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BI-UNIVALENT FUNCTIONS CLASSES DEFINED BY POISSON DISTRIBUTION SERIES

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Abstract. By using Poisson distribution series we introduce and derive different subclasses of regular and bi-univalent functions in the open unit disk. We then present special estimates for the Taylor coefficient inequalities $|a_2|$ and $|a_3|$ of functions belonging to this new subclass. In addition, several consequences of our results are pointed out which are new and have not yet been discussed in association with bounded boundary rotation.

1. Introduction

By ρ we denote the class of functions $\mathrm{Im}(\varsigma)$ in the open unit disk $\mathcal{O} = \{\varsigma : |\varsigma| < 1\}$ and normalized by the conditions $\mathrm{Im}(0) = 0$ and $\mathrm{Im}'(0) = 1$. Each $\mathrm{Im} \in \rho$ has the form

$$\mathrm{Im}(\varsigma) = \varsigma + \sum_{r=2}^{+\infty} x_r \varsigma^r, \quad \varsigma \in \mathcal{O}. \tag{1.1}$$

Indicate by $\mathcal{S}$ the class of all univalent functions in $\mathcal{O}$.

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{J}$ include the class $\mathcal{J}^*(\tau)$ of starlike functions of order τ in $\mathcal{O}$ and the class $k(\tau)$ of convex functions of order τ, $0 \leq \tau < 1$, in $\mathcal{O}$.

It is well known that every function $\mathrm{Im} \in \mathcal{J}$ has an inverse Im^{-1} defined by

$$\mathrm{Im}^{-1}(\mathrm{Im}(\varsigma)) = \varsigma, \quad \varsigma \in \mathcal{O},$$

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and

$$\mathrm{Im}^{-1}(\mathrm{Im}(\varrho)) = \varrho, \quad |\varrho| < n_0(\mathrm{Im}),\ n_0(\mathrm{Im}) \geq 1/4,$$

where

$$\mathrm{Im}^{-1}(\varrho) = \mathcal{G}(\varrho) = \varrho - a_2\varrho^2 + (2x_2^2 - x_3)\varrho^3 - (5x_2^3 - 5x_2x_3 + x_4)\varrho^4 + \cdots . \tag{1.2}$$

A function $\mathrm{Im}(\varsigma) \in \rho$ is said to be bi-univalent in $\mathcal{O}$ if both $\mathrm{Im}(\varsigma)$ and $\mathrm{Im}^{-1}(\varsigma)$ are univalent in $\mathcal{O}$. Indicated by $\mathcal{H}$, the class of bi-univalent functions in $\mathcal{O}$ is given by (1.1).

Brannan and Taha [1] present bi-starlike functions of order τ, $0 \leq \tau < 1$, indicated by $\mathcal{J}^*_{\mathcal{H}}(\tau)$ and bi-convex functions of order τ indicated by $\mathcal{K}_{\mathcal{H}}(\tau)$.

Srivastava et al. [14], Xu et al. [16, 17], Frasin and Aouf [3] and Hayami and Owa [5] defined more interesting examples of bi-univalent function in the class $\mathcal{H}$ (also [2, 3, 15] and [17]).

A regular function $\mathrm{Im}(\varsigma)$ is subordinate to an analytic function $\mathcal{G}(\varsigma)$, written $\mathrm{Im}(\varsigma) \prec \mathcal{G}(\varsigma)$, provided there is an analytic function ϱ defined on $\mathcal{O}$ with $\varrho(0) = 0$ and $|\varrho(\varsigma)| < 1$ satisfying $\mathrm{Im}(\varsigma) = \mathcal{G}(\varrho(\varsigma))$. Ma and Minda [6] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{\varsigma\mathrm{Im}'(\varsigma)}{\mathrm{Im}(\varsigma)}$ or $1+\frac{\varsigma\mathrm{Im}''(\varsigma)}{\mathrm{Im}'(\varsigma)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk $\mathcal{O}$, $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps $\mathcal{O}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike-functions consists of functions $\mathrm{Im}(\varsigma) \in \rho$ satisfying the subordination $\frac{\varsigma\mathrm{Im}'(\varsigma)}{\mathrm{Im}(\varsigma)} \prec \phi(\varsigma)$. Similarly, the class of Ma-Minda convex functions of functions $\mathrm{Im}(\varsigma) \in \rho$ satisfying the subordination $1 + \frac{\varsigma\mathrm{Im}''(\varsigma)}{\mathrm{Im}'(\varsigma)} \prec \phi(\varsigma)$.

A function $\mathrm{Im}(\varsigma)$ is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both $\mathrm{Im}(\varsigma)$ and $\mathrm{Im}^{-1}(\varsigma)$ are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{J}^*_{\mathcal{H}}(\phi)$ and $\mathcal{K}_{\mathcal{H}}(\phi)$. In the sequel, it is assumed that is an analytic function with positive real part in the unit disk $\mathcal{O}$, satisfying $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(\mathcal{O})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(\varsigma) = 1 + V_1\varsigma + V_2\varsigma^2 + V_3\varsigma^3 + \cdots , \quad V_1 > 0. \tag{1.3}$$

A parameter σ is called Poisson distributed taken the values $0, 1, 2, 3, \ldots$, with $e^{-\mu}$, $\mu\frac{e^{-\mu}}{1!}, \mu^2\frac{e^{-\mu}}{2!}, \mu^3\frac{e^{-\mu}}{3!}, \ldots$, respectively, probabilities, where μ is called the parameter. Thus,

$$\mathcal{P}(\sigma = n) = \mu^r\frac{e^{-\mu}}{i!}, \quad i = 0, 1, 2, 3, \ldots \tag{1.4}$$

Lately, in [10] ([7, 8]) Porwal presented

$$\mathcal{K}(\mu, \varsigma) = \varsigma + \sum_{r=2}^{+\infty}\frac{\mu^{r-1}}{(r-1)!}e^{-\mu}\varsigma^r, \quad z \in \mathcal{O}, \tag{1.5}$$

the coefficients of this power series are probabilities of Poisson distribution, where $\mu > 0$.

