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STATISTICAL CONVERGENCE OF A SEQUENCE OF NEUTROSOPHIC RANDOM VARIABLES

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Abstract. In this paper the notions of three kinds of statistical convergence of a sequence of neutrosophic random variables are defined, these types of statistical convergence are called neutrosophic statistical convergence in probability, neutrosophic statistical convergence in the mean of order r and neutrosophic statistical convergence in distribution. Also, some relations and properties among them are investigated.

1. Introduction

Smarandache [4] introduced an innovative philosophical framework termed neutrosophy, which explores the nature, origin, and interactions of neutralities within logical and epistemological systems. A central assertion in neutrosophy is that every concept or proposition carries not only a degree of truth commonly treated in many valued logics but also distinct and independent degrees of falsity and indeterminacy. This triadic characterization captures both subjective and objective aspects of uncertainty, encompassing imprecision, vagueness, and ambiguity. Neutrosophy forms the theoretical foundation for several extended mathematical constructs, including neutrosophic set theory, neutrosophic probability, neutrosophic statistics, and neutrosophic logic. In 2020, Bisher and Hatip [7] extended this conceptual apparatus by incorporating the notion of indeterminacy into classical probability theory. They introduced the idea of neutrosophic random variables and formulated preliminary definitions to support further development. Following this, Granados [8] advanced

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the theory by establishing new results, while Granados and Sanabria [9] investigated independence criteria for neutrosophic random variables. Additional developments include the works of Granados et al. [10,11], who formulated and analyzed both discrete and continuous probability distributions governed by neutrosophic randomness. In parallel, the concept of statistical convergence dates back to the work of A. Zygmund in the 1930s and was formally revisited by Steinhaus [5] and Fast [6] in 1951. Since then, the notion has undergone numerous generalizations and has found fertile ground in the context of neutrosophic mathematics. For example, Kisi et al. [15] introduced statistical convergence within neutrosophic normed spaces, including the notion of statistically Cauchy sequences and statistical completeness. Granados and Dhital [16] extended these ideas to double sequences, offering new criteria for convergence and completeness. Subsequent contributions have expanded the landscape: Gonul [17] explored convergence behaviors linked to difference and lacunary sequences; Kisi [18] focused on ideal convergence and its implications; Khan et al. [19] analyzed lacunary statistically Cauchy sequences and their structural properties in neutrosophic settings. In a related direction, Khan et al. [20] proposed generalizations using λ-statistical convergence, establishing inclusion relationships and completeness conditions. Further, Ali et al. [21] developed a framework for statistical convergence in neutrosophic metric spaces, and Kisi et al. [22] explored triple lacunary Δ-statistical convergence, utilizing lacunary density to elucidate structural connections. Al-Hamido [23] proposed a novel approach to neutrosophic topology, demonstrating that this construction is distinct from classical and previously known neutrosophic topologies, thereby introducing new forms of sets and spatial reasoning. Most recently, Granados and Choudhury [24] introduced quasi-statistical convergence for triple sequences in neutrosophic normed spaces, generalizing prior definitions and providing comprehensive results on equivalence with quasi-statistical Cauchy sequences. Inspired by this growing body of literature and the continuing interest in convergence behaviors under neutrosophic frameworks, this paper introduces three new forms of statistical convergence for sequences of neutrosophic random variables:

(a) Neutrosophic statistical convergence in probability;
(b) Neutrosophic statistical convergence in mean of order r;
(c) Neutrosophic statistical convergence in distribution.

Each mode of convergence is developed with accompanying limit theorems, expanding and enhancing classical convergence theory as well as previously established neutrosophic results (cf. [1–3]).

2. Preliminaries

In this section, we present some well-known notions which will be useful for the development of this paper.

Definition 2.1 ([14])**.** Let $\mathcal{X}$ be a non-empty fixed set. A neutrosophic set $\mathcal{A}$ is an object having the form $\{x, (\mu\mathcal{A}(x), \delta\mathcal{A}(x), \gamma\mathcal{A}(x)) : x \in \mathcal{X}\}$, where $\mu\mathcal{A}(x)$, $\delta\mathcal{A}(x)$

and $\gamma\mathcal{A}(x)$ represent the degree of membership, the degree of indeterminacy, and the degree of non-membership respectively of each element $x \in \mathfrak{X}$ to the set $\mathcal{A}$.

Definition 2.2 ([13])**.** Let $\mathcal{K}$ be a field, the neutrosophic filed generated by $\mathcal{K}$ and I is denoted by $\langle \mathcal{K} \cup I \rangle$ under the operations of $\mathcal{K}$, where I is the neutrosophic element with the property $I^2 = I$.

Definition 2.3 ([12])**.** Classical neutrosophic number has the form $a + bI$, where a, b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers n.

Definition 2.4 ([12])**.** The neutrosophic probability of event $\mathcal{A}$ occurrence is

$$NP(\mathcal{A}) = (ch(\mathcal{A}), ch(neut\mathcal{A}), ch(anti\mathcal{A})) = (T, I, F),$$

where T, I, F are standard or non-standard subsets of the non-standard unitary interval $]^{-}0, 1^{+}[$.

Now, we present some notions of neutrosophic random variables [7].

