Kragujevac Journal of Mathematics
Volume 46(2) (2022), Pages 283–294.

RIESZ LACUNARY SEQUENCE SPACES OF FRACTIONAL DIFFERENCE OPERATOR

KULDIP RAJ[1], KAVITA SAINI[1], AND NEERU SAWHNEY[2]

Abstract. In this paper, we intend to make new approach to introduce and study some fractional difference sequence spaces by Riesz mean associated with infinite matrix and a sequence of modulus functions over n -normed spaces. Various algebraic and topological properties of these newly formed sequence spaces have been explored and some inclusion relations concerning these spaces are also establish. Finally, we make an effort to study the statistical convergence through fractional difference operator.

1. Introduction and Preliminaries

Baliarsingh and Dutta [1] introduced fractional difference operators $\Delta^{\tilde{\gamma}}$, $\Delta^{(\tilde{\gamma})}$, $\Delta^{-\tilde{\gamma}}$, $\Delta^{(-\tilde{\gamma})}$ and discussed some topological results among these operators. Meng and Mei [17] introduced binomial fractional difference sequence spaces by clubbing binomial matrix and fractional difference operator. Recently, Baliarsingh et al. [4] studied approximation theorems and statistical convergence in fractional difference sequence spaces. Also, double difference fractional order sequence spaces has been introduced by Baliarsingh in [5]. In [23] Nayak et al. introduced some weighted mean fractional difference sequence spaces. Kirişci and Kadak [15] proposed almost convergent fractional order difference sequence spaces. The reader can refer to the textbooks Başar [6] and Mursaleen [20] for relevant terminology and required details on the domain of triangles, sequence spaces and related topics. By $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the sets of natural, real and complex numbers respectively. Let w be the space of all real or complex sequences. For a proper fraction $\tilde{\gamma}$, defined the fractional difference operators

Key words and phrases. Fractional difference operator, gamma function, modulus function, Riesz mean, lacunary sequences.

2010 *Mathematics Subject Classification.* Primary: 46A45, 46A35, 40A05.

Received: September 20, 2019.

Accepted: November 25, 2019.

$\Delta^{\tilde{\gamma}}: w \to w$, $\Delta^{(\tilde{\gamma})}: w \to w$ and their inverses are as follows:

$$\Delta^{\tilde{\gamma}}(x_\nu) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\tilde{\gamma}+1)}{i!\Gamma(\tilde{\gamma}+1-i)} x_{\nu+i}, \tag{1.1}$$

$$\Delta^{(\tilde{\gamma})}(x_\nu) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\tilde{\gamma}+1)}{i!\Gamma(\tilde{\gamma}+1-i)} x_{\nu-i}, \tag{1.2}$$

$$\Delta^{-\tilde{\gamma}}(x_\nu) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(1-\tilde{\gamma})}{i!\Gamma(1-\tilde{\gamma}-i)} x_{\nu+i}, \tag{1.3}$$

$$\Delta^{(-\tilde{\gamma})}(x_\nu) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(1-\tilde{\gamma})}{i!\Gamma(1-\tilde{\gamma}-i)} x_{\nu-i}. \tag{1.4}$$

We suppose that the series defined in (1.1)–(1.4) are convergent. For $\tilde{\gamma} = \frac{1}{2}$, we have

- $\Delta^{\frac{1}{2}} x_\nu = x_\nu - \frac{1}{2}x_{\nu+1} - \frac{1}{8}x_{\nu+2} - \frac{1}{16}x_{\nu+3} - \frac{5}{128}x_{\nu+4} - \frac{7}{256}x_{\nu+5} - \cdots;$
- $\Delta^{-\frac{1}{2}} x_\nu = x_\nu + \frac{1}{2}x_{\nu+1} + \frac{3}{8}x_{\nu+2} + \frac{5}{16}x_{\nu+3} + \frac{35}{128}x_{\nu+4} + \frac{63}{256}x_{\nu+5} + \cdots;$
- $\Delta^{(\frac{1}{2})} x_\nu = x_\nu - \frac{1}{2}x_{\nu-1} - \frac{1}{8}x_{\nu-2} - \frac{1}{16}x_{\nu-3} - \frac{5}{128}x_{\nu-4} - \frac{7}{256}x_{\nu-5} - \cdots;$
- $\Delta^{(-\frac{1}{2})} x_\nu = x_\nu + \frac{1}{2}x_{\nu-1} + \frac{3}{8}x_{\nu-2} + \frac{5}{16}x_{\nu-3} + \frac{35}{128}x_{\nu-4} + \frac{63}{256}x_{\nu-5} + \cdots$