In [10], Porwal also defined series

$$\mathfrak{g}(\mu, \varsigma) = 2\varsigma - \mathcal{K}(\mu, \varsigma) = \varsigma - \sum_{r=2}^{+\infty} \frac{\mu^{r-1}}{(r-1)!} e^{-\mu} \varsigma^r, \quad \varsigma \in \mathcal{O}. \tag{1.6}$$

Using the convolution, in [11] Porwal and Kumar introduced a linear operator $\varpi_\mu : \chi \to \chi$ defined by

$$\varpi_\mu \mathrm{Im}(\varsigma) = \mathcal{K}(\mu, \varsigma) * \mathrm{Im}(\varsigma) = \varsigma + \sum_{r=2}^{+\infty} \frac{\mu^{r-1}}{(r-1)!} e_r^{-\mu} x_r \varsigma^r, \quad \varsigma \in \mathcal{O}. \tag{1.7}$$

In [4] Goodman defined uniformly convex $\mathcal{UCV}$ functions so that its regular characterization is $\phi \in \mathcal{UCV}$ if and only if

$$\mathrm{Re}\left\{1 + \frac{\varsigma\phi''(\varsigma)}{\phi'(\varsigma)}\right\} \geq \mathrm{Re}\frac{\zeta\phi''(\varsigma)}{\phi'(\varsigma)}, \quad (\varsigma, \zeta) \in \mathcal{O} \times \mathcal{O}.$$

Ronning [12] defined the class $\mathcal{S}_P$ consisting of functions $\psi(\varsigma) = \varsigma\phi'(\varsigma)$, so $\phi \in \mathcal{UCV}$. By choosing $\zeta = e^{i\gamma}\varsigma$ in suitable way, Rosy et al. [13] wrote the regular description of $\mathcal{S}_P$ as $\psi \in \mathcal{UCV}$ if and only if

$$\mathrm{Re}\left\{\frac{\varsigma\psi'(\varsigma)}{\psi(\varsigma)} + e^{i\gamma}\left(\frac{\varsigma\psi'(\varsigma)}{\psi(\varsigma)} - 1\right)\right\} \geq 0,$$

or equivalently $\mathcal{S}_P$ is the subclass of ρ consisting of functions described in (1.1) satisfying

$$\mathrm{Re}\left\{(1 + e^{i\gamma})\frac{\varsigma \mathrm{Im}'(\varsigma)}{\mathrm{Im}(\varsigma)} - e^{i\gamma}\right\} \geq 0.$$

2. Preparation

Definition 2.1. A function $\mathrm{Im}(\varsigma) \in \mathcal{H}$ described in (1.1) is said to belong to class $\mathcal{S}_\lambda^\mu(\phi, \varsigma)$ if it satisfies

$$(1 + \beta e^{i\gamma})\left\{(1-\lambda)\frac{\varsigma(\varpi_\mu \mathrm{Im}(\varsigma))'}{\varpi_\mu \mathrm{Im}(\varsigma)} + \lambda\left(1 + \frac{\varsigma(\varpi_\mu \mathrm{Im}(\varsigma))''}{(\varpi_\mu \mathrm{Im}(\varsigma))'}\right)\right\} - \beta e^{i\gamma} \prec \phi(\varsigma), \tag{2.1}$$

where $\beta \geq 0,\ 0 \leq \lambda \leq 1,\ -\pi \leq \gamma < \pi,\ \mu > 0$ and $\varsigma \in \mathcal{O}$, and

$$(1 + \beta e^{i\gamma})\left\{(1-\lambda)\frac{\varrho(\varpi_\mu \mathrm{Im}(\varrho))'}{\varpi_\mu \mathrm{Im}(\varrho)} + \lambda\left(1 + \frac{\varrho(\varpi_\mu \mathrm{Im}(\varrho))''}{(\varpi_\mu \mathrm{Im}(\varrho))'}\right)\right\} - \beta e^{i\gamma} \prec \phi(\varrho), \tag{2.2}$$

where $\beta \geq 0,\ 0 \leq \lambda \leq 1,\ -\pi \leq \gamma < \pi,\ \mu > 0$ and $\varrho \in \mathcal{O}$.

