KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 47(1) (2023), PAGES 57–80.

GENERALIZED EXTENDED RIEMANN-LIOUVILLE TYPE FRACTIONAL DERIVATIVE OPERATOR

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ABSTRACT. In this paper, we present new extensions of incomplete gamma, beta, Gauss hypergeometric, confluent hypergeometric function and Appell-Lauricella hypergeometric functions, by using the extended Bessel function due to Boudjelkha [4]. Some recurrence relations, transformation formulas, Mellin transform and integral representations are obtained for these generalizations. Further, an extension of the Riemann-Liouville fractional derivative operator is established.

1. INTRODUCTION

In recent years, incomplete gamma functions have been used in many problems in applied mathematics, statistics, engineering and many other fields including physics and biology. Most generally, special functions became powerful tools to treat all these areas. Classical gamma and Euler's beta functions are defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt, \quad \operatorname{Re}(\alpha) > 0, \tag{1.1}$$

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt, \tag{1.2}$$

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0. \tag{1.3}$$

Key words and phrases. Generalized extended incomplete gamma function, generalized extended beta function, extended Riemann-Liouville fractional derivative, Mellin transform, extended Gauss hypergeometric function, integral representation.

2010 *Mathematics Subject Classification.* Primary: 26A33, 33B15, 33B20, 33C20, 33C65.

Received: January 26, 2020.

Accepted: July 01, 2020.

Using an exponential regulazing term, Chaudhry et al. [9] extended the incomplete gamma function as follows

$$\gamma(\alpha,x;p)=\int_0^x t^{\alpha-1}e^{-t-\frac{p}{t}}dt,\quad \mathrm{Re}(p)>0;\ p=0,\ \mathrm{Re}(\alpha)>0, \tag{1.4}$$

$$\Gamma(\alpha,x;p)=\int_x^\infty t^{\alpha-1}e^{-t-\frac{p}{t}}dt. \tag{1.5}$$

They proved the following recurrence formula

$$\gamma(\alpha,x;p)+\Gamma(\alpha,x;p)=2p^{\alpha/2}K_\alpha(2\sqrt{p}),\quad \mathrm{Re}(p)>0,$$

where $K_\alpha(z)$ is the Macdonald function, known also as modified Bessel function of the third kind, defined for any $\mathrm{Re}(z)>0$ by

$$K_\alpha(z)=\frac{(z/2)^\alpha}{2}\int_0^\infty t^{-\alpha-1}e^{-t-z^2/4t}dt.$$

A first extension of Euler's beta function is given by Chaudhry et al. [8] as follows

$$B(x,y,p)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{\frac{-p}{t(1-t)}}dt,\quad \mathrm{Re}(p)>0;\ p=0,\ \mathrm{Re}(x)>0,\ \mathrm{Re}(y)>0. \tag{1.6}$$

These extensions are useful and provide new connections with error and Whittaker functions. For $p=0$, (1.4), (1.5) and (1.6) will be reduced to known incomplete gamma and beta functions (1.1), (1.2) and (1.3), respectively. Instead of using the exponential function, Chaudhry and Zubair [11] proposed a generalized extension of (1.4), (1.5) in the following form

$$\gamma_\mu(\alpha,x;p)=\sqrt{\frac{2p}{\pi}}\int_0^x t^{\alpha-\frac{3}{2}}e^{-t}K_{\mu+\frac{1}{2}}\left(\frac{p}{t}\right)dt, \tag{1.7}$$

$$\Gamma_\mu(\alpha,x;p)=\sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}K_{\mu+\frac{1}{2}}\left(\frac{p}{t}\right)dt, \tag{1.8}$$

where $\mathrm{Re}(x)>0$, $\mathrm{Re}(p)>0$, $-\infty<\alpha<\infty$.

Nowadays, many authors are developing new extensions of Euler's gamma, beta and hypergeometric functions based on the paper of Chaudhry and Zubair [11] by considering an exponential kernel and some modified special functions (see for more details [13,14,20,22,23,25–27]). Very recently, Agarwal et al. [1] developed an extension of the Euler's beta function as follows

$$B_\mu(x,y;p;m)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right)dt, \tag{1.9}$$

where $x,y\in\mathbb{C}$, $m>0$ and $\mathrm{Re}(p)>0$.

In the present paper, we introduce a new generalized incomplete gamma and Euler's beta functions by substituting in (1.7), (1.8) and (1.9) the Macdonald function $K_\alpha(z)$

by it's extended one developed by Boudjelkha [4], namely

$$R_K(z,\alpha,q,\lambda)=\frac{(z/2)^\alpha}{2}\int_0^\infty t^{-\alpha-1}\frac{e^{-qt-z^2/4t}}{1-\lambda e^{-t}}dt, \tag{1.10}$$

where $|\arg z^2|<\pi/2$, $0<q\leq 1$ and $-1\leq\lambda\leq 1$.

Clearly, when $\lambda=0$ and $q=1$, $R_K(z,\alpha,q,\lambda)$ is reduced to $K_\alpha(z)$. Moreover, Boudjelkha proved that the $R_K(z,-\alpha,q,\lambda)$ function can be expanded in terms of $K_\alpha(z)$ as follows

$$R_K(z,-\alpha,q,\lambda)=\sum_{n=0}^{\infty}\lambda^n\frac{K_\alpha(z\sqrt{q+n})}{(q+n)^{\alpha/2}},\quad \mathrm{Re}(z^2)>0,\ 0<q\leq 1,\ -1\leq\lambda\leq 1,$$

and showed that the behavior of the function $R_K(z,-\alpha,q,\lambda)$ for small values of z is described by the asymptotic formulas:

$$R_K(z,-\alpha,q,\lambda)\sim\begin{cases}\frac{1}{2}\frac{\Gamma(-z)}{(z/2)^{-\alpha}}(1-\lambda)^{-1},\ z\to 0,\ -1<\lambda<1, & \mathrm{Re}(\alpha)<0,\\ \frac{1}{2}\frac{\Gamma(z)}{(z/2)^{\alpha}}\Phi(\lambda,\alpha,q),\ z\to 0,\ -1\leq\lambda\leq 1, & \mathrm{Re}(\alpha)>1,\end{cases}$$

where $\Phi(\lambda,\alpha,q)$ stands for the Lerch function. As for the asymptotic behavior of this function, when $z\to\infty$, it is given by

$$R_K(z,-\alpha,q,\lambda)\sim\sqrt{\frac{\pi}{2z}}\cdot\frac{e^{-z\sqrt{q}}}{q^{\alpha/2+1/4}},\quad \text{as } z\to\infty,\ |\arg z|<\frac{\pi}{4},\ -1\leq\lambda\leq 1.$$

In particular, when $q=1$, we have

$$R_K(z,-\alpha,1,\lambda)\sim\sqrt{\frac{\pi}{2z}}e^{-z},\quad \text{as } z\to\infty,\ |\arg z|<\frac{\pi}{4},$$

which is the same asymptotic formula as that of K_α.

Further, by using the generalized extended beta function we get other extensions of Gauss hypergeometric, confluent hypergeometric, Appell and Lauricella hypergeometric functions and we investigate some of their properties.

Recently, fractional derivative operators become significant research topics due to their wide applications in various areas including mathematical, physical, life sciences and engineering problems. To cite only a few of this operator's applications, we refer to [5–7, 16, 29] and the references therein. The use of fractional derivative operators in obtaining generating relations for some special functions can be found in [22, 28]. There are two important fractional derivatives operators: Riemann-Liouville and Caputo operators. Undoubtedly, the difference between them is very important for applications to differential equations because of required initial conditions which are of different types (see e.g [19] and [31]). It is worth being pointed out that nowadays a great attention is devoted to develop extensions of fractional differential operators, readers may refer to [1–3, 5–7, 17, 18, 21–23, 30]. Making use of the R_K function and inspired by the work of Agarwal et al. [1], we introduce new generalized incomplete Riemann-Liouville fractional derivative operators, and we obtain some generating relations involving generalized extended Gauss hypergeometric function.