For more details about fractional difference operator (see [3]). By $\Gamma(m)$, we denote the Gamma function of a real number m and $m \notin \{0, -1, -2, -3, \dots\}$. Now, by the definition it will be expressed as associate improper integral, i.e.,

$$\Gamma(m) \;=\; \int_0^\infty e^{-t} t^{m-1} dt. \tag{1.5}$$

It is clear from (1.5) if $m \in \mathbb{N}$, the set of nonnegative integers, then $\Gamma(m+1) = m!$. For this reason, Gamma function is considered to be a generalization of elementary factorial function. Currently, we tend to state some properties of Gamma function that are as follows:

(i) if $m \in \mathbb{N}$, then we have $\Gamma(m+1) = m!$;
(ii) if $m \in \mathbb{R} \setminus \{0, -1, -2, -3, \dots\}$, then we have $\Gamma(m+1) = m\Gamma(m)$;
(iii) for particular cases, we have $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2!$, $\Gamma(4) = 3!, \dots$

Let U and V be two sequence spaces and $\mathcal{A} = (a_{n\nu})$ be an infinite matrix of real or complex numbers. Then we say that $\mathcal{A}$ defines a matrix transformation from U into V if for every sequence $x = (x_\nu) \in U$, the sequence $\mathcal{A}x = \{\mathcal{A}_n(x)\}$, the $\mathcal{A}$-transform of x is in V, where

$$\mathcal{A}_n(x) = \sum_\nu a_{n\nu} x_\nu, \quad n \in \mathbb{N}.$$

The idea of n-normed spaces was introduced by Misiak [19]. Let X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $d \geq n \geq 2$ and $n \in \mathbb{N}$. A real valued function $||\cdot, \dots, \cdot||$ on X^n satisfying the following four conditions:

(i) $||\vartheta_1, \vartheta_2, \dots, \vartheta_n|| = 0$ if and only if $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ are linearly dependent in X;
(ii) $||\vartheta_1, \vartheta_2, \dots, \vartheta_n||$ is invariant under permutation;
(iii) $||\beta\vartheta_1, \vartheta_2, \dots, \vartheta_n|| = |\beta| \; ||\vartheta_1, \vartheta_2, \dots, \vartheta_n||$ for any $\beta \in \mathbb{R}$;

(iv) $||\vartheta + \vartheta', \vartheta_2, \dots, \vartheta_n|| \leq ||\vartheta, \vartheta_2, \dots, \vartheta_n|| + ||\vartheta', \vartheta_2, \dots, \vartheta_n||$ is called an n-norm on X and the pair $(X, ||\cdot, \dots, \cdot||)$ is called a n-normed space over the field $\mathbb{R}$. For more definition and results on n-normed spaces see [13, 14, 22]. A sequence (x_ν) in a n-normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{\nu \to \infty} \|(x_\nu - L, z_1, \dots, z_{n-1})\| = 0, \quad \text{for every } z_1, \dots, z_{n-1} \in X.$$

A sequence (x_ν) in a n-normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be Cauchy with respect to the n-norm if

$$\lim_{\nu, p \to \infty} \|(x_\nu - x_p, z_1, \dots, z_{n-1})\| = 0, \quad \text{for every } z_1, \dots, z_{n-1} \in X.$$

In a n-normed space $(X, \|\cdot, \dots, \cdot\|)$, a sequence (x_ν) is said to be bounded if for a positive constant M, $\|(x_\nu, z_1, \dots, z_{n-1})\| \leq M$ for all $z_1, \dots, z_{n-1} \in X$.

Let (p_v) be a sequence of positive real numbers and $P_n = p_1 + p_2 + \dots + p_n$ for all $n \in \mathbb{N}$. Thus, the Riesz transformation of $x = (x_\nu)$ is defined as

$$t_n = \frac{1}{P_n} \sum_{\nu=1}^{n} p_\nu x_\nu. \tag{1.6}$$

If the sequence (t_n) contains a finite limit L, then the sequence (x_ν) is said to be Riesz convergent to L. The set of all Riesz convergent sequence is denoted by (R, P_n). Let us note that if $P_n \to \infty$, as $n \to \infty$. Then Riesz mean is regular. If $p_\nu = 1$ for every natural number ν in (1.6), then Riesz mean reduces to Cesàro mean of order one.

An increasing non-negative integer sequence $\theta = (\nu_r)$ with $\nu_0 = 0$ and $\nu_r - \nu_{r-1} \to \infty$ as $r \to \infty$ is known as lacunary sequence. The intervals determined by θ will be denoted by $I_r = (\nu_{r-1}, \nu_r]$. We write $h_r = \nu_r - \nu_{r-1}$ and q_r denotes the ratio $\frac{\nu_r}{\nu_{r-1}}$. The space of lacunary strongly convergence was defined by Freedman et al. [10] as follows:

$$N_\theta = \Big\{ x = (x_\nu) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{\nu \in I_r} |x_\nu - L| = 0 \text{ for some } L \Big\}.$$