Definition 2.2. A function $\mathrm{Im}(\varsigma) \in \mathcal{H}$, as described in (1.1), is said to belong to the class $\mathcal{K}_\lambda^\mu(\phi, \varsigma)$ if it satisfies

$$(1 + \beta e^{i\gamma})\left\{(1-\lambda)\frac{\varpi_\mu \mathrm{Im}(\varsigma)}{\varsigma} + \lambda(\varpi_\mu \mathrm{Im}(\varsigma))'\right\} - \beta e^{i\gamma} \prec \phi(\varsigma), \tag{2.3}$$

where $\beta \geq 0,\ 0 \leq \lambda \leq 1,\ -\pi \leq \gamma < \pi,\ \mu > 0$ and $\varsigma \in \mathcal{O}$, and

$$(1+\beta e^{i\gamma})\left\{(1-\lambda)\frac{\varpi_\mu \mathrm{Im}(\varrho)}{\varrho}+\lambda(\varpi_\mu \mathrm{Im}(\varrho))'\right\}-\beta e^{i\gamma} \prec \phi(\varrho), \tag{2.4}$$

where $\beta \geq 0,\ 0 \leq \lambda \leq 1,\ -\pi \leq \gamma < \pi,\ \mu > 0$ and $\varrho \in \mathcal{O}$.

By specializing the parameter λ in Definitions 2.1 and 2.2, we obtain the following subfamilies.

Remark 2.1. A function $\mathrm{Im}(\varsigma) \in \mathcal{H}$ described in (1.1) and for $\lambda = 0$, we note that $\mathcal{S}_\lambda^\mu(\phi,\varsigma) \equiv \mathcal{S}^\mu(\phi,\varsigma)$ in Definition 2.1, satisfies the following conditions

$$(1+\beta e^{i\gamma})\left\{\frac{\varsigma(\varpi_\mu \mathrm{Im}(\varsigma))'}{\varpi_\mu \mathrm{Im}(\varsigma)}\right\}-\beta e^{i\gamma} \prec \phi(\varsigma)$$

and

$$(1+\beta e^{i\gamma})\left\{\frac{\varrho(\varpi_\mu \mathrm{Im}(\varrho))'}{\varpi_\mu \mathrm{Im}(\varrho)}\right\}-\beta e^{i\gamma} \prec \phi(\varrho).$$

Remark 2.2. A function $\mathrm{Im}(\varsigma) \in \mathcal{H}$ given by (1.1) and for $\lambda = 1$, we note that $\mathcal{S}_\lambda^\mu(\phi,\varsigma) \equiv \mathcal{S}_1^\mu(\phi,\varsigma)$ in Definition 2.1, satisfies the following conditions

$$(1+\beta e^{i\gamma})\left[1+\frac{\varsigma(\varpi_\mu \mathrm{Im}(\varsigma))''}{(\varpi_\mu \mathrm{Im}(\varsigma))'}\right]-\beta e^{i\gamma} \prec \phi(\varsigma)$$

and

$$(1+\beta e^{i\gamma})\left[1+\frac{\varrho(\varpi_\mu \mathrm{Im}(\varrho))''}{(\varpi_\mu \mathrm{Im}(\varrho))'}\right]-\beta e^{i\gamma} \prec \phi(\varrho).$$

Remark 2.3. A function $\mathrm{Im}(\varsigma) \in \mathcal{H}$ given by (1.1) and for $\lambda = 0$, we note that $\mathcal{K}_\lambda^\mu(\phi,\varsigma) \equiv \mathcal{K}^\mu(\phi,\varsigma)$ in Definition 2.2, satisfies the following conditions

$$(1+\beta e^{i\gamma})\left\{\frac{\varpi_\mu \mathrm{Im}(\varsigma)}{\varsigma}\right\}-\beta e^{i\gamma} \prec \phi(\varsigma)$$

and

$$(1+\beta e^{i\gamma})\left\{\frac{\varpi_\mu \mathrm{Im}(\varrho)}{\varrho}\right\}-\beta e^{i\gamma} \prec \phi(\varrho).$$

Remark 2.4. A function $\mathrm{Im}(\varsigma) \in \mathcal{H}$ given by (1.1) and for $\lambda = 1$, we note that $\mathcal{K}_\lambda^\mu(\phi,\varsigma) \equiv \mathcal{K}_1^\mu(\phi,\varsigma)$ in Definition 2.2, satisfies the following conditions

$$(1+\beta e^{i\gamma})(\varpi_\mu \mathrm{Im}(\varsigma))'-\beta e^{i\gamma} \prec \phi(\varsigma)$$

and

$$(1+\beta e^{i\gamma})(\varpi_\mu \mathrm{Im}(\varrho))'-\beta e^{i\gamma} \prec \phi(\varrho).$$

3. Coefficient Estimates for $\mathcal{S}_\lambda^\mu(\phi,\varsigma)$ and $\mathcal{K}_\lambda^\mu(\phi,\varsigma)$

Before starting and proving our main results, we need the following lemma for deriving our main results.

Lemma 3.1 ([9]). *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions h analytic in $\mathcal{O}$ for which $\mathcal{R}(h(\varsigma)) > 0$ and*

$$h(z) = 1 + c_1\varsigma + c_2\varsigma^2 + \cdots, \quad \text{for } \varsigma \in \mathcal{O}.$$

Define the functions $\mathfrak{s}(\varsigma)$ and $\mathfrak{t}(\varsigma)$ by

$$\mathfrak{s}(\varsigma) = \frac{1+u(\varsigma)}{1-u(\varsigma)} = 1 + \mathfrak{s}_1\varsigma + \mathfrak{s}_2\varsigma^2 + \cdots$$

and

$$\mathfrak{t}(\varsigma) = \frac{1+\upsilon(\varsigma)}{1-\upsilon(\varsigma)} = 1 + \mathfrak{t}_1\varsigma + \mathfrak{t}_2\varsigma^2 + \cdots.$$