The paper is organized as follows. In Section 2, we introduce the generalized extended incomplete Gamma and Euler's beta functions, some of their properties are investigated. Section 3 is devoted to introduce extended hypergeometric and confulent hypergeometric functions by the extended Euler's beta function given in Section 2, their related properties are established. The extended Appell and Lauricella hypergeometric functions are given in Section 4. In Section 5, we give another result which consits to introduce the generalized extended Riemann Liouville fractional derivative operator and establish most important properties such Mellin transform among others. Finally, in the last section, we obtain linear and bilinear generating relations for the generalized extended hypergeometric functions.

2. THE GENERALIZED EXTENDED INCOMPLETE GAMMA AND EULER'S BETA FUNCTIONS

In this section, we define new extended incomplete Gamma and Euler's beta functions based on the extension of Bessel function (1.10) and we give some properties.

2.1. **The generalized extended incomplete Gamma function.**

Definition 2.1. The generalized extended incomplete gamma functions are given by

$$\gamma_\mu(\alpha, x; q; \lambda; p) = \sqrt{\frac{2p}{\pi}} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) dt, \tag{2.1}$$

$$\Gamma_\mu(\alpha, x; q; \lambda; p) = \sqrt{\frac{2p}{\pi}} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} R_K\left(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda\right) dt, \tag{2.2}$$

where $\mathrm{Re}(x) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$ and $\mathrm{Re}(p) > 0$.

Remark 2.1. When $\lambda = 0$ and $q = 1$, (2.1) and (2.2) are respectively reduced to the extended incomplete gamma functions (1.7) and (1.8) defined by Chaudhry and Zubair [10,11].

Proposition 2.1 (Decomposition theorem)**.**

$$\begin{aligned}\Gamma_\mu(\alpha, x; q; \lambda; p) + \gamma_\mu(\alpha, x; q; \lambda; p) = & \frac{\Gamma(\alpha+\mu)}{\sqrt{\pi}} \left(\frac{p}{2}\right)^{-\mu} \Phi_{1-\frac{\alpha+\mu}{2}, \frac{1}{2}-\frac{\alpha+\mu}{2}}\left(\lambda, \mu + \frac{1}{2}, q, \frac{p^2}{16}\right) \\ & + \frac{\Gamma\left(-\frac{\alpha+\mu}{2}\right)}{2\sqrt{\pi}} \left(\frac{p}{2}\right)^{\alpha} \Phi_{\frac{1}{2}, \frac{\alpha+\mu+2}{2}}\left(\lambda, \frac{\mu-\alpha+1}{2}, q, \frac{p^2}{16}\right) \\ & - \frac{\Gamma\left(-\frac{\alpha+\mu+1}{2}\right)}{2\sqrt{\pi}} \left(\frac{p}{2}\right)^{\alpha+1} \Phi_{\frac{3}{2}, \frac{\alpha+\mu+3}{2}}\left(\lambda, \frac{\mu-\alpha}{2}, q, \frac{p^2}{16}\right),\end{aligned}$$

with $\mathrm{Re}(p) > 0$, $-\infty < \alpha < \infty$ *and*

$$\Phi_{b_1,b_2}(\lambda, s, q, \xi) = \int_0^\infty \frac{t^{s-1} e^{-qt}}{1 - \lambda e^{-t}} \, {}_0F_2\left(\begin{matrix} - \\ \\ b_1, b_2 \end{matrix} ; -\tfrac{\xi}{t}\right) dt$$

$$= \int_0^\infty \frac{t^{s-1}e^{-(q-1)t}}{e^t-\lambda}\, {}_0F_2\left(\begin{matrix} - \\ b_1, b_2 \end{matrix}; -\frac{\xi}{t}\right) dt, \tag{2.3}$$

$s \in \mathbb{C}$, $\mathrm{Re}(\xi) > 0$ *and* $b_1, b_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

Proof. We have

$$\begin{aligned}
&\Gamma_\mu(\alpha, x; q; \lambda; p) + \gamma_\mu(\alpha, x; q; \lambda; p)\\
=&\sqrt{\frac{2p}{\pi}} \int_0^\infty t^{\alpha-\frac{3}{2}} e^{-t} R_K\left(\frac{p}{t}, -\mu-\frac{1}{2}, q, \lambda\right) dt\\
=&\frac{1}{\sqrt{\pi}}\left(\frac{p}{2}\right)^{-\mu} \int_0^\infty t^{\alpha+\mu-1}e^{-t}\left(\int_0^\infty \tau^{\mu-\frac{1}{2}} \frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}} d\tau\right) dt\\
=&\frac{1}{\sqrt{\pi}}\left(\frac{p}{2}\right)^{-\mu} \int_0^\infty \frac{\tau^{\mu-\frac{1}{2}}e^{-q\tau}}{1-\lambda e^{-\tau}}\left(\int_0^\infty t^{\alpha+\mu-1}e^{-t}e^{-\frac{p^2}{4t^2\tau}} dt\right) d\tau. \qquad (2.4)
\end{aligned}$$

Using the integral [24, page 31, (6)], we obtain

$$\begin{aligned}
\int_0^\infty t^{\alpha+\mu-1}e^{-t}e^{-\frac{p^2}{4t^2\tau}} dt =&\Gamma(\alpha+\mu)\, {}_0F_2\left(\begin{matrix} - \\ 1-\frac{\alpha+\mu}{2}, \frac{1}{2}-\frac{\alpha+\mu}{2} \end{matrix}; -\frac{p^2}{16\tau}\right)\\
&+\frac{\Gamma\left(-\frac{\alpha+\mu}{2}\right)}{2}\left(\frac{p^2}{4\tau}\right)^{\frac{\alpha+\mu}{2}} {}_0F_2\left(\begin{matrix} - \\ \frac{1}{2}, \frac{\alpha+\mu+2}{2} \end{matrix}; -\frac{p^2}{16\tau}\right)\\
&-\frac{\Gamma\left(-\frac{\alpha+\mu+1}{2}\right)}{2}\left(\frac{p^2}{4\tau}\right)^{\frac{\alpha+\mu+1}{2}} {}_0F_2\left(\begin{matrix} - \\ \frac{3}{2}, \frac{\alpha+\mu+3}{2} \end{matrix}; -\frac{p^2}{16\tau}\right).
\end{aligned} \tag{2.5}$$

Finally, substituting (2.5) in (2.4) and by using the notation (2.3) we get the desired result. ☐

Proposition 2.2 (Recurrence relation)**.**

$$\begin{aligned}
\Gamma_\mu(\alpha+1, x; q; \lambda; p) =&(\alpha+\mu)\Gamma_\mu(\alpha, x; q; \lambda; p) + p\Gamma_{\mu-1}(\alpha-1, x; q; \lambda; p)\\
&+\sqrt{\frac{2p}{\pi}} x^{\alpha-\frac{1}{2}} e^{-x} R_K\left(\frac{p}{x}, -\mu-\frac{1}{2}, q, \lambda\right),
\end{aligned}$$

where $\mathrm{Re}(p) > 0$, $-\infty < \alpha < \infty$.