Definition 2.5. Consider the real valued crisp random variable $\mathfrak{X}$ which is defined as $\mathfrak{X} : \Omega \to \mathbb{R}$, where Ω is the events space. Now, they defined a neutrosophic random variable $\mathfrak{X}_N$ as $\mathfrak{X}_N : \Omega \to \mathbb{R}(I)$ and $\mathfrak{X}_N = X + I$, where I is indeterminacy.

Theorem 2.1. *Consider the neutrosophic random variable $\mathfrak{X}_N = \mathfrak{X} + I$, where cumulative distribution function of $\mathfrak{X}_N$ is $F_{\mathfrak{X}_N}(x) = P(\mathfrak{X}_N \leq x)$. Then, the following statements hold:*

(a) $F_{\mathfrak{X}_N}(x) = F_{\mathfrak{X}}(x - I)$;
(b) $f_{\mathfrak{X}_N}(x) = f_{\mathfrak{X}}(x - I)$,

where $F_{\mathfrak{X}_N}$ and $f_{\mathfrak{X}_N}$ are cumulative distribution function and probability density function of $\mathfrak{X}_N$, respectively.

Theorem 2.2. *Consider the neutrosophic random variable $\mathfrak{X}_N = \mathfrak{X} + I$, expected value can be found as follows:*

$$E(\mathfrak{X}_N) = E(\mathfrak{X}) + I.$$

Next, we define the notion of neutrosophic statistical convergence.

Definition 2.6. The sequence of neutrosophic numbers $\{x_{N_n}\}_{n\in\mathbb{N}}$ is said to be neutrosophic statistically convergent to a neutrosophic number x_N if for each $\varepsilon > 0$

$$\lim_{n\to+\infty} \frac{1}{n} |\{k \leq n : |x_{N_n} - x_N| \geq \varepsilon\}| = 0,$$

and this will be denoted by $x_{N_n} \xrightarrow{st_N} x_N$ or $st_N\text{-}\lim_{n\to+\infty} x_{N_n} = x_N$.

3. Neutrosophic Statistical Convergence in Probability

Definition 3.1. Let $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ be a sequence of neutrosophic random variables where each X_{N_n} is defined on the same event space $\mathcal{S}$ with respect to a given class of subsets of $\mathcal{S}$ as the class Λ of events and a given probability function $\mathcal{P}:\Lambda\to\mathbb{R}$. The sequence $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ is said to be neutrosophic statistically convergent in probability to a neutrosophic random variable $\mathcal{X}$, where $\mathcal{X}:\mathcal{S}\to\mathbb{R}$, if for any $\varepsilon,\delta>0$

$$\lim_{n\to+\infty}\frac{1}{n}|k\le n:\mathcal{P}(|\mathcal{X}_{N_k}-\mathcal{X}_N|\ge\varepsilon)\ge\delta\}|=0, \tag{3.1}$$

or equivalently,

$$\lim_{n\to+\infty}\frac{1}{n}|k\le n:1-\mathcal{P}(|\mathcal{X}_{N_k}-\mathcal{X}_N|<\varepsilon)\ge\delta\}|=0. \tag{3.2}$$

(3.1) and (3.2) can be written as

$$\lim_{n\to+\infty}\frac{1}{n}|k\le n:\mathcal{P}(|\mathcal{X}_k+I_k-\mathcal{X}-I|\ge\varepsilon)\ge\delta\}|=0$$

and

$$\lim_{n\to+\infty}\frac{1}{n}|k\le n:1-\mathcal{P}(|\mathcal{X}_k+I_k-\mathcal{X}-I|<\varepsilon)\ge\delta\}|=0,$$

respectively. In any case, we will denote them as $st_N\text{-}\lim_{n\to+\infty}\mathcal{P}(|\mathcal{X}_{N_n}-\mathcal{X}_N|\ge\varepsilon)=0$ or $st_N\text{-}\lim_{n\to+\infty}\mathcal{P}(|\mathcal{X}_{N_n}-\mathcal{X}_N|<\varepsilon)=1$ or $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N$.

Theorem 3.1. *If a sequence of constants $x_{N_n}\xrightarrow{st_N}x_N$, then regarding a constant as a random variable having one-point distribution at that point, we may also write $x_{N_n}\xrightarrow{st_{p_N}}x_N$.*

Proof. Let ε be any positive real number. Then, $d(U)=1$, where $U=\{n\in\mathbb{N}:|x_{N_n}-u_N|<\varepsilon\}$. So, for any $\delta>0$, $V=\{n\in\mathbb{N}:1-\mathcal{P}(|x_{N_n}-x_N|<\varepsilon)\ge\delta\}\subset\mathbb{N}-U$, which implies $d(U)=0$. Therefore, this proves that $x_{N_n}\xrightarrow{st_{p_N}}x_N$. □

The converse of the above theorem is not always true, as can be seen from the following example.