It follows that

$$u(\varsigma) := \frac{\mathfrak{s}(\varsigma)-1}{\mathfrak{s}(\varsigma)+1} = \frac{1}{2}\left[\mathfrak{s}_1\varsigma + \left(\mathfrak{s}_2 - \frac{\mathfrak{s}_1^2}{2}\right)\varsigma^2 + \cdots\right]$$

and

$$\upsilon(\varsigma) := \frac{\mathfrak{t}(\varsigma)-1}{\mathfrak{t}(\varsigma)+1} = \frac{1}{2}\left[\mathfrak{t}_1\varsigma + \left(\mathfrak{t}_2 - \frac{\mathfrak{t}_1^2}{2}\right)\varsigma^2 + \cdots\right].$$

Then, $\mathfrak{s}(\varsigma)$ and $\mathfrak{t}(\varsigma)$ are analytic in $\mathcal{O}$ with $\mathfrak{s}(0) = \mathfrak{t}(0) = 1$.

Since $u, \upsilon : \mathcal{O} \to \mathcal{O}$, the functions $\mathfrak{s}(\varsigma)$ and $\mathfrak{t}(\varsigma)$ have a positive real part in $\mathcal{O}$, and $|\mathfrak{s}_i| \leq 2$ and $|\mathfrak{t}_i| \leq 2$ for each i.

Now, we will obtain the coefficient estimates for the class $\mathcal{S}_\lambda^\mu(\phi,\varsigma)$ in Theorem 3.1 and the class $\mathcal{K}_\lambda^\mu(\phi,\varsigma)$ in Theorem 3.2.

Theorem 3.1. *If* $\mathrm{Im}(\varsigma)$ *belongs to $\mathcal{S}_\lambda^\mu(\phi,\varsigma)$ and has the series representation described in* (1.1)*, then*

(3.1)

$$|x_2| \leq \frac{V_1\sqrt{V_1}}{\mu\sqrt{(1+\beta e^{i\gamma})e^{-\mu}\left\{[(1+2\lambda)-(1+3\lambda)e^{-\mu}]\,V_1^2 + (V_1-V_2)(1+\beta e^{i\gamma})(1+\lambda)^2e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{V_1}{(1+\beta e^{i\gamma})(1+2\lambda)\mu^2e^{-\mu}} + \frac{V_1^2}{(1+\beta e^{i\gamma})^2(1+\lambda)^2\mu^2e^{-2m}}. \tag{3.2}$$

Proof. From (2.1) and (2.2) we can get

$$(1+\beta e^{i\gamma})\left\{(1-\lambda)\frac{\varsigma(\varpi_\mu\mathrm{Im}(\varsigma))'}{\varpi_\mu\mathrm{Im}(\varsigma)} + \lambda\left[1 + \frac{\varsigma(\varpi_\mu\mathrm{Im}(\varsigma))''}{(\varpi_\mu\mathrm{Im}(\varsigma))'}\right]\right\} - \beta e^{i\gamma} = \phi(u(\varsigma)) \tag{3.3}$$

and

$$(3.4)\quad (1+\beta e^{i\gamma})\left\{(1-\lambda)\frac{\varrho(\varpi_\mu \mathrm{Im}(\varrho))'}{\varpi_\mu \mathrm{Im}(\varrho)}+\lambda\left[1+\frac{\varrho(\varpi_\mu \mathrm{Im}(\varrho))''}{(\varpi_\mu \mathrm{Im}(\varrho))'}\right]\right\}-\beta e^{i\gamma}=\phi(\upsilon(\varrho)),$$

where $\mathfrak{s}(\varsigma), \mathfrak{t}(\varsigma) \in \mathcal{P}$ and

$$\phi(u(\varsigma))=\phi\left(\frac{1}{2}\left[\mathfrak{s}_1\varsigma+\left(\mathfrak{s}_2-\frac{\mathfrak{s}_1^2}{2}\right)\varsigma^2+\cdots\right]\right)$$

and

$$\phi(\upsilon(\varrho))=\phi\left(\frac{1}{2}\left[\mathfrak{t}_1\varrho+\left(\mathfrak{t}_2-\frac{\mathfrak{t}_1^2}{2}\right)\varrho^2+\cdots\right]\right).$$

Now, from (3.3) and (3.4), we get

$$(3.5)\quad (1+\beta e^{i\gamma})(1+\lambda)\mu e_2^{-\mu}x_2=\frac{1}{2}V_1\mathfrak{s}_1,$$

$$(3.6)\quad \begin{aligned}&(1+\beta e^{i\gamma})(1+2\lambda)\mu^2e^{-\mu}x_3-(1+\beta e^{i\gamma})(1+3\lambda)\mu^2e^{-2\mu}x_2^2\\&=\frac{1}{2}V_1\left(\mathfrak{s}_2-\frac{\mathfrak{s}_1^2}{2}\right)+\frac{1}{4}V_2^2\mathfrak{s}_1,\end{aligned}$$

$$(3.7)\quad -(1+\beta e^{i\gamma})(1+\lambda)\mu e^{-\mu}x_2=\frac{1}{2}V_1\mathfrak{t}_1$$

and

$$(3.8)\quad \begin{aligned}&(1+\beta e^{i\gamma})(1+2\lambda)\mu^2e^{-\mu}(2x_2^2-x_3)-(1+\beta e^{i\gamma})(1+3\lambda)\mu^2e^{-2\mu}x_2^2\\&=\frac{1}{2}V_1\left(\mathfrak{t}_2-\frac{\mathfrak{t}_1^2}{2}\right)+\frac{1}{4}V_2^2\mathfrak{t}_1.\end{aligned}$$

