+1}} = \{a_1, a_2, \dots, a_n\}$, hence $G/\rho = \{\rho(a_1)\}$ and so $|G/\rho| = 1$.

(iii) Let $|G/\rho| = 1$. Then for any $1 \leq i, j \leq n$, $|\rho(a_i)| = |\rho(a_j)|$. If for some $1 \leq i \leq n, |\rho(a_i)| = n$, then $(G, \{E_i\}_{i=1}^{n+1})$ is a discrete complete hypergraph, but if for any $1 \leq i \leq n$, $|\rho(a_i)| < n$, since $(G, \{E_i\}_{i=1}^{n+1})$ is complete, we get there exists a hyperedge E such that $E = \{a_1, a_2, \dots, a_n\}$. Therefore, $(G, \{E_i\}_{i=1}^{n+1})$ is a discrete complete hypergraph.

(iv), (v) Let $|G/\rho| \in \{1, n\}$. Then by (i), (ii), $(G, \{E_x\}_{x\in G})$ is a joint complete hypergraph or discrete complete hypergraph. Now by Theorem 5.5, there exists a hyperedge as E, in such a way $E = G$. Thus there exists $x, y \in G$ so that $x \circ y = G$ so there exists an element $a \in G$ so that $\beta^*(a) = G$. Therefore, $G^{\uparrow}/\beta^* = \{\beta^*(a)\}$ and so $|G^{\uparrow}/\beta^*| = 1$. □

Example 5.5. Let $G = \{a, b, c, d, e\}$. Then $(G, \circ)$ is a hypergroup as follows:

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{c\}$	$\{d\}$	$\{d\}$	$\{d,e\}$
b	$\{b\}$	$\{a,b\}$	$\{d,e\}$	$\{c,d,e\}$	$\{c,d,e\}$
c	$\{c\}$	$\{d,e\}$	$\{a,c\}$	$\{b,d,e\}$	$\{b,d,e\}$
d	$\{d\}$	$\{c,d,e\}$	$\{b,d,e\}$	G	G
e	$\{d,e\}$	$\{c,d,e\}$	$\{b,d,e\}$	G	G

Obviously, $(G, \circ)$ is a hypergroup, $|G/\beta^*| = 1$ and $|G^{\downarrow}/\rho| = 2$. So the converse of Theorem 5.6, (iv) is not necessarily true.

Example 5.6. (i) Consider the discrete complete hypergraph $(G, \{E_x\}_{x\in G})$ and joint complete hypergraph $(G', \{E_x\}_{x\in G'})$ that are defined in Example 5.4. It is easy to see that $|G/\rho| = 4, |G^{\uparrow}/\beta^*| = 1, |G'/\rho| = 1$ and $|G'^{\uparrow}/\beta^*| = 1$.

(ii) Let $G = \{a, b, c, d, e\}$. Then $(G, \{E_x\}_{x\in G})$ is a discrete complete hypergraph as Figure 16. Clearly $|G/\rho| = 1$ and $|G^{\uparrow}/\beta^*| = 1$.

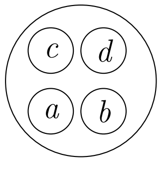


FIGURE 16. Discrete hypergraph

Theorem 5.7. *Let $G = \{a_1, a_2, \dots, a_n\}$. Then*

(i) *if G is a hypergroup and $|G/\beta^*| = n$, then $|G^{\downarrow}/\rho| = 1$;*
(ii) *if $|G/\beta^*| = 1$, then $G^{\downarrow}$ is a complete hypergraph;*
(iii) *if $|G/\beta^*| = 1$, then $|G^{\downarrow}/\rho| \in \{1, 2, \dots, n\}$.*

Proof. (i) Since G is a hypergroup, by Theorem 4.2, $G^{\downarrow}$ is a hypergraph and $|G/\beta^*| = n$ implies that for any $x \in G, |\beta^*(x)| = 1$. Hence, for any $x \in G^{\downarrow}, |\rho(x)| = 1$ and so $|G^{\downarrow}/\rho| = 1$.
(ii) Since $|G/\beta^*| = 1$, then for any $x, y \in G$, $\beta^*(x) = \beta^*(y)$. It follows that for any $x, y \in G$ there exists a hyperedge $E_{x,y}$ so that $\{x, y\} \subseteq E_{x,y}$, hence $G^{\downarrow}$ is a complete hypergraph.
(iii) Since $|G/\beta^*| = 1$, by (ii) we get that $G^{\downarrow}$ is a complete hypergraph. Now $G^{\downarrow}$ is a complete hypergraph, so by Theorem 5.3, $|G^{\downarrow}/\rho| = m$, where $m \leq n$. □

Example 5.7. Let $G = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$. Then $(G, \circ)$ is a hypergroup as follows:

$\circ$	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_1	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_4\}$	$\{a_5\}$	$\{a_6, a_7\}$	$\{a_6, a_7\}$
a_2	$\{a_2\}$	$\{a_1\}$	$\{a_5\}$	$\{a_6, a_7\}$	$\{a_3\}$	$\{a_4\}$	$\{a_4\}$
a_3	$\{a_3\}$	$\{a_6, a_7\}$	$\{a_1\}$	$\{a_5\}$	$\{a_4\}$	$\{a_2\}$	$\{a_2\}$
a_4	$\{a_4\}$	$\{a_5\}$	$\{a_6, a_7\}$	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_3\}$
a_5	$\{a_5\}$	$\{a_4\}$	$\{a_2\}$	$\{a_3\}$	$\{a_6, a_7\}$	$\{a_1\}$	$\{a_1\}$
a_6	$\{a_6, a_7\}$	$\{a_3\}$	$\{a_4\}$	$\{a_2\}$	$\{a_1\}$	$\{a_5\}$	$\{a_5\}$
a_7	$\{a_6, a_7\}$	$\{a_3\}$	$\{a_4\}$	$\{a_2\}$	$\{a_1\}$	$\{a_5\}$	$\{a_5\}$

It is easy to see that $|G/\beta^*| = 6$ but $G^{\downarrow}$ is not a complete hypergraph and $|G^{\downarrow}/\rho| = 2$. Hence the converse of Theorem 5.7, (iii) is not necessarily true.

Theorem 5.8. *Let G be a nonempty set and $|G| = n$. Then*

(i) *if $(G, \{E_x\}_{x \in G})$ is a complete hypergraph that is not a complete graph, then $(G^{\uparrow})^{\downarrow}$ is a complete hypergraph;*
(ii) *if $(G, \{E_x\}_{x \in G})$ is a complete hypergraph in such a way is not a complete graph, then $(G^{\uparrow})^{\downarrow} = G$;*
(iii) *if $(G, \circ)$ is a hypergroup so that $G^{\downarrow}$ is a complete hypergraph and is not a complete graph, then $(G^{\downarrow})^{\uparrow} = G$.*

Proof. (i) If $(G, \{E_x\}_{x \in G})$ is a complete hypergraph in such a way is not a complete graph, then by Theorems 4.4 and 4.5, $G^{\uparrow}$ is a commutative hypergroup in such a way for any $x \in G, x \circ x \in \{\{x\}, G\}$, and so $(G^{\uparrow})^{\downarrow}$ is a complete hypergraph.
(ii), (iii) By (i) are immediate. □

Example 5.8. (i) Consider the hypergroup which is defined in Example 5.5, Then $((G^{\downarrow})^{\uparrow}, \circ)$ is obtained as follows:

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	G	G
b	$\{a,b\}$	$\{b\}$	G	$\{b,d,e\}$	$\{b,d,e\}$
c	$\{a,c\}$	G	$\{c\}$	$\{c,d,e\}$	$\{c,d,e\}$
d	G	$\{b,d,e\}$	$\{c,d,e\}$	$\{d\}$	$\{e,d\}$
e	G	$\{b,d,e\}$	$\{c,d,e\}$	$\{d,e\}$	$\{d,e\}$

.

It is easy to see that $((G^{\downarrow})^{\uparrow},\circ) \neq (G,\circ)$ and $((G^{\downarrow})^{\uparrow},\circ) \ncong (G,\circ)$.

(ii) Let G be a total hypergroup. Then $G^{\downarrow} = G$ and so $(G^{\downarrow})^{\uparrow} = G^{\uparrow} = G$.

(iii) Let $(G,\circ)$ be a hypergroup in such a way that for any $x, y \in G, x \circ y = \{x, y\}$. It is easy to see that $G^{\downarrow}$ is a complete hypergraph and so $(G^{\downarrow})^{\uparrow} = G$.

6. Conclusion

The current paper has considered the notion of hypergraphs and hypergroupoid and has investigated some of their properties. An equivalence relation as ρ on any hypergraph is defined and the concept of fundamental graph is introduced via this relation. We have defined the concept of complete hypergraph and showed that:

(i) on any nonempty set, we can construct a hypergraph;

(ii) using the inclusion-exclusion principle, we computed the number of hypergraph on any finite set;

(iii) a hypergraph was obtained from any quasihypergroup, a hypergroup was obtained from any complete hypergraph and any hypergraph gives a hypergroupoid;

(iv) via the equivalence relation ρ on any hypergraph G, we proved that the quotient G/ρ is a graph;

(v) through the concept of fundamental graph, it was shown that any graph is a fundamental graph and is shown there exists a lower bound for enumerating of complete hypergraphs;

(vi) with respect to the relations β^* and ρ there exists a two sided connection between fundamental group and fundamental graph.

We hope that these results are helpful for further studies in graph theory. In our future studies, we hope to obtain more results regarding graphs, hypergraphs and their applications.

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[1]Department Of Mathematics,
University Of Payame Noor,
Tehran, Iran
Email address: m.hamidi@pnu.ac.ir

[2] Department of Pure Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman,
Kerman, Iran
Email address: arsham@uk.ac.ir