Proof. We have

$$\frac{d}{dt}R_K\left(\frac{p}{t}, -\mu-\frac{1}{2}, q, \lambda\right) = \frac{d}{dt}\left[\frac{\left(\frac{p}{2t}\right)^{-\mu-\frac{1}{2}}}{2}\int_0^\infty \tau^{\mu-\frac{1}{2}} \frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}} d\tau\right]$$

$$=\frac{\mu+\frac{1}{2}}{t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)+\frac{p}{t^2}R_K\left(\frac{p}{t},-\mu+\frac{1}{2},q,\lambda\right). \tag{2.6}$$

Differentiating $t^{\alpha-\frac{1}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)$ with respect to t and by using (2.6), we get

$$\frac{d}{dt}\left[t^{\alpha-\frac{1}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right] \tag{2.7}$$

$$=(\alpha+\mu)t^{\alpha-\frac{3}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)+p\,t^{\alpha-\frac{5}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu+\frac{1}{2},q,\lambda\right) \tag{2.8}$$
$$-t^{\alpha-\frac{1}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right).$$

Multiplying both sides of (2.7) by $\sqrt{\frac{2p}{\pi}}$ and integrating from x to ∞ and using (2.2), we find

$$\begin{aligned}&0-\sqrt{\frac{2p}{\pi}}x^{\alpha-\frac{1}{2}}e^{-x}R_K\left(\frac{p}{x},-\mu-\frac{1}{2},q,\lambda\right)\\ =&(\alpha+\mu)\Gamma_\mu(\alpha,x;q;\lambda;p)+p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)-\Gamma_\mu(\alpha+1,x;q;\lambda;p),\end{aligned}$$

which can be also written as

$$\begin{aligned}\Gamma_\mu(\alpha+1,x;q;\lambda;p)=&(\alpha+\mu)\Gamma_\mu(\alpha,x;q;\lambda;p)+p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)\\ &+\sqrt{\frac{2p}{\pi}}x^{\alpha-\frac{1}{2}}e^{-x}R_K\left(\frac{p}{x},-\mu-\frac{1}{2},q,\lambda\right).\end{aligned}$$ □

Proposition 2.3. *The following formula holds*

$$\begin{aligned}&\Gamma_{\mu-1}(\alpha,x;1;\lambda;p)-\Gamma_{\mu+1}(\alpha,x;1;\lambda;p)+\frac{2\mu+1}{p}\Gamma_\mu(\alpha+1,x;1;\lambda;p)\\ =&\lambda\frac{\partial}{\partial\lambda}\Gamma_{\mu+1}(\alpha,x;1;\lambda;p),\end{aligned}$$

where $\mathrm{Re}(p)>0$, $-\infty<\alpha<\infty$.

Proof. By using (2.2), for $q=1$ and the following relation [4, (22)], we get

$$R_K(z,-\alpha+1,1,\lambda)-R_K(z,-\alpha-1,1,\lambda)+\frac{2\alpha}{z}R_K(z,-\alpha,1,\lambda)=\lambda\frac{\partial}{\partial\lambda}R_K(z,-\alpha-1,1,\lambda).$$

□

Proposition 2.4 (Laplace transform)**.** *Let*

$$H(\tau)=\begin{cases}1, & \tau>0,\\ 0, & \tau<0,\end{cases}$$

be the Heaviside unit step function and $\mathcal{L}$ *be the Laplace transform operator. Then*

$$\mathcal{L}\left\{t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)H(t-x);s\right\}=\sqrt{\frac{\pi}{2p}}s^{-\alpha}\Gamma_\mu(\alpha,sx;q;\lambda;sp), \tag{2.9}$$

$$\mathcal{L}\left\{t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)H(t-x)H(t);s\right\}=\sqrt{\frac{\pi}{2p}}s^{-\alpha}\gamma_\mu(\alpha,sx;q;\lambda;sp), \tag{2.10}$$

where $x>0$, $\operatorname{Re}(p)>0$, $-\infty<\alpha<\infty$.

Proof. We have

$$\begin{aligned}&\mathcal{L}\left\{t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)H(t-x);s\right\}\\&=\int_0^\infty t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)e^{-st}H(t-x)dt\\&=\int_x^\infty t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)e^{-st}dt.\end{aligned}$$

Substituting $t=\frac{\tau}{s}$, $dt=\frac{d\tau}{s}$, we get

$$\begin{aligned}&\int_x^\infty t^{\alpha-\frac{3}{2}}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)e^{-st}dt\\=&s^{-\alpha+\frac{1}{2}}\int_{sx}^\infty \tau^{\alpha-\frac{3}{2}}e^{-\tau}R_K\left(\frac{sp}{\tau},-\mu-\frac{1}{2},q,\lambda\right)dt=\sqrt{\frac{\pi}{2p}}s^{-\alpha}\Gamma_\mu(\alpha,sx;q;\lambda;sp).\end{aligned}$$

The proof of (2.10) is omitted since it is quite similar as that of (2.9). ☐

Proposition 2.5 (Parametric differentiation)**.**

$$\frac{\partial}{\partial p}\left(\Gamma_\mu(\alpha,x;q;\lambda;p)\right)=-\frac{1}{p}\left[\mu\Gamma_\mu(\alpha,x;q;\lambda;p)+p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)\right].$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial p}\left(\Gamma_\mu(\alpha,x;q;\lambda;p)\right)=&\frac{1}{2p}\sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)dt\\&+\sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}\frac{\partial}{\partial p}\left(R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right)dt.\end{aligned} \tag{2.11}$$

We have

$$\begin{aligned}\frac{\partial}{\partial p}\left(R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right)=&-\frac{\mu+\frac{1}{2}}{p}\frac{(p/2t)^{-\mu-\frac{1}{2}}}{2}\int_0^\infty \tau^{\mu-\frac{1}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau\\&-\frac{1}{t}\frac{(p/2t)^{-\mu+\frac{1}{2}}}{2}\int_0^\infty \tau^{\mu-\frac{3}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau\\=&-\frac{\mu+\frac{1}{2}}{p}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\\&-\frac{1}{t}R_K\left(\frac{p}{t},-\mu+\frac{1}{2},q,\lambda\right),\end{aligned} \tag{2.12}$$

Finally, by substituting (2.12) into (2.11) we get the desired result. ☐

2.2. **The generalized extended beta function.**

Definition 2.2. The generalized extended beta function is given by
(2.13)

$$B_\mu(x,y;q;\lambda;p;m)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,$$

where $x,y\in\mathbb{C}$, $0<q\leq 1$, $-1\leq\lambda\leq 1$, $m>0$ and $\operatorname{Re}(p)>0$.

Remark 2.2. Taking $\lambda=0$ and $q=1$, (2.13) is reduced to the extended Euler's beta function (1.9) defined by Agarwal et al. [1].

Proposition 2.6 (Functional relations)**.** 1. *The following formula holds*

$$B_\mu(x,y;q;\lambda;p;m)=B_\mu(x+1,y;q;\lambda;p;m)+B_\mu(x,y+1;q;\lambda;p;m). \tag{2.14}$$

2. *Let $n\in\mathbb{N}$. Then the following summation formula holds*

$$B_\mu(x,y;q;\lambda;p;m)=\sum_{k=0}^{n}B_\mu(x+k,y+n-k;q;\lambda;p;m). \tag{2.15}$$

Proof. 1. The right-hand side of (2.14) yields to

$$\sqrt{\frac{2p}{\pi}}\int_0^1\left\{t^{x-\frac{1}{2}}(1-t)^{y-\frac{3}{2}}+t^{x-\frac{3}{2}}(1-t)^{y-\frac{1}{2}}\right\}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,$$

which, after simplification, implies

$$\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,$$

which is equal to the left-hand side of (2.14).

2. The case $n=0$ of (2.15) holds easily. The case $n=1$ of (2.15) is just (2.14). For the other cases we can easily proceed by induction on n. □

Proposition 2.7. *The following formula holds*

$$B_\mu(x,1-y;q;\lambda;p;m)=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}B_\mu(x+n,1;q;\lambda;p;m). \tag{2.16}$$

Proof. We have
(2.17)

$$B_\mu(x,1-y;q;\lambda;p;m)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{-y-\frac{1}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt.$$

By substituting the formula

$$(1-t)^{-y}=\sum_{n=0}^{\infty}(y)_n\frac{t^n}{n!},\quad |t|<1,\ y\in\mathbb{C},$$

in the right-hand of (2.17) and after interchanging the order of integral and summation, we get (2.16). □

Proposition 2.8. *The following formula holds*

$$B_\mu(x,y;q;\lambda;p;m)=\sum_{n=0}^{\infty}B_\mu(x+n,y+1;q;\lambda;p;m).$$

Proof. By substituting again the formula

$$(1-t)^{y-1}=(1-t)^y\sum_{n=0}^{\infty}t^n,\quad |t|<1,$$

in the right-hand of (2.13) and similarly as in the proof of Proposition 2.7 we get the desired result. ☐

Lemma 2.1. *Let $\mathcal{M}$ be the Mellin transform operator. Then*

$$\mathcal{M}\{R_K(z,-\alpha,q,\lambda),z\to s\}=2^{s-2}\Gamma\left(\frac{s-\alpha}{2}\right)\Gamma\left(\frac{s+\alpha}{2}\right)\Phi\left(\lambda,\frac{s+\alpha}{2},q\right),$$

where $0<q\le 1$*, or* $-1\le\lambda<1$*,* $\mathrm{Re}(s)>|\mathrm{Re}(\alpha)|$ *or* $\lambda=1$*,* $\mathrm{Re}(s)>\max\{\mathrm{Re}(\alpha),2-\mathrm{Re}(\alpha)\}$ *and* $\Phi\left(\lambda,\frac{s+\alpha}{2},q\right)$ *stands for the Lerch function* (*see* [12,15]).