From (3.5) and (3.7), we get

$$(3.9)\quad \mathfrak{s}_1=-\mathfrak{t}_1$$

and

$$(3.10)\quad 8(1+\beta e^{i\gamma})^2(1+\lambda)^2\mu^2e^{-2\mu}x_2^2=V_1^2(\mathfrak{s}_1^2+\mathfrak{t}_1^2).$$

Now, from (3.6), (3.8) and (3.10), we obtain

$$(3.11)\quad \begin{aligned}&4(1+\beta e^{i\gamma})\mu^2e^{-\mu}\left\{\left[(1+2\lambda)-(1+3\lambda)e^{-\mu}\right]V_1^2\right.\\&\left.+(V_1-V_2)(1+\beta e^{i\gamma})(1+\lambda)^2e^{-\mu}\right\}x_2^2=V_1^3(\mathfrak{s}_2+\mathfrak{t}_2).\end{aligned}$$

By using Lemma 3.1 to the coefficients $\mathfrak{s}_2$ and $\mathfrak{t}_2$, we have

$$|x_2|\leq\frac{V_1\sqrt{V_1}}{\mu\sqrt{(1+\beta e^{i\gamma})e^{-\mu}\{[(1+2\lambda)-(1+3\lambda)e^{-\mu}]V_1^2+(V_1-V_2)(1+\beta e^{i\gamma})(1+\lambda)^2e^{-\mu}\}}}.$$

To find $|x_3|$, we subtract (3.6) from (3.8) and using (3.9), we have

$$2(1+\beta e^{i\gamma})(1+2\lambda)\mu^2e^{-\mu}(x_3-x_2^2)=\frac{1}{2}V_1(\mathfrak{s}_2-\mathfrak{t}_2).$$

Upon substituting the value of x_2^2 from (3.10), we get

$$x_3 = \frac{V_1(\mathfrak{s}_2 - \mathfrak{t}_2)}{4(1+\beta e^{i\gamma})(1+2\lambda)\mu^2 e^{-\mu}} + \frac{V_1^2(\mathfrak{s}_1^2 + \mathfrak{t}_1^2)}{8(1+\beta e^{i\gamma})^2(1+\lambda)^2\mu^2 e^{-2\mu}}.$$

By using Lemma 3.1 once again to the coefficients $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{t}_1$ and $\mathfrak{t}_2$, we get

$$|x_3| \leq \frac{V_1}{(1+\beta e^{i\gamma})(1+2\lambda)\mu^2 e^{-\mu}} + \frac{V_1^2}{(1+\beta e^{i\gamma})^2(1+\lambda)^2\mu^2 e^{-2\mu}},$$

we obtain the estimate given by (3.2). □

Theorem 3.2. *If* $\mathrm{Im}(\varsigma)$ *belongs to* $\mathcal{K}_\lambda^\mu(\phi,\varsigma)$ *and has the series representation described in* (1.1), *then*

$$|x_2| \leq \frac{V_1\sqrt{V_1}}{\sqrt{2\mu(1+\beta e^{i\gamma})e^{-\mu}\left\{(2\lambda+1)V_1^2 + 2\mu(V_1 - V_2)(1+\beta e^{i\gamma})(\lambda+1)^2 e^{-\mu}\right\}}} \tag{3.12}$$

and

$$|x_3| \leq \frac{2V_1}{\mu(1+\beta e^{i\gamma})(2\lambda+1)e^{-\mu}} + \frac{V_1^2}{\mu^2(1+\beta e^{i\gamma})^2(\lambda+1)^2 e^{-2\mu}}. \tag{3.13}$$

Proof. From (2.3) and (2.4), we have

$$(1+\beta e^{i\gamma})\left\{(1-\lambda)\frac{\varpi_\mu \mathrm{Im}(\varsigma)}{\varsigma} + \lambda\varpi_\mu \mathrm{Im}(\varsigma))'\right\} - \beta e^{i\gamma} = \phi(u(\varsigma)) \tag{3.14}$$

and

$$(1+\beta e^{i\gamma})\left\{(1-\lambda)\frac{\varpi_\mu \mathrm{Im}(\varrho)}{\varrho} + \lambda(\varpi_\mu \mathrm{Im}(\varrho))'\right\} - \beta e^{i\gamma} = \phi(v(\varrho)), \tag{3.15}$$

where $\mathfrak{s}(\varsigma), \mathfrak{t}(\varsigma) \in \mathcal{P}$ and

$$\phi(u(\varsigma)) = \phi\left(\frac{1}{2}\left[\mathfrak{s}_1\varsigma + \left(\mathfrak{s}_2 - \frac{\mathfrak{s}_1^2}{2}\right)\varsigma^2 + \cdots\right]\right)$$

and

$$\phi(v(\varrho)) = \phi\left(\frac{1}{2}\left[\mathfrak{t}_1\varrho + \left(\mathfrak{t}_2 - \frac{\mathfrak{t}_1^2}{2}\right)\varrho^2 + \cdots\right]\right).$$