The space N_θ is a BK-space with the norm

$$\|x\| = \sup \Bigg(\frac{1}{h_r} \sum_{\nu \in I_r} |x_\nu| \Bigg).$$

Let $\theta = (\nu_r)$ be a lacunary sequence and (p_ν) be a sequence of positive real numbers such that $H_r = \sum_{\nu \in I_r} p_\nu$, $P_{\nu_r} = \sum_{\nu \in (0, \nu_r]} p_\nu$, $P_{\nu_{r-1}} = \sum_{\nu \in (0, \nu_{r-1}]} p_\nu$, $Q_r = \frac{P_{\nu_r}}{P_{\nu_{r-1}}}$, $P_0 = 0$. Clearly, $H_r = P_{\nu_r} - P_{\nu_{r-1}}$ and the intervals determine by θ and (p_ν) are denoted by $I'_r = (P_{\nu_{r-1}}, P_{\nu_r}]$. If we take $p_\nu = 1$ for all $\nu \in \mathbb{N}$, then H_r, P_{ν_r}, $P_{\nu_{r-1}}$, Q_r and I'_r reduce to h_r, ν_r, ν_{r-1}, q_r and I_r, respectively.

A function $\psi : X \to \mathbb{R}$ is termed as paranorm, where X be a linear metric space, if following conditions are satisfied

(i) $\psi(x) \geq 0$ for all $x \in X$;

(ii) $\psi(-x) = \psi(x)$ for all $x \in X$;

(iii) $\psi(x + y) \leq \psi(x) + \psi(y)$ for all $x, y \in X$;

(iv) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $\psi(x_n - x) \to 0$ as $n \to \infty$, then $\psi(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A function $\mathfrak{f} : [0, \infty) \to [0, \infty)$ is said to be modulus function if

(i) $\mathfrak{f}(\upsilon) = 0$ if and only if $\upsilon = 0$;

(ii) $\mathfrak{f}(\upsilon_1 + \upsilon_2) \leq f(\upsilon_1) + f(\upsilon_2)$ for all υ_1, υ_2;

(iii) $\mathfrak{f}$ is increasing;

(iv) $\mathfrak{f}$ is continuous from the right at 0.

The modulus function may be bounded or unbounded. Later, modulus function has been discussed in [21, 25–27, 29] and references therein.

Lemma 1.1. *Consider* $\mathfrak{f} = (\mathfrak{f}_\nu)$ *be a sequence of modulus functions and* $0 < \rho < 1$. *Then for each* $x > \rho$, *we have*

$$\mathfrak{f}_\nu(x) \leq \frac{2\mathfrak{f}_\nu(1)(x)}{\rho}.$$

For a proper fraction $\tilde{\gamma}$, let $\mathfrak{f} = (\mathfrak{f}_\nu)$ be a sequence of modulus functions, $q = (q_\nu)$ be a bounded sequence of strictly positive real numbers, $\mu = (\mu_\nu)$ be a sequence of strictly positive real numbers and θ be a lacunary sequence. In this paper we define the following sequence spaces as follows:

$$\begin{aligned}
&[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0 \\
=&\left\{x = (x_\nu) \in w : \lim_{r \to \infty} \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \left[a_{n\nu} \left[\mathfrak{f}_\nu \left(\left\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\right\|\right)\right]^{q_\nu}\right] = 0\right\}, \\
&[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \\
=&\left\{x = (x_\nu) \in w : \lim_{r \to \infty} \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \left[a_{n\nu} \left[\mathfrak{f}_\nu \left(\left\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1}\right\|\right)\right]^{q_\nu}\right] = 0,\right. \\
&\left.\text{for some } L > 0\right\}
\end{aligned}$$

and

$$\begin{aligned}
&[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty \\
=&\left\{x = (x_\nu) \in w : \sup_r \frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \left[a_{n\nu} \left[\mathfrak{f}_\nu \left(\left\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\right\|\right)\right]^{q_\nu}\right] < \infty\right\}.
\end{aligned}$$

If the sequence $x = (x_\nu)$ is convergent to the limit L in

$$[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$$

we denote it by $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] - \lim x = L$.

Suppose $\mathfrak{f}(x) = x$. Then above spaces reduces to $[\mathcal{R}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$, $[\mathcal{R}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$ and $[\mathcal{R}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty$.

By taking $q = (q_\nu) = 1$ for all $\nu \in \mathbb{N}$, then we get the spaces $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$, $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$ and $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty$.