Example 3.1. Let $\mathcal{X}_{N_n}$ be a neutrosophic random variable with neutrosophic probability density

$$f_n(x-I)=\begin{cases}1, & \text{if } I<x<1+I,\\ 0, & \text{otherwise}, \qquad \text{if } n=2^m,\ \text{where } m\in\mathbb{N},\\ \\ \frac{n(x-I)^{n-1}}{2^n}, & \text{where } I<x<2+I;\\ 0, & \text{otherwise}, \qquad \text{if } n\ne 2^m,\ \text{where } m\in\mathbb{N}.\end{cases}$$

Let $0<\varepsilon$, $\delta>1$. Then,

$$\mathcal{P}(|\mathcal{X}_{N_n}-(2+I)|\geq\varepsilon)=\begin{cases}1, & \text{if } n=2^m,\ \text{where } m\in\mathbb{N},\\ 1-\mathcal{P}(|\mathcal{X}_{N_n}-(2+I)|<\varepsilon) & \\ =1-\left\{1-\left(\frac{2-\varepsilon+I}{2+I}\right)^n\right\} & \\ =\left(1-\frac{\varepsilon}{2}\right) & \text{if } n\neq 2^m,\ \text{where } m\in\mathbb{N}.\end{cases}$$

Then,

$$\lim_{n\to+\infty}\frac{1}{n}|\{k\leq n:\mathcal{P}(|\mathcal{X}_{N_k}-(2+I)|\geq\varepsilon)\geq\delta\}|$$
$$\leq\lim_{n\to+\infty}\frac{1}{n}|\{(2+I)^0,(2+I)^1,\ldots\}\cup\mathcal{A}|=0,$$

where $\mathcal{A}$ is a finite subset of $\mathbb{N}$. Therefore,

$$\lim_{n\to+\infty}\frac{1}{n}|\{k\leq n:\mathcal{P}(|\mathcal{X}_{N_k}-(2+I)|\geq\varepsilon)\geq\delta\}|=0.$$

Theorem 3.2. *The following properties are satisfied for neutrosophic statistical convergence in probability:*

(a) *if* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N$ *and* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}\mathcal{Y}_N$, *then* $\mathcal{P}(\mathcal{X}_N=\mathcal{Y}_N)=1$;

(b) $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N$ *if and only if* $\mathcal{X}_{N_n}-\mathcal{X}_N\xrightarrow{st_{p_N}}0$;

(c) *if* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N$, *then* $c\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}c\mathcal{X}_N$, *where* $c\in\mathbb{R}$;

(d) *if* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N$ *and* $\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}\mathcal{Y}_N$, *then* $\mathcal{X}_{N_n}+\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N+\mathcal{Y}_N$;

(e) *if* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N$ *and* $\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}\mathcal{Y}_N$, *then* $\mathcal{X}_{N_n}-\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N-\mathcal{Y}_N$;

(f) *if* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}x+I$, *then* $\mathcal{X}_{N_n}^2\xrightarrow{st_{p_N}}(x+I)^2$;

(g) *if* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}x+I_1$ *and* $\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}y+I_2$, *then* $\mathcal{X}_{N_n}\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}(x+I_1)(y+I_2)$, *where* $I_1\neq I_2$;

(h) *if* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}x+I_1$ *and* $\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}y+I_2$, *then* $\mathcal{X}_{N_n}\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}[xy+I(1+x+y)]$, *where* $I_1=I_2=I$;

(i) *if* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}x+I_1$ *and* $\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}y+I_2$, *then* $\frac{\mathcal{X}_{N_n}}{\mathcal{Y}_{N_n}}\xrightarrow{st_{p_N}}\frac{x+I_1}{y+I_2}$ *provided* $y\neq-I_2$;

(j) *if* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N$ *and* $\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}\mathcal{Y}_N$, *then* $\mathcal{X}_{N_n}\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N\mathcal{Y}_N$;

(k) *if* $0\leq\mathcal{X}_{N_n}\leq\mathcal{Y}_{N_n}$ *and* $\mathcal{Y}_{N_n}\xrightarrow{st_{p_N}}0$, *then* $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}0$;

(l) *if* $X_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N$, *then for each* $\varepsilon,\delta>0$, *there exists* $k\in\mathbb{N}$ *such that*

$$d(\{n\in\mathbb{N}:\mathcal{P}(|\mathcal{X}_{N_n}-\mathcal{X}_{N_k}|\geq\varepsilon)\geq\delta\})=0.$$

This will be called the neutrosophic statisical Cauchy condition in probability.

Proof. Let δ and ε be two positive real numbers.

(a) Let $k\in\{n\in\mathbb{N}:\mathcal{P}(|\mathcal{X}_{N_n}-\mathcal{X}_N|\geq\frac{1}{2}\varepsilon)<\frac{1}{2}\delta\}\cap\{n\in\mathbb{N}:\mathcal{P}(|\mathcal{X}_{N_n}-\mathcal{Y}|\geq\frac{1}{2}\varepsilon)<\frac{1}{2}\delta\}$. Then, $\mathcal{P}(|\mathcal{X}_N-\mathcal{Y}_N|\geq\varepsilon)\leq\mathcal{P}(|\mathcal{X}_{N_k}-\mathcal{X}_N|\geq\frac{1}{2}\varepsilon)+\mathcal{P}(|\mathcal{X}_{N_k}-\mathcal{Y}|\geq\frac{1}{2}\varepsilon)<\delta$. Therefore, this implies $\mathcal{P}\{\mathcal{X}_N=\mathcal{Y}_N\}=1$.

(b), (c), (d) and (e) are followed directly from the definitions, hence their proofs are omitted.