Proof.

$$\begin{aligned}\mathcal{M}\{R_K(z,-\alpha,q,\lambda),z\to s\}&=\int_0^\infty z^{s-1}R_K(z,-\alpha,q,\lambda)dz\\&=2^{\alpha-1}\int_0^\infty z^{s-\alpha-1}\left(\int_0^\infty t^{\alpha-1}\frac{e^{-qt-z^2/4t}}{1-\lambda e^{-t}}dt\right)dz\\&=2^{\alpha-1}\int_0^\infty t^{\alpha-1}\frac{e^{-qt}}{1-\lambda e^{-t}}\left(\int_0^\infty z^{s-\alpha-1}e^{-z^2/4t}dz\right)dt\\&=2^{s-2}\Gamma\left(\frac{s-\alpha}{2}\right)\int_0^\infty t^{\frac{s+\alpha}{2}-1}\frac{e^{-qt}}{1-\lambda e^{-t}}dt\\&=2^{s-2}\Gamma\left(\frac{s-\alpha}{2}\right)\Gamma\left(\frac{s+\alpha}{2}\right)\Phi\left(\lambda,\frac{s+\alpha}{2},q\right).\end{aligned}$$ ☐

Proposition 2.9 (Mellin transform). *The following expression holds true*

$$\begin{aligned}\mathcal{M}\{B_\mu(x,y;q;\lambda;p;m),p\to s\}=&\frac{2^{s-1}}{\sqrt{\pi}}B\left(x+ms+\frac{m-1}{2},y+ms+\frac{m-1}{2}\right)\\&\times\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right),\end{aligned}$$

where $x,y\in\mathbb{C}$*,* $m>0$ *and* $0<q\le 1$ *or* $1\le\lambda<1$*,*

$$\mathrm{Re}(s)>\max\left\{\mathrm{Re}(\mu),-1-\mathrm{Re}(\mu),-\frac{1}{2}+\frac{1}{2m}-\frac{\mathrm{Re}(x)}{m},-\frac{1}{2}+\frac{1}{2m}-\frac{\mathrm{Re}(y)}{m}\right\},$$

or $\lambda=1$*,*

$$\mathrm{Re}(s)>\max\left\{\mathrm{Re}(\mu),1-\mathrm{Re}(\mu),-\frac{1}{2}+\frac{1}{2m}-\frac{\mathrm{Re}(x)}{m},-\frac{1}{2}+\frac{1}{2m}-\frac{\mathrm{Re}(y)}{m}\right\}.$$

Proof.

$$
\begin{aligned}
&\mathcal{M}\{B_\mu(x,y;q;\lambda;p;m),p\to s\}\\
=&\int_0^\infty p^{s-1}B_\mu(x,y;q;\lambda;p;m)dp\\
=&\int_0^\infty p^{s-1}\sqrt{\frac{2p}{\pi}}\left(\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt\right)dp\\
=&\sqrt{\frac{2}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}\left(\int_0^\infty p^{s+\frac{1}{2}-1}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dp\right)dt\\
=&\sqrt{\frac{2}{\pi}}\int_0^1 t^{x+m(s+\frac{1}{2})-\frac{3}{2}}(1-t)^{y+m(s+\frac{1}{2})-\frac{3}{2}}dt\int_0^\infty u^{s+\frac{1}{2}-1}R_K\left(u,-\mu-\frac{1}{2},q,\lambda\right)du\\
=&\sqrt{\frac{2}{\pi}}B\left(x+ms+\frac{m-1}{2},y+ms+\frac{m-1}{2}\right)\int_0^\infty u^{s+\frac{1}{2}-1}R_K\left(u,-\mu-\frac{1}{2},q,\lambda\right)du.
\end{aligned}
$$

Finally, by using Lemma 2.1 we get the desired result. ☐

3. Extended Gauss Hypergeometric and Confluent Hypergeometric Functions

We use the generalized extended beta function (2.13) to extend hypergeometric and confluent hypergeometric functions, respectively, as follows.

Definition 3.1. The extended Gauss hypergeometric function $F_\mu(a,b;c;z;q;\lambda;p;m)$ and the confluent hypergeometric function $\Phi_\mu(b;c;z;q;\lambda;p;m)$ are respectively defined by

$$F_\mu(a,b;c;z;q;\lambda;p;m)=\sum_{n=0}^{\infty}(a)_n\frac{B_\mu(b+n,c-b;q;\lambda;p;m)}{B(b,c-b)}\cdot\frac{z^n}{n!},\tag{3.1}$$

$|z|<1$, $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, $0<q\le 1$, $-1\le\lambda\le 1$, $m>0$, $\mathrm{Re}(p)>0$,

$$\Phi_\mu(b;c;z;q;\lambda;p;m)=\sum_{n=0}^{\infty}\frac{B_\mu(b+n,c-b;q;\lambda;p;m)}{B(b,c-b)}\cdot\frac{z^n}{n!},$$

$z\in\mathbb{C}$, $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, $-1\le\lambda\le 1$, $m>0$, $\mathrm{Re}(p)>0$.

Remark 3.1. Taking $\lambda=0$ and $q=1$, (3.1) reduces to the extended Gauss hypergeometric function defined by Agarwal et al. [1, Definition 2.8].

Proposition 3.1 (Integral representation)**.** 1. *The following integral representation for the extended Gauss hypergeometric function $F_\mu(a,b;c;z;q;\lambda;p;m)$ is valid*

$$
\begin{aligned}
F_\mu(a,b;c;z;q;\lambda;p;m)=&\sqrt{\frac{2p}{\pi}}\frac{1}{B(b,c-b)}\int_0^1 t^{b-\frac{3}{2}}(1-t)^{c-b-\frac{3}{2}}(1-zt)^{-a}\\
&\times R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,
\end{aligned}\tag{3.2}
$$

where $\arg(1-z) < \pi$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

2. *The following integral representation for the extended confluent hypergeometric function* $\Phi_\mu(b;c;z;q;\lambda;p;m)$ *is valid*

$$\begin{aligned}\Phi_\mu(b;c;z;q;\lambda;p;m) =&\sqrt{\frac{2p}{\pi}}\frac{1}{B(b,c-b)}\int_0^1 t^{b-\frac{3}{2}}(1-t)^{c-b-\frac{3}{2}}e^{zt}\\ &\times R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,\end{aligned} \tag{3.3}$$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

Proof. 1. By using (2.13) and the generalized binomial expansion

$$(1-zt)^{-a} = \sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!}, \quad |zt| < 1,$$

we get the required result.