Suppose $p_\nu = 1$ for all $\nu \in \mathbb{N}$, then we get the spaces as follows:

$$\begin{aligned}
&[\mathcal{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0 \\
=&\Big\{x = (x_\nu) \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{\nu \in I_r} a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu} = 0\Big\}, \\
&[\mathcal{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \\
=&\Big\{x = (x_\nu) \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{\nu \in I_r} a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu} = 0\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&[\mathcal{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty \\
=&\Big\{x = (x_\nu) \in w : \sup_r \frac{1}{h_r} \sum_{\nu \in I_r} a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu} < \infty\Big\}.
\end{aligned}$$

Suppose (p_ν) be a sequence of positive numbers and $P_n = p_1 + p_2 + ... + p_n$. Now, we define the sequence spaces as follows:

$$\begin{aligned}
&[\mathcal{R}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0 \\
=&\Big\{x = (x_\nu) \in w : \lim_{n\to\infty} \frac{1}{P_n} \sum_{\nu=1}^{n} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] = 0\Big\}, \\
&[\mathcal{R}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \\
=&\Big\{x = (x_\nu) \in w : \lim_{n\to\infty} \frac{1}{P_n} \sum_{\nu=1}^{n} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] = 0\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&[\mathcal{R}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty \\
=&\Big\{x = (x_\nu) \in w : \sup_n \frac{1}{P_n} \sum_{\nu=1}^{n} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu \Delta^{(\tilde{\gamma})} x_\nu, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] < \infty\Big\}.
\end{aligned}$$

If $0 < q_\nu \le \sup q_\nu = D$, $C = \max\{1, 2^{D-1}\}$. Then

$$|c_\nu + d_\nu|^{q_\nu} \le C(|c_\nu|^{q_\nu} + |d_\nu|^{q_\nu}), \tag{1.7}$$

for every natural number ν and $c_\nu, d_\nu \in \mathbb{R}$.

The main purpose of this paper is to introduce and study some lacunary convergent sequence spaces defined by Riesz mean via modulus functions over $n-$normed spaces. We shall make an effort to study some interesting algebraic and topological properties of concerning sequence spaces. Also, we examine some interrelations between these sequence spaces.

2. Main Results

Theorem 2.1. *Suppose $\mathfrak{f} = (\mathfrak{f}_\nu)$ be a sequence of modulus functions, $\Delta^{(\tilde{\gamma})}$ be a fractional difference operator, $\mu = (\mu_\nu)$ be a sequence of positive real numbers and $q = (q_\nu)$ be a bounded sequence of positive real numbers. Then the sequence spaces $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$, $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$ and $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_\infty$ are linear spaces over the field $\mathbb{R}$ of real numbers.*

Proof. Consider $x = (x_\nu), y = (y_\nu) \in [\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$ and $\alpha, \beta \in \mathbb{R}$. Since f is additive and by using inequality (1.7), we have

$$\begin{aligned}
&\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}(\alpha x_\nu+\beta y_\nu), z_1, \ldots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\Big\}\\
\leq&\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(|\alpha|\left\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu, z_1, \ldots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\\
&+\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(|\beta|\left\|\mu_\nu\Delta^{(\tilde{\gamma})}y_\nu, z_1, \cdots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\\
\leq&C\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu, z_1, \ldots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\\
&+C\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}y_\nu, z_1, \ldots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\\
\to&0 \quad \text{as} \quad r\to\infty.
\end{aligned}$$

Hence, $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$ is a linear space. Similarly, we can prove others. ☐

Theorem 2.2. *Let $\mathfrak{f} = (\mathfrak{f}_\nu)$ be a sequence of modulus functions and $q = (q_\nu)$ be a bounded sequence of strictly positive real numbers. Then the sequence space $[\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$ is a paranormed space with respect to the paranorm*

$$\psi(x) = \sup_r\left(\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu, z_1, \cdots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\right)^{\frac{1}{M}},$$

where $M = \max\{1, \sup_\nu q_\nu < \infty\}$.

Proof. Consider $x = (x_\nu), y = (y_\nu) \in [\mathcal{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]_0$. Clearly, $\psi(x) \geq 0$ and $\psi(0) = 0$. Now, by using Minkowski's inequality, we get

$$\begin{aligned}
&\left(\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}(x_\nu+y_\nu), z_1, \cdots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\right)^{\frac{1}{M}}\\
\leq&\left(\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu, z_1, \cdots, z_{n-1}\right\|\right)\right]^{q_\nu}\right]\right)^{\frac{1}{M}}
\end{aligned}$$

$$+\left(\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}y_\nu, z_1,\cdots,z_{n-1}\right\|\right)\right]^{q_\nu}\right]\right)^{\frac{1}{M}}.$$

Hence, $\psi(x+y)\le\psi(x)+\psi(y)$.