(f) If $\mathcal{Z}_{N_n} \xrightarrow{st_{p_N}} 0$, then $\mathcal{Z}^2_{N_n} \xrightarrow{st_{p_N}} 0$ for $\{n \in \mathbb{N} : \mathcal{P}(|\mathcal{Z}^2_{N_n} - 0| \geq \varepsilon) \geq \delta\} = \{n \in \mathbb{N} : |\mathcal{Z}_{N_n} - 0| \geq \sqrt{\varepsilon}) \geq \delta\}$. Next,

$$\mathcal{X}^2_{N_n} = (\mathcal{X}_{N_n} - (x+I))^2 + 2(x+I)(\mathcal{X}_{N_n} - (x+I)) + (x+I)^2 \xrightarrow{st_{p_N}} (x+I)^2.$$

(g) We get $\mathcal{X}_{N_n}\mathcal{Y}_{N_n} = \frac{1}{4}\{(\mathcal{X}_{N_n} + \mathcal{Y}_{N_n})^2 - (\mathcal{X}_{N_n} - \mathcal{Y}_{N_n})^2\} \xrightarrow{st_{p_N}} \frac{1}{4}\{(x + I_1 + y + I_2)^2 - (x + I_1 - (y + I_2))^2 = (x + I_1)(y + I_2)$.

(h) We have $\mathcal{X}_{N_n}\mathcal{Y}_{N_n} = \frac{1}{4}\{(\mathcal{X}_{N_n} + \mathcal{Y}_{N_n})^2 - (\mathcal{X}_{N_n} - \mathcal{Y}_{N_n})^2\} \xrightarrow{st_{p_N}} \frac{1}{4}\{(x+y+2I)^2 - (x-y)^2 = [xy + I(1+x+y)]$.

(i) Let $\mathcal{A}$ and $\mathcal{B}$ be events of $|\mathcal{Y}_{N_n} - (y+I_2)| < |y+I_2|$, $\left|\frac{1}{\mathcal{Y}_{N_n}} - \frac{1}{y+I_2}\right| \geq \varepsilon$, respectively. Now,

$$\begin{aligned}\left|\frac{1}{\mathcal{Y}_{N_n}} - \frac{1}{y+I_2}\right| &= \frac{|\mathcal{Y}_{N_n} - (y+I_2)|}{|(y+I_2)\mathcal{Y}_{N_n}} \\ &= \frac{|\mathcal{Y}_{N_n} - (y+I_2)}{|y+I_2| \cdot |(y+I_2) + (\mathcal{Y}_{N_n} - (y+I_2)|} \\ &\leq \frac{|\mathcal{Y}_{N_n} - (y+I_2)}{|y+I_2| \cdot |(|y+I_2| - |\mathcal{Y}_{N_n} - (y+I_2)|)|}.\end{aligned}$$

If $\mathcal{A}$ and $\mathcal{B}$ occur simultaneously, then

$$|\mathcal{Y}_{N_n} - (y+I_2)| \geq \frac{\varepsilon|y+I_2|^2}{1+\varepsilon|y+I_2|},$$

follows from the above inequality. Next, let $\varepsilon_0 = \frac{\varepsilon|y+I_2|^2}{1+\varepsilon|y+I_2|}$ and let $\mathcal{C}$ be the event $|\mathcal{Y}_{N_n} - (y+I_2)| \geq \varepsilon_0$. This implies $\mathcal{AB} \subset \mathcal{C}$ and then, $\mathcal{P}(\mathcal{B}) \leq \mathcal{C} + \mathcal{P}(\overline{\mathcal{A}})$, where the bar represents the set of complement. This implies

$$\begin{aligned}&\left\{n \in \mathbb{N} : \mathcal{P}\left(\left|\frac{1}{\mathcal{Y}_{N_n}} - \frac{1}{y+I_2}\right| \geq \varepsilon\right) \geq \delta\right\} \\ &\subset \left\{n \in \mathbb{N} : \mathcal{P}(|\mathcal{Y}_{N_n} - (y+I_2)| \geq \varepsilon_0) \geq \frac{1}{2}\delta\right\} \\ &\cup \left\{n \in \mathbb{N} : \mathcal{P}(\mathcal{Y}_{N_n} - (y+I_2)| \geq |y+I_2|) \geq \frac{1}{2}\delta\right\}.\end{aligned}$$

Therefore, $\frac{1}{\mathcal{Y}_{N_n}} \xrightarrow{st_{p_N}} \frac{1}{y+I_2}$ provided $y \neq I_2$. Consequently, $\frac{\mathcal{X}_{N_n}}{\mathcal{Y}_{N_n}} \xrightarrow{st_{p_N}} \frac{x+I_1}{y+I_2}$ provided $y \neq I_2$. If $I_2 = I_1 = I$, it can be seen that $\frac{\mathcal{X}_{N_n}}{\mathcal{Y}_{N_n}} \xrightarrow{st_{p_N}} \frac{x+I}{y+I}$, provided $y \neq I$.

(j) First at all, we should prove that if $\mathcal{X}_{N_n} \xrightarrow{st_{p_N}} \mathcal{X}_N$ and $\mathcal{Z}_N$ is a neutrosophic random variable, then $\mathcal{X}_{N_n}\mathcal{Z}_N \xrightarrow{st_{p_N}} \mathcal{X}_N\mathcal{Z}_N$. Since $\mathcal{Z}_N$ is a neutrosophic random variable, given $\delta > 0$, there exists an $\alpha > 0$ such that $\mathcal{P}(|\mathcal{Z}_N| > \alpha) \leq \frac{1}{2}\delta$. Then, for any $\varepsilon > 0$,

$$\begin{aligned}\mathcal{P}(\mathcal{X}_{N_n}\mathcal{Z}_N - \mathcal{X}_N\mathcal{Z}_N| \geq \varepsilon) =&\mathcal{P}(|\mathcal{X}_{N_n} - \mathcal{X}_N| \cdot |\mathcal{Z}_N| \geq \varepsilon, |\mathcal{Z}_N > \alpha)\\ &+ \mathcal{P}(|\mathcal{X}_{N_n} - \mathcal{X}_N| \cdot |\mathcal{Z}_N| \geq \varepsilon, |\mathcal{Z}_N \leq \alpha)\\ \leq& \frac{1}{2}\delta + \mathcal{P}\left(|\mathcal{X}_{N_n} - \mathcal{X}_N| \geq \frac{\varepsilon}{\alpha}\right).\end{aligned}$$