2. Similarly as in the proof of 1. □

Proposition 3.2 (Differentiation formula)**.** (a) *For* $n \in \mathbb{N}$

$$\frac{d^n}{dz^n}\{F_\mu(a,b;c;z;q;\lambda;p;m)\} = \frac{(a)_n(b)_n}{(c)_n}F_\mu(a+n,b+n;c+n;z;q;\lambda;p;m), \tag{3.4}$$

where $|z| < 1$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

(b) *For* $n \in \mathbb{N}$

$$\frac{d^n}{dz^n}\{\Phi_\mu(b;c;z;q;\lambda;p;m)\} = \frac{(b)_n}{(c)_n}\Phi_\mu(b+n;c+n;z;q;\lambda;p;m),$$

where $z \in \mathbb{C}$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

Proof. (a) For $n = 1$, we have

$$\begin{aligned}\frac{d}{dz}\{F_\mu(a,b;c;z;q;\lambda;p;m)\} &= \sum_{n=1}^{\infty}(a)_n\frac{B_\mu(b+n,c-b;q;\lambda;p;m)}{B(b,c-b)}\cdot\frac{z^{n-1}}{(n-1)!}\\ &= \sum_{n=0}^{\infty}(a)_{n+1}\frac{B_\mu(b+n+1,c-b;q;\lambda;p;m)}{B(b,c-b)}\cdot\frac{z^n}{n!}.\end{aligned} \tag{3.5}$$

Using identities $B(b,c-b) = \frac{c}{b}B(b+1,c-b)$ and $(a)_{n+1} = a(a+1)_n$ in (3.5), we get

$$\begin{aligned}\frac{d}{dz}\{F_\mu(a,b;c;z;q;\lambda;p;m)\} &= \frac{ab}{c}\sum_{n=0}^{\infty}(a+1)_n\frac{B_\mu(b+n+1,c-b;q;\lambda;p;m)}{B(b+1,c-b)}\cdot\frac{z^n}{n!}\\ &= \frac{ab}{c}F_\mu(a+1,b+1;c+1;z;q;\lambda;p;m),\end{aligned}$$

and hence

$$\frac{d}{dz}\{F_\mu(a,b;c;z;q;\lambda;p;m)\} = \frac{ab}{c}F_\mu(a+1,b+1;c+1;z;q;\lambda;p;m). \tag{3.6}$$

Then, by using (3.6) repeatedly, we get (3.4).

The proof of part (b) is similar as that of part (a). □

Proposition 3.3 (Transformation formulas)**.**

1. *For* $\arg(1-z) < \pi$ *we have*

$$F_\mu(a,b;c;z;q;\lambda;p;m) = (1-z)^{-a}F_\mu\left(a, c-b; c; \frac{z}{z-1}; q;\lambda;p;m\right),$$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

2. $\Phi_\mu(b;c;z;q;\lambda;p;m) = e^z\Phi_\mu(c-b;c;-z;q;\lambda;p;m)$, *where* $z \in \mathbb{C}$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

Proof. Replacing t by $1-t$ in the integral representations (3.2) and (3.3). □

4. Extended Appell and Lauricella Hypergeometric Functions

Definition 4.1. Extended Appell hypergeometric functions $F_{1,\mu}$, $F_{2,\mu}$ and the Lauricella hypergeometric function $F^3_{D,\mu}$ are, respectively, defined by

$$F_{1,\mu}(a,b,c;d;x,y;q;\lambda;p;m) = \sum_{n,k=0}^{\infty}(b)_n(c)_k\frac{B_\mu(a+n+k,d-a;q;\lambda;p;m)}{B(a,d-a)}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!}, \tag{4.1}$$

where $|x| < 1$, $|y| < 1$, $\mathrm{Re}(d) > \mathrm{Re}(a) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$,

$$\begin{aligned} F_{2,\mu}(a,b,c;d,e;x,y;q;\lambda;p;m) = \sum_{n,k=0}^{\infty}(a)_{n+k}&\frac{B_\mu(b+n,d-b;q;\lambda;p;m)}{B(b,d-b)} \\ &\times\frac{B_\mu(c+k,e-c;q;\lambda;p;m)}{B(c,e-c)}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!}, \end{aligned} \tag{4.2}$$

where $|x| + |y| < 1$, $\mathrm{Re}(d) > \mathrm{Re}(b) > 0$, $\mathrm{Re}(e) > \mathrm{Re}(c) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$,

$$\begin{aligned} &F^3_{D,\mu}(a,b,c,d;e;x,y,z;q;\lambda;p;m) \\ &\quad= \sum_{n,k,r=0}^{\infty}(b)_n(c)_k(d)_r\frac{B_\mu(a+n+k+r,e-a;q;\lambda;p;m)}{B(a,e-a)}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!}\cdot\frac{z^r}{r!}, \end{aligned} \tag{4.3}$$

where $|x| < 1$, $|y| < 1$, $|z| < 1$, $\mathrm{Re}(e) > \mathrm{Re}(a) > 0$, $0 < q \leq 1$, $-1 \leq \lambda \leq 1$, $m > 0$, $\mathrm{Re}(p) > 0$.

Remark 4.1. Taking $\lambda = 0$ and $q = 1$, (4.1), (4.2) and (4.3) are reduced to extended Appell hypergeometric functions $F_{1,\mu}$, $F_{2,\mu}$ and the Lauricella hypergeometric function $F^3_{D,\mu}$, defined by Agarwal et al. [1, Definitions 2.9, 2.10, 2.11].

Proposition 4.1 (Integral representation)**.** *The following integral representations for the extended Appell hypergeometric functions $F_{1,\mu}$, $F_{2,\mu}$ and the Lauricella hypergeometric function $F^3_{D,\mu}$ are, respectively, valid*

$$\begin{aligned}
&F_{1,\mu}(a,b,c;d;x,y;q;\lambda;p;m)\\
=&\sqrt{\frac{2p}{\pi}}\frac{1}{B(a,d-a)}\int_0^1 t^{a-\frac{3}{2}}(1-t)^{d-a-\frac{3}{2}}(1-xt)^{-b}\times(1-yt)^{-c}\\
&\times R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt,\\
&F_{2,\mu}(a,b,c;d;x,y;q;\lambda;p;m)\\
=&\frac{2p}{\pi}\cdot\frac{1}{B(b,d-b)B(c,e-c)}\\
&\times\int_0^1\int_0^1 t^{b-\frac{3}{2}}(1-t)^{d-b-\frac{3}{2}}\times w^{b-\frac{3}{2}}(1-w)^{e-c-\frac{3}{2}}(1-xt-yw)^{-a}\\
&\times R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)R_K\left(\frac{p}{w^m(1-w)^m},-\mu-\frac{1}{2},q,\lambda\right)dtdw,\\
&F^3_{D,\mu}(a,b,c,d;e;x,y,z;q;\lambda;p;m)\\
=&\sqrt{\frac{2p}{\pi}}\frac{1}{B(a,e-a)}\int_0^1 t^{a-\frac{3}{2}}(1-t)^{e-a-\frac{3}{2}}(1-xt)^{-b}\times(1-yt)^{-c}(1-zt)^{-d}\\
&\times R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt.
\end{aligned}$$

Proof. The proofs are very similar to those of Theorems 2.13, 2.15 and 2.16 in [1]. ☐

5. The Generalized Extended Riemann-Liouville Fractional Derivative Operator

The classical Riemann-Liouville fractional derivative operator is defined by

$$D_z^\delta f(z):=\frac{1}{\Gamma(-\delta)}\int_0^z(z-t)^{-\delta-1}f(t)dt, \tag{5.1}$$

where $\mathrm{Re}(\delta)<0$. It coincides with the fractional integral of order $-\delta$. In the case $n-1<\mathrm{Re}(\delta)<n$, $n\in\mathbb{N}$, we write

$$D_z^\delta f(z):=\frac{d^n}{dz^n}D_z^{\delta-n}f(z)=\frac{d^n}{dz^n}\left\{\frac{1}{\Gamma(n-\delta)}\int_0^z(z-t)^{n-\delta-1}f(t)dt\right\}.$$

Definition 5.1. The generalized extended Riemann-Liouville fractional derivative is defined as follows

$$D_z^{\delta,\mu;p;q;\lambda;m}f(z):=\frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z(z-t)^{-\delta-1}f(t)R_K\left(\frac{pz^{2m}}{t^m(z-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt, \tag{5.2}$$

where $\mathrm{Re}(\delta) < 0$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(m) > 0$, $\mathrm{Re}(\mu) \geq 0$ and $0 < q \leq 1$, $-1 \leq \lambda \leq 1$.