Finally, we prove that the scalar multiplication is continuous. Let γ be any complex number. Then

$$\psi(\gamma x)=\sup_r\left(\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}\gamma x_\nu, z_1,\cdots,z_{n-1}\right\|\right)\right]^{q_\nu}\right]\right)^{\frac{1}{M}}\le K_\gamma^{\frac{D}{M}}\psi(x),$$

where K_γ is a positive integer such that $\gamma\le K_\gamma$. Now, let $\gamma\to 0$ for any fixed x with $\psi(x)\ne 0$. So, by using definition of $\mathfrak{f}$ for $|\gamma|<1$, we have

$$\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}\gamma x_\nu, z_1,\ldots,z_{n-1}\right\|\right)\right]^{q_\nu}<\epsilon,\quad\text{for } r>r_0(\epsilon).\tag{2.1}$$

Since $\mathfrak{f}$ is continuous and taking γ small enough, for $1\le r\le r_0$, we have

$$\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}\gamma x_\nu, z_1,\ldots,z_{n-1}\right\|\right)\right]^{q_\nu}<\epsilon.\tag{2.2}$$

Now, by combining (2.1) and (2.2) implies that $\psi(\gamma x)\to 0$ as $\gamma\to 0$. Thus, the space $[\mathcal{R},\theta,\mathfrak{f},\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]_0$ is a paranormed space with respect to the paranorm $\psi(\cdot)$. □

Theorem 2.3. *Suppose $\mathfrak{f}=(\mathfrak{f}_\nu)$ be a sequence of modulus functions, $q=(q_\nu)$ be a bounded sequence of positive real numbers, $\mu=(\mu_\nu)$ be a sequence of positive real numbers and $\theta=(\nu_r)$ be a lacunary sequence such that* $\limsup_r Q_r<\infty$. *Then* $[\mathcal{R},\theta,\mathfrak{f},\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]\subseteq[\mathcal{R},\mathfrak{f},\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]$.

Proof. Let $x=(x_\nu)\in[\mathcal{R},\theta,\mathfrak{f},\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]$. Then for every $\epsilon>0$ there exists i_0 such that for every $i>i_0$

$$\mathbf{A}_i=\frac{1}{H_i}\sum_{\nu\in I_i} p_\nu\left[a_{n\nu}\left[\mathfrak{f}_\nu\left(\left\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu-L, z_1,\ldots,z_{n-1}\right\|\right)\right]^{q_\nu}\right]<\epsilon.\tag{2.3}$$

Then, there is some positive constant N such that

$$\mathbf{A}_i\le N,\quad\text{for all } i.\tag{2.4}$$

Now, $\limsup_r Q_r<\infty$. Then, there exists some positive number K such that

$$Q_r\le K,\quad\text{for all } r\ge 1.\tag{2.5}$$

Therefore, for $\nu_{r-1}<n\le\nu_r$ and by (2.3), (2.4) and (2.5), we have

$$\begin{aligned}\frac{1}{P_n}\sum_{\nu=1}^{n}p_\nu y_\nu\le&\frac{1}{P_{\nu_{r-1}}}\sum_{\nu=1}^{\nu_r}p_\nu y_\nu\\ =&\frac{1}{P_{\nu_{r-1}}}\left(\sum_{\nu\in I_1}p_\nu y_\nu+\sum_{\nu\in I_2}p_\nu y_\nu+\cdots+\sum_{\nu\in I_{i_0}}p_\nu y_\nu+\sum_{\nu\in I_{i_0+1}}p_\nu y_\nu+\right.\end{aligned}$$

$$
\begin{aligned}
&\cdots + \sum_{\nu \in I_r} p_\nu y_\nu \Big) \\
=&\frac{1}{P_{\nu_{r-1}}} \left(\mathbf{A}_1 H_1 + \mathbf{A}_2 H_2 + \cdots + \mathbf{A}_{i_0} H_{i_0} + \mathbf{A}_{i_0+1} H_{i_0+1} + \cdots + \mathbf{A}_r H_r \right) \\
\leq&\frac{N}{P_{\nu_{r-1}}} \left(H_1 + H_2 + \cdots + H_{i_0} \right) + \frac{\epsilon}{P_{\nu_{r-1}}} \left(H_{i_0+1} + H_{i_0+2} + \cdots + H_r \right) \\
=&\frac{N}{P_{\nu_{r-1}}} \Big(P_{\nu_1} - P_{\nu_0} + P_{\nu_2} - P_{\nu_1} + \cdots + P_{\nu_{i_0}} - P_{\nu_{i_0-1}} \Big) \\
&+ \frac{\epsilon}{P_{\nu_{r-1}}} \Big(P_{\nu_{i_0+1}} - P_{\nu_{i_0}} + P_{\nu_{i_0+2}} - P_{\nu_{i_0+1}} \cdots + P_{\nu_r} - P_{\nu_{r-1}} \Big) \\
=&\frac{N P_{\nu_{i_0}}}{P_{\nu_{r-1}}} + \frac{\epsilon (P_{\nu_r} - P_{\nu_{i_0}})}{P_{\nu_{r-1}}} \\
\leq&\frac{N P_{\nu_{i_0}}}{P_{\nu_{r-1}}} + \epsilon K,
\end{aligned}
$$

where $y_\nu = a_{n\nu} \Big[\mathfrak{f}_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{q_\nu}$. Now, $P_{\nu_{r-1}} \to \infty$ as $r \to \infty$, then we have $x \in [\mathfrak{R}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$. This completes the proof. $\square$

Corollary 2.1. *Let (p_ν) be sequence of positive numbers. If $1 < \liminf_r Q_r \leq \limsup_r Q_r < \infty$. Then*

$$[\mathfrak{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] = [\mathfrak{R}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|].$$

Theorem 2.4. *The following inclusions are true.*

(*i*) *If $p_\nu < 1$ for all $\nu \in \mathbb{N}$, then*

$$[\mathfrak{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \subset [\mathfrak{R}, \theta, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|],$$

with $[\mathfrak{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] - \lim x = [\mathfrak{R}, \mathfrak{f}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] - \lim x = L$.