This implies,

$$\{n \in \mathbb{N} : \mathcal{P}(|\mathcal{X}_{N_n}\mathcal{Z}_N - \mathcal{X}_N\mathcal{Z}_N| \geq \varepsilon) \geq \delta\} \subset \left\{n \in \mathbb{N} : \mathbb{P}(|\mathcal{X}_{N_n} - (x+I)| \geq \varepsilon/\alpha) \geq \frac{1}{2}\delta\right\}.$$

Thus, $(\mathcal{X}_{N_n} - \mathcal{X}_N)(\mathcal{Y}_{N_n} - \mathcal{Y}_N) \xrightarrow{st_{p_N}} 0$. Therefore, this implies $\mathcal{X}_{N_n}\mathcal{Y}_{N_n} \xrightarrow{st_{p_N}} \mathcal{X}_N\mathcal{Y}_N$.

(k) Proof is straightforward and hence omitted.

(l) Take $k \in \mathbb{N}$ such that $\mathcal{P}(|\mathcal{X}_{N_k} - \mathcal{X}_N| \geq \frac{1}{2}\varepsilon) < \frac{1}{2}\delta$. Then, $\{n \in \mathbb{N} : \mathcal{P}(|\mathcal{X}_{N_n} - \mathcal{X}_{N_k}| \geq \varepsilon) \geq \delta\} \subset \{n \in \mathbb{N} : \mathcal{P}(|\mathcal{X}_{N_n} - \mathcal{X}_N| \geq \frac{1}{2}\varepsilon) \geq \frac{1}{2}\delta\}$. □

Theorem 3.3. *Let $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ be a sequence of identically and independently distributed neutrosophic random variables and $\mathcal{Y}_{N_n} = \frac{\mathcal{S}_{N_n} - E(\mathcal{S}_{N_n})}{n}$, where $\mathcal{S}_{N_n} = \mathcal{X}_{N_1} + \cdots + \mathcal{X}_{N_n}$ and $E(\cdot)$ is the expected value. Then, $\mathcal{Y}_{N_n} \xrightarrow{st_{p_N}} 0$ if and only if st_N-$\lim_{n\to+\infty} E\left\{\frac{\mathcal{Y}^2_{N_n}}{1+\mathcal{Y}^2_{N_n}}\right\} = 0$.*

Proof. Let $st_N\text{-}\lim\limits_{n\to+\infty} E\left\{\frac{\mathcal{Y}^2_{N_n}}{1+\mathcal{Y}_{N_n}}\right\} = 0$ and $\varepsilon > 0$. Then, $|\mathcal{Y}_{N_n}| \geq \varepsilon$ and $|\mathcal{Y}_{N_n}|^2 \geq \varepsilon^2$. Consequently, $\mathcal{Y}^2_{N_n} + \mathcal{Y}^2_{N_n}\varepsilon^2 \geq \varepsilon^2 + \mathcal{Y}^2_{N_n}\varepsilon^2$, hence

$$\frac{\dfrac{\mathcal{Y}^2_{N_n}}{1+\mathcal{Y}^2_{N_n}}}{\dfrac{\varepsilon^2}{1+\varepsilon^2}} \geq 1.$$

Therefore, by neutrosophic Markov's inequality,

$$\mathcal{P}(|\mathcal{Y}_{N_n}| \geq \varepsilon) \leq \mathcal{P}\left\{\frac{\dfrac{\mathcal{Y}^2_{N_n}}{1+\mathcal{Y}^2_{N_n}}}{\dfrac{\varepsilon^2}{1+\varepsilon^2}}\right\} \leq E\left\{\frac{\dfrac{\mathcal{Y}^2_{N_n}}{1+\mathcal{Y}^2_{N_n}}}{\dfrac{\varepsilon^2}{1+\varepsilon^2}}\right\}.$$

This implies, $\mathcal{Y}_{N_n} \xrightarrow{st_{p_N}} 0$. Conversely, let us consider that $\mathcal{X}_{N_i}$ are continuous and $\mathcal{Y}_{N_n}$ have the neutrosophic probability density function $f_{N_n}(y) = f_n(y - I_n)$. Then,

$$\begin{aligned}E\left\{\frac{\mathcal{Y}^2_{N_n}}{1+\mathcal{Y}^2_{N_n}}\right\} &= \int_{-\infty}^{+\infty} \frac{(y+I_n)^2_n}{1+(y+I_n)^2_n} f_{N_n}(y)dy\\ &= \int_{|\mathcal{Y}_{N_n}|\geq\varepsilon} \frac{(y+I_n)^2_n}{1+(y+I_n)^2_n} f_{N_n}(y)dy + \int_{|\mathcal{Y}_{N_n}|<\varepsilon} \frac{(y+I_n)^2_n}{1+(y+I_n)^2_n} f_{N_n}(y)dy\\ &\leq \int_{|\mathcal{Y}_{N_n}|\geq\varepsilon} f_{N_n}(y)dy + \int_{|\mathcal{Y}_{N_n}|<\varepsilon} (y+I_n)^2_n f_{N_n}(y)dy\end{aligned}$$