From (3.14) and (3.15), we get

$$(1+\beta e^{i\gamma})(\lambda+1)\mu e^{-\mu}x_2 = \frac{1}{2}V_1\mathfrak{s}_1, \tag{3.16}$$

$$\mu(1+\beta e^{i\gamma})(2\lambda+1)e^{-\mu}x_3 = V_1\mathfrak{s}_2 + \left(\frac{V_2 - V_1}{2}\right)\mathfrak{s}_1^2, \tag{3.17}$$

$$-(1+\beta e^{i\gamma})(\lambda+1)\mu e^{-\mu}x_2 = \frac{1}{2}V_1\mathfrak{t}_1 \tag{3.18}$$

and

$$\mu(1+\beta e^{i\gamma})(2\lambda+1)e^{-\mu}(2x_2^2 - x_3) = V_1\mathfrak{t}_2 + \left(\frac{V_2 - V_1}{2}\right)\mathfrak{t}_1^2. \tag{3.19}$$

From (3.16) and (3.18), we get

$$\mathfrak{s}_1 = -\mathfrak{t}_1 \tag{3.20}$$

and

$$8(1+\beta e^{i\gamma})^2(\lambda+1)^2\mu^2 e^{-2\mu}x_2^2 = V_1^2(\mathfrak{s}_1^2+\mathfrak{t}_1^2). \tag{3.21}$$

Now, from (3.17), (3.19) and (3.11), we obtain

$$\begin{aligned} &2(1+\beta e^{i\gamma})\mu e^{-\mu}\left\{(2\lambda+1)V_1^2 + 2\mu(V_1-V_2)(1+\beta e^{i\gamma})\ (\lambda+1)^2 e^{-\mu}\right\}x_2^2 \\ &=V_1^3(\mathfrak{s}_2+\mathfrak{t}_2). \end{aligned} \tag{3.22}$$

By using Lemma 3.1 to the coefficients $\mathfrak{s}_2$ and $\mathfrak{t}_2$, we have

$$|x_2| \leq \frac{V_1\sqrt{V_1}}{\sqrt{2\mu(1+\beta e^{i\gamma})e^{-\mu}\left\{(2\lambda+1)V_1^2 + 2\mu(V_1-V_2)(1+\beta e^{i\gamma})(\lambda+1)^2 e^{-\mu}\right\}}}.$$

To find $|x_3|$, we subtract (3.17) from (3.19) and using (3.20), we have

$$2\mu(1+\beta e^{i\gamma})(2\lambda+1)e^{-\mu}(x_3-x_2^2) = \frac{1}{2}V_1(\mathfrak{s}_2-\mathfrak{t}_2).$$

Upon replacing the value of x_2^2 from (3.21), we get

$$x_3 = \frac{V_1(\mathfrak{s}_2-\mathfrak{t}_2)}{2\mu(1+\beta e^{i\gamma})(2\lambda+1)e^{-\mu}} + \frac{V_1^2(\mathfrak{s}_1^2+\mathfrak{t}_1^2)}{8\mu^2(1+\beta e^{i\gamma})^2(\lambda+1)^2 e^{-2\mu}}.$$

By using Lemma 3.1 once again to the coefficients $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{t}_1$ and $\mathfrak{t}_2$, we have

$$|x_3| \leq \frac{2V_1}{\mu(1+\beta e^{i\gamma})(2\lambda+1)e^{-\mu}} + \frac{V_1^2}{\mu^2(1+\beta e^{i\gamma})^2(\lambda+1)^2 e^{-2\mu}},$$

we obtain the estimate given by (3.13). □

By specializing the parameter $\lambda = 0$ in Theorems 3.1 and 3.2, we get the coefficients for the functions in $\mathcal{S}^\mu(\phi,\varsigma)$ and $\mathcal{K}^\mu(\phi,\varsigma)$ defined in Remark 2.1 and Remark 2.3.

Corollary 3.1. *If* $\mathrm{Im}(\varsigma)$ *belongs to the class* $\mathcal{S}^\mu(\phi,\varsigma)$ *and has the series representation described in* (1.1), *then*

$$|x_2| \leq \frac{V_1\sqrt{V_1}}{\mu\sqrt{(1+\beta e^{i\gamma})e^{-\mu}\left\{(1-e^{-\mu})V_1^2 + (V_1-V_2)(1+\beta e^{i\gamma})e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{V_1}{(1+\beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{V_1^2}{(1+\beta e^{i\gamma})^2\mu^2 e^{-2\mu}}.$$

Corollary 3.2. *If* $\mathrm{Im}(\varsigma)$ *belongs to the class* $\mathcal{K}^{\mu}(\phi,\varsigma)$ *and has the series representation described in* (1.1), *then*

$$|x_2| \leq \frac{V_1\sqrt{V_1}}{\sqrt{2\mu(1+\beta e^{i\gamma})e^{-\mu}\left\{V_1^2+2\mu(V_1-V_2)(1+\beta e^{i\gamma})e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{2V_1}{\mu(1+\beta e^{i\gamma})e^{-\mu}} + \frac{V_1^2}{\mu^2(1+\beta e^{i\gamma})^2 e^{-2\mu}}.$$

Putting $\lambda = 1$ in Theorems 3.1 and 3.2, the coefficient estimates for the functions in $\mathcal{S}_1^{\mu}(\phi,\varsigma)$ and $\mathcal{K}_1^{\mu}(\phi,\varsigma)$ was defined in Remark 2.2 and 2.4.