(*ii*) *If $p_\nu > 1$ for all $\nu \in \mathbb{N}$ and $\frac{H_r}{h_r}$ be upper bounded. Then*

$$[\mathfrak{R}, \mathfrak{f}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] \subset [\mathfrak{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|],$$

with $[\mathfrak{R}, \mathfrak{f}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] - \lim x = [\mathfrak{C}_{\theta_r}, f, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|] - \lim x = L$.

Proof. (*i*) Let $p_\nu < 1$ for all $\nu \in \mathbb{N}$, then $H_r < h_r$ for all $r \in \mathbb{N}$. So, there exists a constant M_1 such that $0 < M_1 \leq \frac{H_r}{h_r} < 1$ for all $r \in \mathbb{N}$. Let $x = (x_\nu)$ be a sequence which converges to the limit L in $[\mathfrak{C}_{\theta_r}, \mathfrak{f}, \Delta^{(\tilde{\gamma})}, \mu, q, \mathcal{A}, \|\cdot, \ldots, \cdot\|]$. Then for $\epsilon > 0$ we get

$$
\begin{aligned}
&\frac{1}{H_r} \sum_{\nu \in I_r} p_\nu \Big[a_{n\nu} \Big[\mathfrak{f}_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{q_\nu} \Big] \\
<&\frac{1}{M_1 h_r} \sum_{\nu \in I_r} a_{n\nu} \Big[\mathfrak{f}_\nu \Big(\Big\| \mu_\nu \Delta^{(\tilde{\gamma})} x_\nu - L, z_1, \ldots, z_{n-1} \Big\| \Big) \Big]^{q_\nu}.
\end{aligned}
$$

Now, we get the desired result by taking the limit as $r \to \infty$.

(ii) It is easy so we omit it. □

Theorem 2.5. *Suppose $\mathfrak{f}$ and $\mathfrak{f}'$ be two sequences of modulus functions. Then the following inclusions hold:*

(i) $[\mathcal{R}, \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \dots, \cdot\|] \subset [\mathcal{R}, \mathfrak{f} \circ \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \dots, \cdot\|]$;

(ii) $[\mathcal{R}, \mathfrak{f}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \dots, \cdot\|] \cap [\mathcal{R}, \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \dots, \cdot\|] \subset \mathcal{R}, \mathfrak{f} + \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \dots, \cdot\|]$.

Proof. Suppose $x = (x_\nu) \in [\mathcal{R}, \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \dots, \cdot\|]$. For given $\epsilon > 0$, choose $\rho \in (0,1)$ such that $\mathfrak{f}_\nu(t) < \epsilon$ for all $0 < t < \rho$. Then we have

$$\begin{aligned}
&\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu \circ \mathfrak{f}'_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] \\
=&\frac{1}{H_r}\sum_{\nu\in I_r, \Big[\mathfrak{f}'_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{q_\nu} < \rho} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu \circ \mathfrak{f}'_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \dots, \\
&z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] \\
&+\frac{1}{H_r}\sum_{\nu\in I_r, \Big[\mathfrak{f}'_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{q_\nu} \geq \rho} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu \circ \mathfrak{f}'_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \dots, \\
&z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] \\
\leq&(\epsilon)^D + \max\left\{1, \left(\frac{2\mathfrak{f}_\nu(1)}{\rho}\right)\right\}\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}'_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big].
\end{aligned}$$

Thus, we get $x = (x_\nu) \in [\mathcal{R}, \mathfrak{f} \circ \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \dots, \cdot\|]$. This completes the proof.

(ii) Let

$$x = (x_\nu) \in [\mathcal{R}, \mathfrak{f}, \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \dots, \cdot\|] \cap [\mathcal{R}, \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \dots, \cdot\|].$$

Then, we have

$$\begin{aligned}
&\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu + \mathfrak{f}'_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] \\
\leq&C\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] \\
&+C\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}'_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big] \\
\to&0 \quad \text{as} \quad r \to \infty.
\end{aligned}$$

Therefore, $(x_\nu) \in [\mathcal{R}, \mathfrak{f} + \mathfrak{f}', \theta, \Delta^{(\tilde{\gamma})}, \mu, p, q, \mathcal{A}, \|\cdot, \dots, \cdot\|]$. This completes the proof. □

3. Statistical Convergence

The concept of statistical convergence was introduced independently by Fast [9] and Steinhaus [28]. Statistical convergence has been further studied by Connor [8], Fridy ([11], [12]), Miller [18], Balcerzak et al. [2], Y. Q. Cao and Xiaofei Qu [7] and others. In this section, we introduce some inclusion relation between $S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}$ and $[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]$.