$$\leq \mathcal{P}(|\mathcal{Y}_{N_n}| \geq \varepsilon) + \varepsilon \int_{|\mathcal{Y}_{N_n}|<\varepsilon} f_{N_n}(y)dy$$
$$= \mathcal{P}(|\mathcal{Y}_{N_n}| \geq \varepsilon) + \varepsilon^2 \mathcal{P}(|\mathcal{Y}_{N_n}| < \varepsilon)$$
$$\leq \mathcal{P}(|\mathcal{Y}_{N_n}| \geq \varepsilon) + \varepsilon^2.$$

Since $\mathcal{Y}_{N_n} \xrightarrow{st_{p_N}} 0$ and ε^2 is an arbitrarily small positive real number, we obtain $st_N\text{-}\lim_{n\to+\infty} E\left\{\frac{\mathcal{Y}^2_{N_n}}{1+\mathcal{Y}^2_{N_n}}\right\} = 0.$ □

Remark 3.1. The result of Theorem 3.3 holds even if $E(\mathcal{X}_{N_i})$ does not exist. In this case we simply define $\mathcal{Y}_{N_n} = \frac{\mathcal{S}_{N_n}}{n}$ rather than $\frac{\mathcal{S}_{N_n}-E(\mathcal{S}_{N_n})}{n}$.

4. Neutrosophic Statistical Convergence in Mean of Order r

Definition 4.1. A sequence of neutrosophic random variables $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ is said to be statistically convergent in the r^{th}-mean, where $r > 0$, to a neutrosophic random variable $\mathcal{X}_N$ if for any $\delta > 0$

$$d(\{n \in \mathbb{N} : E(|\mathcal{X}_{N_n} - \mathcal{X}_N|^r) \geq \delta\}) = 0$$

provided $E(|\mathcal{X}_{N_n}|^r)$ exists for every $n \in \mathbb{N}$ and if $E(|\mathcal{X}_{N_n}|^r)$ exists we write

$$st_N\text{-}\lim_{n\to+\infty} E(|\mathcal{X}_{N_n} - \mathcal{X}_N|^r) = 0$$

or $\mathcal{X}_{N_n} \xrightarrow{st_{rm_N}} \mathcal{X}_N$.

Remark 4.1. Neutrosophic statistical convergence in the mean of orders one and two are called neutrosophic statistical convergence in mean and neutrosophic statistical convergence in quadratic mean (or mean square), respectively.

Remark 4.2. A slightly stronger concept of neutrosophic statistical convergence in probability was defined by neutrosophic statistical convergence in the mean of order r (see Definition 4.1) and it is proved in Theorem 4.1.

Theorem 4.1. *Let $\mathcal{X}_{N_n} \xrightarrow{st_{rm_N}} \mathcal{X}_N$. Then, $\mathcal{X}_{N_n} \xrightarrow{st_{p_N}} \mathcal{X}_N$, i.e., neutrosophic statistical convergence in r^{th}-mean implies neutrosophic statistical convergence in probability.*

Proof. The proof is easily carried using the neutrosophic Bienayme-Tchebycheff inequality, i.e., $\mathcal{P}(|\mathcal{X}_N - m|^r| \geq \varepsilon) \leq \frac{E\{|\mathcal{X}_N - m|^r\}}{\varepsilon^r}$. □

The following example shows that in general the converse of Theorem 4.1 need not be true.

Example 4.1. Let us consider the sequence of neutrosophic random variables $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ given as follows $\mathcal{X}_{N_n} \in \{-I_n, n - I_n\}$ with $\mathcal{P}(X_n = -I_n) = 1 - \frac{1}{(n-I_n)^r}$ and $\mathcal{P}(\mathcal{X}_n = n - I_n) = \frac{1}{(n-I_n)^r}$, where $r > 0$, $n \in \mathbb{N}$. For any $\varepsilon > 0$, $\mathcal{P}(|\mathcal{X}_{N_n} + I_n| \geq \varepsilon) = \mathcal{P}(\mathcal{X}_n = m - I_n)$ if $0 < \varepsilon \leq n - I_n$ and $\mathcal{P}(|\mathcal{X}_{N_n} + I_n| \geq \varepsilon) = 0$ if $\varepsilon > n - I_n$.

Therefore, for any $\delta > 0$, $\{n \in \mathbb{N} : \mathcal{P}(|\mathcal{X}_{N_n} + I_n| \geq \varepsilon) \geq \delta\}$ is equal to a finite set. Thus, $st_N\text{-}\lim_{n\to+\infty} \mathcal{P}(|\mathcal{X}_{N_n} + I_n| \geq \varepsilon) = 0$. But, $E(|\mathcal{X}_{N_n}|^r = 1$ for all $n \in \mathbb{N}$, then $\{n \in \mathbb{N} : E(|\mathcal{X}_{N_n} + I_n|^r) \geq \frac{1}{2}\}$. This implies, $\{E(|\mathcal{X}_{N_n} + I_n|^r)\}_{n\in\mathbb{N}}$ is not neutrosophic statistically convergent to $-I_n$.