For $n - 1 < \mathrm{Re}(\delta) < n$, $n \in \mathbb{N}$, we have

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m} f(z) :=& \frac{d^n}{dz^n} D_z^{\delta-n,\mu;p;q;\lambda;m} f(z) = \frac{d^n}{dz^n}\left\{ \frac{1}{\Gamma(n-\delta)} \sqrt{\frac{2p}{\pi}} \int_0^z (z-t)^{n-\delta-1} f(t) \right. \\ & \left. \times R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu - \frac{1}{2}, q, \lambda \right) dt \right\}. \end{aligned}$$

Remark 5.1. 1. Taking $\lambda = 0$ and $q = 1$, the generalized extended Riemann-Liouville fractional derivative operator (5.2) is reduced to the extended Riemann-Liouville fractional derivative operator given by Agarwal et al. [1]

$$D_z^{\delta,\mu;p;m} f(z) := \frac{1}{\Gamma(-\delta)} \sqrt{\frac{2p}{\pi}} \int_0^z (z-t)^{-\delta-1} f(t) K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m} \right) dt,$$

where $\mathrm{Re}(\delta) < 0$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(m) > 0$, $\mathrm{Re}(\mu) > 0$.

2. If $\lambda = 0$, $q = 1$, $m = 0$, $\mu = 0$ and $p \to 0$, then the generalized extended Riemann-Liouville fractional derivative operator (5.2) reduces to the classical Riemann-Liouville fractional derivative operator (5.1).

In order to calculate generalized extended fractional derivatives for some functions, we give two results concerning the generalized extended Riemann-Liouville fractional derivative operator of some elementary functions which will be useful in the sequel.

Lemma 5.1. *Let* $\mathrm{Re}(\delta) < 0$*. Then we have*

$$D_z^{\delta,\mu;p;q;\lambda;m}\{z^\beta\} = \frac{z^{\beta-\delta}}{\Gamma(-\delta)} B_\mu\left(\beta + \frac{3}{2}, -\delta + \frac{1}{2}; p; q; \lambda; m \right).$$

Proof. Using Definition 5.1 and a local setting $t = zu$, we obtain

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{z^\beta\} =& \frac{1}{\Gamma(-\delta)} \sqrt{\frac{2p}{\pi}} \int_0^z (z-t)^{-\delta-1} t^\beta R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu - \frac{1}{2}, q, \lambda \right) dt \\ =& \frac{z^{\beta-\delta}}{\Gamma(-\delta)} \sqrt{\frac{2p}{\pi}} \int_0^1 (1-u)^{(-\delta+\frac{1}{2})-\frac{3}{2}} u^{(\beta+\frac{3}{2})-\frac{3}{2}} \\ & \times R_K\left(\frac{p}{u^m(1-u)^m}, -\mu - \frac{1}{2}, q, \lambda \right) du \\ =& \frac{z^{\beta-\delta}}{\Gamma(-\delta)} B_\mu(\beta + \frac{3}{2}, -\delta + \frac{1}{2}; p; q; \lambda; m). \end{aligned}$$ □

More generally, we give the generalized extended Riemann-Liouville fractional derivative of an analytic function $f(z)$ at the origin.

Lemma 5.2. *Let* $\mathrm{Re}(\delta) < 0$*. If a function* $f(z)$ *is analytic at the origin, then*

$$D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\}.$$

Proof. Since f is analytic at the origin, its Maclaurin expansion is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (for $|z| < \rho$ with $\rho \in \mathbb{R}^+$ is the convergence radius). By substituting entire power series in Definition 5.1, we obtain

$$D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} = \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1} \times R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}; q; \lambda\right)\sum_{n=0}^{\infty} a_n t^n dt.$$

By virtue of the uniform continuity on the convergence disk, we can do integration term by term in the equation above. Thus

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{f(z)\} =& \sum_{n=0}^{\infty} a_n \left\{\frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1}\right. \\ & \left.\times R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}; q; \lambda\right) t^n dt\right\} \\ =& \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\}. \end{aligned}$$

□

Corollary 5.1.

$$D_z^{\delta,\mu;p;q;\lambda;m}\{(1-z)^{-\alpha}\} = \frac{z^{-\delta}}{\Gamma(-\delta)} B\left(\frac{3}{2}, -\delta+\frac{1}{2}\right) F_\mu\left(\alpha, \frac{3}{2}, -\delta+2; z; q; \lambda; p; m\right),$$

where $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\delta) < 0$.

Proof. Using binomial theorem for $(1-z)^{-\alpha}$ and Lemma 5.1, we obtain:

$$\begin{aligned} D_z^{\delta,\mu;p;q;\lambda;m}\{(1-z)^{-\alpha}\} &= D_z^{\delta,\mu;p;q;\lambda;m}\left\{\sum_{n=0}^{\infty}(\alpha)_n \frac{z^n}{n!}\right\} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!} D_z^{\delta,\mu;p;q;\lambda;m}\{z^n\} \\ &= \frac{z^{-\delta}}{\Gamma(-\delta)}\sum_{n=0}^{\infty}(\alpha)_n B_\mu\left(n+\frac{3}{2}, -\delta+\frac{1}{2}; p, q; \lambda; m\right)\frac{z^n}{n!}. \end{aligned}$$

Hence, the result. □

Combining previous lemmas, we obtain the generalized extended derivative of the product of analytic function with a power function.

Theorem 5.1. *Let* $\mathrm{Re}(\delta) < 0$. *Suppose that a function* $f(z)$ *is analytic at the origin with its Maclaurin expansion given by* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < \rho$, *for some* $\rho \in \mathbb{R}^+$. *Then we have*

$$D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu;p;q;\lambda;m}\{z^{\beta+n-1}\}$$

$$= \frac{z^{\beta-\delta-1}}{\Gamma(-\delta)} \sum_{n=0}^{\infty} a_n B_\mu \left(\beta + n + \frac{1}{2}, -\delta + \frac{1}{2}; p; q; \lambda; m \right) z^n.$$

A subsequent result can be given as follows.

Theorem 5.2. *For* $\mathrm{Re}(\delta) > \mathrm{Re}(\beta) > -\frac{1}{2}$, *we have*

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\}$$
$$= \frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B\left(\beta + \frac{1}{2}, \delta - \beta + \frac{1}{2}\right) F_\mu\left(\alpha, \beta + \frac{1}{2}; \delta + 1; z; q; \lambda; p; m\right),$$

where $|z| < 1$, $\alpha \in \mathbb{C}$.

Proof. The result is easily established by taking $f(z) = (1-z)^{-\alpha}$, so we have

$$\begin{aligned} D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} &= D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}\sum_{k=0}^{\infty}(\alpha)_k \frac{z^k}{k!}\right\} \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta+k-1}\} \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{B_\mu(\beta + k + \frac{1}{2}, \delta - \beta + \frac{1}{2}; p; q; \lambda; m)}{\Gamma(\delta-\beta)} z^{\delta+k-1}. \end{aligned}$$

By the expression (3.1), we get

$$\begin{aligned} D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} =& \frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B\left(\beta + \frac{1}{2}, \delta - \beta + \frac{1}{2}\right) \\ & \times F_\mu\left(\alpha, \beta + \frac{1}{2}; \delta + 1; z; q; \lambda; p; m\right). \end{aligned}$$

□

Theorem 5.3. *For* $\mathrm{Re}(\delta) > \mathrm{Re}(\beta) > -\frac{1}{2}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\gamma) > 0$, $|az| < 1$ *and* $|bz| < 1$. *Then, the following generating relation holds true*

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\}$$
$$= \frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B\left(\beta + \frac{1}{2}, \delta - \beta + \frac{1}{2}\right) F_{1,\mu}\left(\beta + \frac{1}{2}, \alpha, \gamma; \delta + 1; az, bz; q; \lambda; p; m\right).$$