Definition 3.1. A sequence $x=(x_\nu)$ is said to be $S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}$-convergent to L if for every $\epsilon>0$,

$$\frac{1}{H_r}\Big|\{\nu\in I_r : p_\nu(\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\|)\geq\epsilon\}\Big| = 0.$$

In this case, we write $S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}-\lim x = L$ or $x_\nu\to LS_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}$.

Theorem 3.1. *Let $\mathfrak{f}=(\mathfrak{f}_\nu)$ be a sequence of modulus functions and $0<\inf_\nu q_\nu\leq q_\nu\leq\sup_\nu q_\nu = D<\infty$. Then $[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]\subset S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}$.*

Proof. Consider $x=(x_\nu)\in[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]$ and given $\epsilon>0$. Then for each $z_1,\ldots,z_{n-1}$, we have

$$\begin{aligned}
&\frac{1}{H_r}\sum_{\nu\in I_r}p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
=&\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\ldots,z_{n-1}\|\geq\epsilon}p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
&+\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\ldots,z_{n-1}\|<\epsilon}p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
\geq&\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\ldots,z_{n-1}\|\geq\epsilon}p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
\geq&\frac{1}{H_r}\sum_{\nu\in I_r}[\mathfrak{f}_\nu(\epsilon)]^{q_\nu}\\
\geq&\frac{1}{H_r}\sum_{\nu\in I_r}\min\Big\{[\mathfrak{f}_\nu(\epsilon)]^{\inf q_\nu},[\mathfrak{f}_\nu(\epsilon)]^D\Big\}\\
\geq&R\frac{1}{H_r}\Big|\{\nu\in I_r : p_\nu(\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\ldots,z_{n-1}\|)\geq\epsilon\}\Big|,
\end{aligned}$$

where $R=\min\Big\{[\mathfrak{f}_\nu(\epsilon)]^{\inf q_\nu},[\mathfrak{f}_\nu(\epsilon)]^D\Big\}$. Thus, $(x_\nu)\in S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}$. ☐

Theorem 3.2. *Let $\mathfrak{f}=(\mathfrak{f}_\nu)$ be a bounded sequence of modulus functions and $q=(q_\nu)$ be a bounded sequence of positive real numbers. If $0<\inf_\nu q_\nu\leq q_\nu\leq\sup_\nu q_\nu = D<\infty$, then $S_{[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]}\subset[\mathcal{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\ldots,\cdot\|]$.*

Proof. Suppose $x = (x_\nu) \in S_{[\mathfrak{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\dots,\cdot\|]}$ and $\epsilon > 0$ be given. Since $\mathfrak{f}$ is bounded, then there exists an integer J such that $\mathfrak{f}(x) < J$ for all $x > 0$, then we have

$$\begin{aligned}
&\frac{1}{H_r}\sum_{\nu\in I_r} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
=&\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\dots,z_{n-1}\|\geq\epsilon} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
&+\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\dots,z_{n-1}\|<\epsilon} p_\nu\Big[a_{n\nu}\Big[\mathfrak{f}_\nu\Big(\Big\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{q_\nu}\Big]\\
\leq&\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\dots,z_{n-1}\|\geq\epsilon} \max\Big\{J^{\inf q_\nu}, J^D\Big\}\\
&+\frac{1}{H_r}\sum_{\nu\in I_r,\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L,z_1,\dots,z_{n-1}\|<\epsilon} [\mathfrak{f}(\epsilon)]^{q_\nu}\\
\leq&\max\Big\{J^{\inf q_\nu}, J^D\Big\}\frac{1}{H_r}\Big|\{\nu\in I_r : p_\nu(\|\mu_\nu\Delta^{(\tilde{\gamma})}x_\nu - L, z_1,\dots,z_{n-1}\|)\geq\epsilon\}\Big|\\
&+\max\Big\{[\mathfrak{f}_\nu(\epsilon)]^{\inf q_\nu}, [\mathfrak{f}_\nu(\epsilon)]^D\Big\}.
\end{aligned}$$

Thus, $(x_\nu) \in [\mathfrak{R},\mathfrak{f},\theta,\Delta^{(\tilde{\gamma})},\mu,p,q,\mathcal{A},\|\cdot,\dots,\cdot\|]$. This completes the proof. $\square$

Acknowledgements. The authors thank the referee for providing valuable suggestions which improved the presentation of the paper. The corresponding author thanks the Council of Scientific and Industrial Research (CSIR), India for partial support under Grant No. 25(0288)/18/EMR-II, dated 24/05/2018.