Proposition 4.1. *Let $\{a_{N_n}\}_{n\in\mathbb{N}}, \{b_{N_n}\}_{n\in\mathbb{N}}$ and $\{c_{N_n}\}_{n\in\mathbb{N}}$ be three sequences of neutrosophic numbers such that $d(\{n \in \mathbb{N} : a_{N_n} \leq b_{N_n} \leq c_{N_n}\}) = 1$ and $st_N\text{-}\lim_{n\to+\infty} a_{N_n} = st_N\text{-}\lim_{n\to+\infty} c_{N_n} = x + yI$. Then, $st_N\text{-}\lim_{n\to+\infty} b_{N_n} = x + yI$, where $x, y \in \mathbb{N}$.*

Proof. The proof is straightforward and hence omitted. □

Proposition 4.2. *Let $\{a_{N_n}\}_{n\in\mathbb{N}}$ be a sequence of non-negative neutrosophic numbers such that $st_N\text{-}\lim_{n\to+\infty} a_{N_n} = a_N$ and $a \geq 0$. Then, $st_N\text{-}\lim_{n\to+\infty} (a_{N_n})^q = a_N^q$, where $a_N = a + bI$ and $a, b \in \mathbb{N}$.*

Proof. Let $q = m/r \in \mathbb{Q}^+$. If $a_N = 0$, the result is obvious. Take $a, b \geq 0$. It is sufficient to show $st_N\text{-}\lim_{n\to\infty} a_{N_n} = a_N$ implies $st_N\text{-}\lim_{n\to\infty} (a_{N_n})^{1/r} = a_N^{1/r}$, where $r \in \mathbb{N} - \{1\}$. Then, $\{n \in \mathbb{N} : |a_{N_n} - a_N| < \frac{1}{2}a_N\} \subset n \in \mathbb{N} : a_{N_n} > \frac{1}{2}a_N\} = \mathcal{M}$ (say, then $d(\mathcal{M}) = 1$). If $n \in \mathcal{M}$, then

$$\begin{aligned} |a_{N_n} - a_N| &= \left|(a_{N_n})^{1/r} - a_N^{1/r}\right| \cdot \left|\{(a_{N_n})^{(r-1)/r} + \cdots + a_N^{(r-1)/r}\right| \\ &\geq \mathcal{L} \left|(a_{N_n})^{1/r} - a_N^{1/r}\right|, \end{aligned}$$

where $\mathcal{L} = \frac{a_{N_n}^{(r-1)/r}}{2\left(1-\frac{1}{\sqrt[r]{2}}\right)}$, i.e., $\mathcal{M} \subset \{n \in \mathbb{N} : 0 \leq \left|(a_{N_n})^{1/r} - a_N^{1/r}\right| \leq \mathcal{L}^{-1}|a_{N_n} - a_N|\}$. Therefore, by Proposition 4.1, $st_N\text{-}\lim_{n\to+\infty} \sqrt[r]{a_{N_n}} = \sqrt[r]{a_N}$. □

Theorem 4.2. *Let $p > 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$E|\mathcal{X}_N\mathcal{Y}_N| \leq (E|\mathcal{X}_N|^p)^{1/p}(E|\mathcal{Y}_N|^q)^{1/q}.$$

Proof. By Holder's inequality, taking $x = \mathcal{X}_N\{E|\mathcal{X}_N|^p\}^{-1/p}$, $y = \mathcal{Y}_N\{E|\mathcal{Y}_N|^q\}^{-1/q}$, we obtain

$$|\mathcal{X}_N\mathcal{Y}_N| \leq p^{-1}|\mathcal{X}_N|^p\{E|\mathcal{X}_N|^p\}^{1/p-1}\{E|\mathcal{Y}_N|^q\}^{1/q} + q^{-1}|\mathcal{Y}_N|^p\{E|\mathcal{Y}_N|^q\}^{1/q-1}\{E|\mathcal{X}_N|^p\}^{1/p}.$$

Taking the expectation on both sides leads to the result. □

Theorem 4.3. *For $p \geq 1$,*

$$\{E|X_N + Y_N|^p\}^{1/p} \leq \{E|\mathcal{X}_N|^p\}^{1/p} + \{E|\mathcal{Y}_N|^p\}^{1/p}.$$

Proof. For $p > 1$, we have

$$|\mathcal{X}_N + \mathcal{Y}_N| \leq |\mathcal{X}_N| \cdot |\mathcal{X}_N + \mathcal{Y}_N|^{p-1} + |\mathcal{Y}_N| \cdot |\mathcal{X}_N + \mathcal{Y}_N|^{p-1}.$$

Choosing expectations and using Holder's inequality with $\mathcal{Y}_N$ replaced by $|\mathcal{X}_N+\mathcal{Y}_N|^{p-1}$, we get

$$\begin{aligned}E|\mathcal{X}_N+\mathcal{Y}_N|^p \leq &\{E|\mathcal{X}_N|^p\}^{1/p}\{E|\mathcal{X}_N+\mathcal{Y}_N|^{(p-1)q}\}^{1/q}\\ &+\{E|\mathcal{Y}_N|^p\}^{1/p}\{E|\mathcal{X}_N+\mathcal{Y}_N|^{(p-1)q}\}^{1/q}\\ =&[\{E|\mathcal{X}_N|^p\}^{1/p}+\{E|\mathcal{Y}_N|^p\}^{1/p}]\{E|\mathcal{X}_N+\mathcal{Y}_N|^{(p-1)q}\}^{1/q}.\end{aligned}$$