Proof. By applying the binomial Theorem to $(1-az)^{-\alpha}$ and $(1-bz)^{-\gamma}$ and making use of Lemmas 5.1 and 5.2, we obtain

$$\begin{aligned} & D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\} \\ =& D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(\alpha)_k(\gamma)_r \frac{(az)^k}{k!} \cdot \frac{(bz)^r}{r!}\right\} \\ =& \sum_{k,r=0}^{\infty}(\alpha)_k(\gamma)_r D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta+k+r-1}\}\frac{a^k}{k!} \cdot \frac{b^r}{r!} \end{aligned}$$

$$=z^{\delta-1}\sum_{k,r=0}^{\infty}(\alpha)_k(\gamma)_r\frac{B_\mu(\beta+k+r+\frac{1}{2},\delta-\beta+\frac{1}{2};p;q;\lambda;m)}{\Gamma(\delta-\beta)}\cdot\frac{(az)^k}{k!}\cdot\frac{(bz)^r}{r!}.$$

By using (4.1), we can get

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\}$$
$$=\frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B\left(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2}\right)F_{1,\mu}\left(\beta+\frac{1}{2},\alpha,\gamma;\delta+1;az,bz;q;\lambda;p;m\right). \qquad \square$$

Theorem 5.4. *For* $\mathrm{Re}(\delta)>\mathrm{Re}(\beta)>-\frac{1}{2}$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\gamma)>0$, $\mathrm{Re}(\tau)>0$, $|az|<1$, $|bz|<1$ *and* $|cz|<1$, *we have*

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}(1-cz)^{-\tau}\}$$
$$=\frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B\left(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2}\right)F_{D,\mu}^{3}\left(\beta+\frac{1}{2},\alpha,\gamma,\tau;\delta+1;az,bz;q;\lambda;p;m\right).$$

Proof. The proof is similar to that of Theorem 5.3, it is sufficient to use the binomial Theorem for $(1-az)^{-\alpha}$, $(1-bz)^{-\gamma}$, $(1-cz)^{-\tau}$, then applying Lemmas 5.1 and 5.2. $\square$

Theorem 5.5. *For* $\mathrm{Re}(\delta)>\mathrm{Re}(\beta)>-\frac{1}{2}$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\tau)>\mathrm{Re}(\gamma)>0$, $\left|\frac{x}{1-z}\right|<1$ *and* $|x|+|z|<1$, *we have*

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}F_\mu\left(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m\right)\right\}$$
$$=z^{\delta-1}\frac{B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})}{\Gamma(\delta-\beta)}F_{2,\mu}\left(\alpha,\gamma,\beta+\frac{1}{2},\tau;\delta+1;x,z;q;\lambda;p;m\right).$$

Proof. By the binomial formula and according to Definition 3.1, we expand $z^{\beta-1}(1-z)^{-\alpha}F_\mu(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m)$ to get

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}F_\mu\left(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m\right)\right\}$$
$$=D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\cdot\frac{B_\mu(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}\left(\frac{x}{1-z}\right)^n\right\}$$
$$=\sum_{n=0}^{\infty}(\alpha)_n\frac{B_\mu(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha-n}\}\frac{x^n}{n!}.$$

In order to exhibit $F_{2,\mu}$, we apply Theorem 5.2 for $D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha-n}\}$ and substitute the extended hypergeometric function F_μ by its series representation, we obtain

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}F_\mu\left(\alpha,\gamma;\tau;\frac{x}{1-z};q;\lambda;p;m\right)\right\}$$
$$=\frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B\left(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2}\right)\sum_{n,k=0}^{\infty}(\alpha)_{n+k}\frac{B_\mu(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}$$

$$\times \frac{B_\mu(\beta+k+\frac{1}{2},\delta-\beta+\frac{1}{2};q;\lambda;p;m)}{B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})}\cdot\frac{x^n z^k}{n!z!}$$
$$=\frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B\left(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2}\right)F_{2,\mu}\left(\alpha,\gamma,\beta+\frac{1}{2},\tau;\delta+1;x,z;q;\lambda;p;m\right).$$

This completes the proof. □

Proposition 5.1 (Mellin transform)**.** *The following expression holds true*

$$\begin{aligned}\mathcal{M}\{D_z^{\delta,\mu,p;q;\lambda;m}z^\beta,p\to s\}=&2^{s-1}z^{\beta-\delta}\frac{1}{\sqrt{\pi}}B\left(\beta+m\left(s+\frac{1}{2}\right)+1,-\delta+m\left(s+\frac{1}{2}\right)\right)\\&\times\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right),\end{aligned}$$

for $\mathrm{Re}(\mu)\geq 0$*,* $m>0$ *and* $\mathrm{Re}(s)>\max\left\{\mathrm{Re}(\mu),-\frac{1}{2}-\frac{1}{m}-\frac{\mathrm{Re}(\beta)}{m},\frac{\mathrm{Re}(\delta)}{m}-\frac{1}{2}\right\}$.

Proof. We can prove this result by applying Mellin transform and using Lemma 5.1.

$$\begin{aligned}\mathcal{M}\{D_z^{\delta,\mu,p;q;\lambda;m}z^\beta,p\to s\}&=\frac{1}{\Gamma(-\delta)}\int_0^\infty p^{s-1}z^{\beta-\delta}B_\mu\left(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m\right)dp\\&=\frac{z^{\beta-\delta}}{\Gamma(-\delta)}\int_0^\infty p^{s-1}B_\mu\left(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m\right)dp.\end{aligned}$$

As the last integral is the Mellin transform of $B_\mu(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m)$, the result immediately follows via Proposition 2.9. □

Proposition 5.2. *The following expression holds true*

$$\begin{aligned}&\mathcal{M}\{D_z^{\delta,\mu,p;q;\lambda;m}(1-z)^{-\beta},p\to s\}\\&=2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}B\left(m\left(s+\frac{1}{2}\right)+1,-\delta+m\left(s+\frac{1}{2}\right)\right)\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\\&\times\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right){}_2F_1\left(\beta,m\left(s+\frac{1}{2}\right)+1;-\delta+m(2s+1)+1;z\right),\end{aligned}$$

where $\mathrm{Re}(\mu)\geq 0$*,* $\mathrm{Re}(\delta)<0$*,* $m>0$*,* $|z|<1$*,* $\mathrm{Re}(s)>\max\left\{\mathrm{Re}(\mu),-\frac{1}{2}+\frac{1}{m},\frac{\delta}{m}-\frac{1}{2}\right\}$ *and* ${}_2F_1$ *is the well-known Gauss hypergeometric function.*

Proof. The result can be proved using the Binomial theorem for $(1-z)^{-\alpha}$ and the Mellin transform of the general term. Indeed,

$$\begin{aligned}&\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}\{(1-z)^{-\alpha}\},p\to s\}\\&=\mathcal{M}\left\{D_z^{\delta,\mu;p;q;\lambda;m}\left\{\sum_{n=0}^\infty(\alpha)_n\frac{z^n}{n!}\right\},p\to s\right\}\\&=\sum_{n=0}^\infty\frac{(\alpha)_n}{n!}\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}z^n,p\to s\}\end{aligned}$$

$$
\begin{aligned}
&=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}2^{s-1}z^{n-\delta}\frac{1}{\sqrt{\pi}}B\left(n+m\left(s+\frac{1}{2}\right)+1,-\delta+m\left(s+\frac{1}{2}\right)\right)\\
&\quad\times\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right).\\
&=2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right)\\
&\quad\times\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}B\left(n+m\left(s+\frac{1}{2}\right)+1,-\delta+m\left(s+\frac{1}{2}\right)\right)z^n\\
&=2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}B\left(m\left(s+\frac{1}{2}\right)+1,-\delta+m\left(s+\frac{1}{2}\right)\right)\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)\\
&\quad\times\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right){}_2F_1\left(\beta,m\left(s+\frac{1}{2}\right)+1;-\delta+m(2s+1)+1;z\right)\check{\mathrm{R}}.\qquad\square
\end{aligned}
$$

6. Generating Function Involving the Extended Generalized Gauss Hypergeometric Function

In this section, we establish some generating functions for the generalized Gauss hypergeometric functions.