References

[1] P. Baliarsingh and S. Dutta, *A unifying approach to the difference operators and their applications*, Bol. Soc. Parana. Mat. **33**(1) (2015), 49–57.

[2] M. Balcerzak, K. Dems and A. Komisarski, *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl. **328**(1) (2007), 715–729.

[3] P. Baliarsingh, *Some new difference sequence spaces of fractional order and their dual spaces*, Appl. Math. Comput. **219**(18) (2013), 9737–9742.

[4] P. Baliarsingh, U. Kadak and M. Mursaleen, *On statistical convergence of difference sequences of fractional order and related Korovkin type approximation theorems*, Quaest. Math. **41**(8) (2018), 1117–1133.

[5] P. Baliarsingh, *On difference double sequence spaces of fractional order*, Indian J. Math. **58**(3) (2016), 287–310.

[6] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, İstanbul, 2012.

[7] Y. Cao and X. Qu, *On possibility distribution*, Fuzzy Sets and Systems **69**(2) (1995), 171–179.

[8] J. Connor, *Two valued measure and summability*, Analysis **10**(4) (1990), 373–385.

[9] H. Fast, *Sur le convergence*, Colloq. Math. **2**(3-4) (1951), 241–244.

[10] A. R. Freedman, J. J. Sember and M. Raphael, *Some Cesàro-type summability spaces*, Proc. Lond. Math. Soc. **37**(3) (1978), 508–520.

[11] J. A. Fridy, *On statistical convergent*, Analysis **5**(4) (1985), 301–313.

[12] J. A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. **118**(4) (1993), 1187–1192.

[13] H. Gunawan, *On n-inner product, n-norms, and the Cauchy-Schwartz inequality*, Sci. Math. Jpn. **5**(1) (2001), 47–54.

[14] H. Gunawan, *The space of p-summable sequence and its natural n-norm*, Bull. Aust. Math. Soc. **64**(1) (2001), 137–147.

[15] M. Kirişci and U. Kadak, *The method of almost convergence with operator of the form fractional order and applications*, J. Nonlinear Sci. Appl. **10**(2) (2017), 828–842.

[16] I. J. Maddox, *Spaces of strongly summable sequences*, Q. J. Math. **18**(1) (1967), 345–355.

[17] J. Meng and L. Mei, *Binomial difference sequence spaces of fractional order*, J. Inequal. Appl. (2018), Paper ID 274, 8 pages.

[18] H. I. Miller and C. Orhan, *On almost convergent and statistically convergent subsequences*, Acta Math. Hungar. **93**(1-2) (2001), 131–151.

[19] A. Misiak, *n-inner product spaces*, Math. Nachr. **140**(1) (1989), 299–319.

[20] M. Mursaleen, *Applied Summability Methods*, Springer Briefs, Heidelberg, New York, 2014.

[21] M. Mursaleen and M. Başarir, *On some new sequence spaces of fuzzy numbers*, Indian J. Pure Appl. Math. **34**(9) (2003), 1351–1353.

[22] M. Mursaleen and K. Raj, *Some vector valued sequence spaces of Musielak-Orlicz functions and their operator ideals*, Math. Slovaca **68**(1) (2018), 115–134.

[23] L. Nayak, M. Et and P. Baliarsingh, *On certain generalized weighted mean fractional difference sequence spaces*, Proc. Nat. Acad. Sci. India Sect. A **89**(1) (2019), 163–170.

[24] H. Polat and F. Başar, *Some Euler spaces of difference sequence spaces and their topological properties*, Journal of New Results in Science **6**(1) (2014), 1–14.

[25] K. Raj and A. Kiliçman, *On certain generalized paranormed spaces*, J. Inequal. Appl. (2015), Paper ID 37, 12 pages.

[26] K. Raj and C. Sharma, *Applications of strongly convergent sequences to Fourier series by means of modulus functions*, Acta Math. Hungar. **150**(2) (2016), 396–411.

[27] K. Raj and S. Jamwal, *Applications of statistical convergence to n-normed spaces*, Adv. Pure Appl. Math. **7**(3) (2016), 197–204.

[28] H. Steinhaus, *Sur la convergence ordinarie et la convergence asymptotique*, Colloq. Math. **2**(1) (1951), 73–74.

[29] B. C. Tripathy and R. Goswami, *Statistically convergent multiple sequences in probabilistic normed spaces*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **78**(4) (2016), 83–94.

[1]School of Mathematics,
Shri Mata Vaishno Devi University,
Katra-182320, J & K (India)
Email address: kuldipraj68@gmail.com
Email address: kavitasainitg3@gmail.com

[2]Department of Physics,
G. G. M. Science College,
Jammu, J & K (India)
Email address: neerusawhney217@gmail.com