Corollary 3.3. *If* $\mathrm{Im}(\varsigma)$ *belongs to the class* $\mathcal{S}_1^{\mu}(\phi,\varsigma)$ *and has the series representation described in* (1.1), *then*

$$|x_2| \leq \frac{V_1\sqrt{V_1}}{\mu\sqrt{(1+\beta e^{i\gamma})e^{-\mu}\left\{(3-4e^{-\mu})V_1^2+4(V_1-V_2)(1+\beta e^{i\gamma})e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{V_1}{3(1+\beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{V_1^2}{4(1+\beta e^{i\gamma})^2\mu^2 e^{-2\mu}}.$$

Corollary 3.4. *Let* $\mathrm{Im}(\varsigma)$ *given by* (1.1) *be in the class* $\mathcal{K}_1^{\mu}(\phi,\varsigma)$. *Then,*

$$|x_2| \leq \frac{V_1\sqrt{V_1}}{\sqrt{2\mu(1+\beta e^{i\gamma})e^{-\mu}\left\{3V_1^2+8\mu(V_1-V_2)(1+\beta e^{i\gamma})e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{2V_1}{3\mu(1+\beta e^{i\gamma})e^{-\mu}} + \frac{V_1^2}{4\mu^2(1+\beta e^{i\gamma})^2 e^{-2\mu}}.$$

4. Corollaries and Consequences

For the function ϕ, given by

$$\phi(\varsigma) = \left(\frac{1+\varsigma}{1-\varsigma}\right)^{\alpha} = 1 + 2\alpha\varsigma + 2\alpha^2\varsigma^2 + \cdots, \quad 0 < \alpha \leq 1, \tag{4.1}$$

we have $V_1 = 2\alpha$ and $V_2 = 2\alpha^2$.

On the other hand if we take

$$\phi(\varsigma) = \frac{1+(1-2\zeta)\varsigma}{1-\varsigma} = 1 + 2(1-\zeta)\varsigma + 2(1-\zeta)\varsigma^2 + \cdots, \quad 0 \leq \zeta < 1, \tag{4.2}$$

then $V_1 = V_2 = 2(1-\zeta)$.

Corollary 4.1. *By using $\phi(\varsigma)$ of the form* (4.1) *for functions* $\mathrm{Im}(\varsigma) \in \mathcal{S}_\lambda^\mu(\phi, \varsigma)$ *as given by Theorem* 3.1, *we state the following results:*

$$|x_2| \leq \frac{2\alpha}{\mu\sqrt{(1+\beta e^{i\gamma})e^{-\mu}\left\{2\alpha\left[(1+2\lambda)-(1+3\lambda)e^{-\mu}\right]+(1-\alpha)(1+\beta e^{i\gamma})(1+\lambda)^2 e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{2\alpha}{(1+\beta e^{i\gamma})(1+2\lambda)\mu^2 e^{-\mu}} + \frac{4\alpha^2}{(1+\beta e^{i\gamma})^2(1+\lambda)^2\mu^2 e^{-2\mu}}.$$

Corollary 4.2. *By choosing $\phi(\varsigma)$ of the form* (4.1) *for functions* $\mathrm{Im}(\varsigma) \in \mathcal{K}_\lambda^\mu(\phi, \varsigma)$ *as given by Theorem* 3.2, *we state the following results:*

$$|x_2| \leq \frac{2\alpha}{\sqrt{2\mu(1+\beta e^{i\gamma})e^{-\mu}\left\{2\alpha(2\lambda+1)+2(1-\alpha)\mu(1+\beta e^{i\gamma})(\lambda+1)^2 e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{4\alpha}{\mu(1+\beta e^{i\gamma})(2\lambda+1)e^{-\mu}} + \frac{4\alpha^2}{\mu^2(1+\beta e^{i\gamma})^2(\lambda+1)^2 e^{-2\mu}}.$$

Remark 4.1. From Corollary 4.1, taken $\lambda = 0$, we can get

$$|x_2| \leq \frac{2\alpha}{\mu\sqrt{(1+\beta e^{i\gamma})e^{-\mu}\left\{2\alpha(1-e^{-\mu})+(1-\alpha)(1+\beta e^{i\gamma})e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{2\alpha}{(1+\beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{4\alpha^2}{(1+\beta e^{i\gamma})^2\mu^2 e^{-2\mu}}.$$

Remark 4.2. From Corollary 4.2, taken $\lambda = 0$, we obtain the following results:

$$|x_2| \leq \frac{2\alpha}{\sqrt{2\mu(1+\beta e^{i\gamma})e^{-\mu}\left\{2\alpha+2(1-\alpha)\mu(1+\beta e^{i\gamma})e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{4\alpha}{\mu(1+\beta e^{i\gamma})e^{-\mu}} + \frac{4\alpha^2}{\mu^2(1+\beta e^{i\gamma})^2 e^{-2\mu}}.$$

Remark 4.3. From Corollary 4.1, taken $\lambda = 1$, we obtain the following results:

$$|x_2| \leq \frac{2\alpha}{\mu\sqrt{(1+\beta e^{i\gamma})e^{-\mu}\left\{2\alpha\left(3-4e^{-\mu}\right)+4(1-\alpha)(1+\beta e^{i\gamma})e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{2\alpha}{3(1+\beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{\alpha^2}{(1+\beta e^{i\gamma})^2\mu^2 e^{-2\mu}}.$$