Theorem 6.1. *Let* $\mathrm{Re}(\beta)>0$ *and* $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>-\frac{1}{2}$*. Then we have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;p;\lambda;m\right)t^n\\
&\quad=(1-t)^{-\beta}F_\mu\left(\beta,\alpha+\frac{1}{2};\gamma+1;\frac{z}{1-t};q;p;\lambda;m\right),
\end{aligned}
\tag{6.1}
$$

where $|z|<\min\{1,|1-t|\}$.

Proof. By considering the following elementary identity

$$(1-z)^{-\beta}\left(1-\frac{t}{1-z}\right)^{-\beta}=(1-t)^{-\beta}\left(1-\frac{z}{1-t}\right)^{-\beta}$$

and expanding its left-hand side to give

$$(1-z)^{-\beta}\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}\left(\frac{t}{1-z}\right)^n=(1-t)^{-\beta}\left(1-\frac{z}{1-t}\right)^{-\beta},\quad\text{for } |t|<|1-z|. \tag{6.2}$$

Multiplying both sides of (6.2) by $z^{\alpha-1}$ and applying the extended Riemann-Liouville fractional derivative operator $D^{\alpha-\gamma;\mu;q;p;\lambda;m}$, we find

$$D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{\sum_{n=0}^{\infty}\frac{(\beta)_n t^n}{n!}z^{\alpha-1}(1-z)^{-\beta-n}\right\}$$

$$=D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{(1-t)^{-\beta}z^{\alpha-1}\left(1-\frac{z}{1-t}\right)^{-\beta}\right\}.$$

Uniform convergence of the involved series allows us to permute the summation and fractional derivative operator to get

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{z^{\alpha-1}(1-z)^{-\beta-n}\}t^n$$

$$=(1-t)^{-\beta}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{z^{\alpha-1}\left(1-\frac{z}{1-t}\right)^{-\beta}\right\}.$$

The result easily follows using Theorem 5.2. □

Theorem 6.2. *Let* $\operatorname{Re}(\beta)>0$, $\operatorname{Re}(\tau)>0$ *and* $\operatorname{Re}(\gamma)>\operatorname{Re}(\alpha)>-\frac{1}{2}$. *Then we have*

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_{\mu}\left(\beta-n,\alpha+\frac{1}{2};\gamma+1;z;q;p;\lambda;m\right)t^n$$

$$=(1-t)^{-\beta}F_{1,\mu}\left(\alpha+\frac{1}{2},\tau,\beta;\gamma+1;z;\frac{-zt}{1-t};q;p;\lambda;m\right),$$

where $|z|<1$, $|t|<|1-z|$ *and* $|z||t|<|1-t|$.

Proof. By considering the following identity

$$[1-(1-z)t]^{-\beta}=(1-t)^{-\beta}\left(1+\frac{zt}{1-t}\right)^{-\beta},$$

and expanding its left-hand side as power series, we get

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}(1-z)^nt^n=(1-t)^{-\beta}\left(1-\frac{-zt}{1-t}\right)^{-\beta},\quad \text{for } |t|<|1-z|.$$

Multiplying both sides by $z^{\alpha-1}(1-z)^{-\tau}$ and applying the definition of the extended Riemann-Liouville fractional derivative operator $D_z^{\alpha-\gamma;\mu;q;p;\lambda;m}$ on both sides, we find

$$D_z^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}z^{\alpha-1}(1-z)^{-\tau}(1-z)^nt^n\right\}$$

$$=D_z^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{(1-t)^{-\beta}z^{\alpha-1}(1-z)^{-\tau}\left(1-\frac{-zt}{1-t}\right)^{-\beta}\right\}.$$

Interchanging the order of the summation and fractional derivative under the given conditions, we obtain

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{z^{\alpha-1}(1-z)^{-\tau+n}\}t^n$$

$$=(1-t)^{-\beta}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{z^{\alpha-1}(1-z)^{-\tau}\left(1-\frac{z}{1-t}\right)^{-\beta}\right\}.$$

Finally, the desired result follows by Theorems 5.2 and 5.3. □

Theorem 6.3. *Let* $\mathrm{Re}(\xi)>\mathrm{Re}(\upsilon)>-\frac{1}{2}$, $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>-\frac{1}{2}$ *and* $\mathrm{Re}(\beta)>0$. *Then we have*

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m\right)F_\mu\left(-n,\upsilon+\frac{1}{2};\xi+1;u;q;\lambda;p;m\right)t^n$$
$$=(1-t)^{-\beta}F_{2,\mu}\left(\beta,\alpha+\frac{1}{2},\upsilon+\frac{1}{2};\gamma+1,\xi+1;\frac{z}{1-t},\frac{-ut}{1-t};q;\lambda;p;m\right),$$

where $|z|<1$, $|\frac{1-u}{1-z}t|<1$ *and* $|\frac{z}{1-t}|+|\frac{ut}{1-t}|<1$.

Proof. By replacing t by $(1-u)t$ in (6.1) and multiplying both sides of the resulting identity by $u^{\upsilon-1}$, we get

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m\right)u^{\upsilon-1}(1-u)^nt^n \tag{6.3}$$
$$=u^{\upsilon-1}[1-(1-u)t]^{-\beta}F_\mu\left(\beta,\alpha+\frac{1}{2};\gamma+1;\frac{z}{1-(1-u)t};q;\lambda;p;m\right),$$

where $\mathrm{Re}(\beta)>0$ and $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>-\frac{1}{2}$.

Next, applying the fractional derivative $D^{\upsilon-\xi,\mu;q;\lambda;p;m}$ to both sides of (6.3) and changing the order of the summation and the fractional derivative under conditions $|z|<1$, $|\frac{1-u}{1-z}t|<1$ and $|\frac{z}{1-t}|+|\frac{ut}{1-t}|<1$, yields

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m\right)D^{\upsilon-\xi,\mu;q;\lambda;p;m}\{u^{\upsilon-1}(1-u)^n\}t^n$$
$$=D^{\upsilon-\xi,\mu;q;\lambda;p;m}\left\{u^{\upsilon-1}[1-(1-u)t]^{-\beta}F_\mu\left(\beta,\alpha+\frac{1}{2};\gamma+1;\frac{z}{1-(1-u)t};q;\lambda;p;m\right)\right\},$$

The last identity can be written as follows:

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m\right)D^{\upsilon-\xi,\mu;q;\lambda;p;m}\{u^{\upsilon-1}(1-u)^n\}t^n$$
$$=(1-t)^{-\beta}D^{\upsilon-\xi,\mu;q;\lambda;p;m}\left\{u^{\upsilon-1}\left[1-\frac{-ut}{1-t}\right]^{-\beta}\right.$$
$$\left.\times F_\mu\left(\beta+n,\alpha+\frac{1}{2};\gamma+1;\frac{\frac{z}{1-t}}{1-\frac{-ut}{1-t}};q;\lambda;p;m\right)\right\}.$$

Thus, by using Theorems 5.2 and 5.5 in the resulting identity, we obtain the desired result. □

7. Concluding remarks

In this paper, by using an extension of macdonald given by Boudjekha function we developed a generalized extension of some special functions namely: incomplete gamma, beta, hypergeomtric and confluent functions and we obtained a new extended Riemann-Liouville fractional derivative operator. We conclude first, for $\lambda = 0$ and $q = 1$, that extended incomplete gamma functions are respectively reduced to incomplete gamma functions (see [9]) and all the results established here will coincide with those obtained in [1]. Finally, if we letting $\lambda = m = \mu = 0$, $q = 1$ and $p \to 0$ then all the results established in this paper will reduce to the results associated with classical Riemann-Liouville fractional derivative operator (see [16]).

We intend to investigate aslo some other extensions based on Lerch and Hurwitz functions and Pochhammer Symbol, recently initiated in [25, 27].

Acknowledgements. The authors are very grateful to the anonymous referees for their valuable comments and suggestions which helped to improve this work.

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