Remark 4.4. From Corollary 4.2, taken $\lambda = 1$, we obtain the following results:

$$|x_2| \leq \frac{\alpha}{\sqrt{\mu(1+\beta e^{i\gamma})e^{-\mu}\left\{3\alpha + 4\mu(1-\alpha)(1+\beta e^{i\gamma})e^{-\mu}\right\}}}$$

and

$$|x_3| \leq \frac{4\alpha}{3\mu(1+\beta e^{i\gamma})e^{-\mu}} + \frac{\alpha^2}{\mu^2(1+\beta e^{i\gamma})^2 e^{-2\mu}}.$$

Corollary 4.3. *By using $\phi(\varsigma)$ of the form* (4.2) *for functions* $\mathrm{Im}(\varsigma) \in \mathcal{S}_\lambda^\mu(\phi, \varsigma)$ *as given by Theorem* 3.1*, we acquire the results:*

$$|x_2| \leq \frac{1}{\mu}\sqrt{\frac{2(1-\zeta)}{(1+\beta e^{i\gamma})e^{-\mu}\left[(1+2\lambda) - (1+3\lambda)e^{-\mu}\right]}}$$

and

$$|x_3| \leq \frac{2(1-\zeta)}{(1+\beta e^{i\gamma})(1+2\lambda)\mu^2 e^{-\mu}} + \frac{4(1-\zeta)^2}{(1+\beta e^{i\gamma})^2(1+\lambda)^2\mu^2 e^{-2\mu}}.$$

Corollary 4.4. *By using $\phi(\varsigma)$ of the form* (4.2) *for functions* $\mathrm{Im}(\varsigma) \in \mathcal{K}_\lambda^\mu(\phi, \varsigma)$ *as given by Theorem* 3.2*, we acquire the following results:*

$$|x_2| \leq \sqrt{\frac{2(1-\zeta)}{2\mu(1+\beta e^{i\gamma})e^{-\mu}(2\lambda+1)}}$$

and

$$|x_3| \leq \frac{4(1-\zeta)}{\mu(1+\beta e^{i\gamma})(2\lambda+1)e^{-\mu}} + \frac{4(1-\zeta)^2}{\mu^2(1+\beta e^{i\gamma})^2(\lambda+1)^2 e^{-2\mu}}.$$

Remark 4.5. From Corollary 4.3, taken $\lambda = 0$, we obtain the following results:

$$|x_2| \leq \frac{1}{\mu}\sqrt{\frac{2(1-\zeta)}{(1+\beta e^{i\gamma})e^{-\mu}\left(1-e^{-\mu}\right)}}$$

and

$$|x_3| \leq \frac{2(1-\zeta)}{(1+\beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{4(1-\zeta)^2}{(1+\beta e^{i\gamma})^2\mu^2 e^{-2\mu}}.$$

Remark 4.6. From Corollary 4.4, taken $\lambda = 0$, we obtain the results:

$$|x_2| \leq \sqrt{\frac{2(1-\zeta)}{2\mu^2(1+\beta e^{i\gamma})e^{-\mu^2}}}$$

and

$$|x_3| \leq \frac{4(1-\zeta)}{\mu(1+\beta e^{i\gamma})e^{-\mu}} + \frac{4(1-\zeta)^2}{\mu^2(1+\beta e^{i\gamma})^2 e^{-2\mu}}.$$

Remark 4.7. From Corollary 4.3, taken $\lambda = 1$, we acquire the following results:

$$|x_2| \leq \frac{1}{\mu}\sqrt{\frac{2(1-\zeta)}{(1+\beta e^{i\gamma})e^{-\mu}\left(3-4e^{-\mu}\right)}}$$

and

$$|x_3| \leq \frac{2(1-\zeta)}{3(1+\beta e^{i\gamma})\mu^2 e^{-\mu}} + \frac{(1-\zeta)^2}{(1+\beta e^{i\gamma})^2\mu^2 e^{-2\mu}}.$$

Remark 4.8. From Corollary (4.4), taken $\lambda = 1$, we obtain the following results:

$$|x_2| \leq \sqrt{\frac{2(1-\zeta)}{6\mu(1+\beta e^{i\gamma})e^{-\mu}}}$$

and

$$|x_3| \leq \frac{4(1-\zeta)}{3\mu(1+\beta e^{i\gamma})e^{-\mu}} + \frac{(1-\zeta)^2}{\mu^2(1+\beta e^{i\gamma})^2 e^{-2\mu}}.$$

Notation

There are many applications of Poisson distribution such as a space-time spectral order since collocation method for the fourth-order nonlocal heat model arising in viscoelasticity, a high-order and efficient numerical technique for reactors, a robust error analysis of the OSC method for multi-term fourth-order sub-diffusion equation, an efficient ADI difference scheme for the nonlocal evolution problem in three-dimensional space.

Conclusion

We successfully used the Poisson distribution series to introduce and derive different subclasses of regular and bi-univalent functions in the open unit disk in part two. In part three we present special estimates for the Taylor coefficients' inequalities $|a_2|$ and $|a_3|$ of the functions belonging to the new two classes $\mathcal{S}_{\lambda}^{\mu}(\phi,\varsigma)$ and $\mathcal{K}_{\lambda}^{\mu}(\phi,\varsigma)$. Also, in part four we investigated certain corollaries and consequences of the results by choosing $V2081 = 23b1$, $V2082 = 23b1$, $V2081 = V2082 = 2(1-\zeta)$. The results presented in this paper have been beneficial supplement for the research of geometric function theory of complex analysis.

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