Excluding the trivial case in which $E|\mathcal{X}_N+\mathcal{Y}_N|^p=0$, and noting that $(p-1)q=p$, we obtain, after dividing both sides of the last inequality by $E|\mathcal{X}_N+\mathcal{Y}_N|^p\}^{1/q}$,

$$\{E|X_N+Y_N|^p\}^{1/p}\leq\{E|\mathcal{X}_N|^p\}^{1/p}+\{E|\mathcal{Y}_N|^p\}^{1/p},\quad p>1.$$

The case $p=1$ is trivial. □

Proposition 4.3. *Let $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ be a sequence of neutrosophic random variables such that $\mathcal{X}_{N_n}\xrightarrow{st_{2m_N}}\mathcal{X}_N$. Then, $st_N\text{-}\lim_{n\to+\infty}E(\mathcal{X}_{N_n})=E(\mathcal{X}_N)$ and $st_N\text{-}\lim_{n\to+\infty}E(\mathcal{X}^2_{N_n})=E(\mathcal{X}^2_N)$*

Proof. The proof follows from Theorems 4.2 and 4.3. □

5. Neutrosophic Statistical Convergence in Distribution

Definition 5.1. Let $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ be a sequence of neutrosophic random variables, where $F_{N_n}(x)$ is the neutrosophic distribution function of $\mathcal{X}_{N_n}$ for $n\in\mathbb{N}$. If there exists a neutrosophic random variable $\mathcal{X}_N$ whose distribution function is $F_N(x)=F(x-I)$ such that $st_N\text{-}\lim_{n\to+\infty}F_{N_n}(x)=F_N(x)=F(x-I)$ at every point of continuity $x-I$ of $F_N(x)=F(x-I)$, then $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ is said to be neutrosophic statistically convergent in distribution to $\mathcal{X}_N$ and we write $\mathcal{X}_{N_n}\xrightarrow{st_{d_N}}\mathcal{X}_N$.

Proposition 5.1. *Let $\{a_{N_n}\}_{n\in\mathbb{N}}$ and $\{b_{N_n}\}_{n\in\mathbb{N}}$ be two sequences of neutrosophic numbers such that $a_{N_n}\leq b_{N_n}$ for all $n\in\mathbb{N}$. Then,*

$$st_N-\underline{\lim}a_{N_n}\leq st-\underline{\lim}b_{N_n}$$

and

$$st_N-\overline{\lim}a_{N_n}\leq st-\overline{\lim}b_{N_n}$$

Proof. The proof is straightforward and hence omitted. □

Theorem 5.1. *Let $\mathcal{X}_{N_n}\xrightarrow{st_{p_N}}\mathcal{X}_N$. Then, $\mathcal{X}_{N_n}\xrightarrow{st_{d_N}}\mathcal{X}_N$.*

Proof. Let $F_{N_n}(x)$ and $F_N(x)$ be the neutrosophic distribution functions of $\mathcal{X}_{N_n}$ and $\mathcal{X}_N$, respectively. Now, for any two real numbers x and y with $x<y$, we get

$$(\mathcal{X}_N\leq x)=(\mathcal{X}_{N_n}\leq y,\mathcal{X}_N\leq x)+(\mathcal{X}_{N_n}>y,\mathcal{X}_N\leq x).$$

Since $(\mathcal{X}_{N_n}\leq y,\mathcal{X}_N\leq y)\subset(\mathcal{X}_{N_n}\leq y)$, we obtain

$$(\mathcal{X}_N\leq x)\subset(\mathcal{X}_{N_n}\leq y)+(\mathcal{X}_{N_n}>y,\mathcal{X}_N\leq x).$$

Hence,

$$\begin{aligned}\mathcal{P}(\mathcal{X}_N \le x) &\le \mathcal{P}\{(\mathcal{X}_{N_n} \le y) + (\mathcal{X}_{N_n} > y, \mathcal{X}_N \le x)\} \\ &\le \mathcal{P}(\mathcal{X}_{N_n} \le y) + \mathcal{P}(\mathcal{X}_{N_n} > y, \mathcal{X}_N \le x)\end{aligned} \tag{5.1}$$

implies $F_{N_n}(y) \ge F_N(x) - \mathcal{P}(\mathcal{X}_{N_n} > y, \mathcal{X}_N \le x)$.

Next, if $\mathcal{X}_{N_n} > y$, $\mathcal{X}_N \le x$ occur simultaneity, then $\mathcal{X}_{N_n} > y$, $-\mathcal{X}_N \ge -x$, thus $\mathcal{X}_{N_n} - \mathcal{X}_N > y - x$. This implies,

$$\mathcal{P}(\mathcal{X}_{N_n} > y, \mathcal{X}_N \le x) \le \mathcal{P}(|\mathcal{X}_{N_n} - \mathcal{X}_N| > y - x).$$

Now, from (5.1) and Proposition 5.1, st_N-$\underline{\lim} F_{N_n}(y) \ge F_N(x)$. Similarly, if $y < z$, then

$$(\mathcal{X}_{N_n} \le y) = (\mathcal{X}_N \le z, \mathcal{X}_{N_n} \le y) + (\mathcal{X}_N > z, \mathcal{X}_{N_n} \le y).$$

Then, $F_{N_n}(y) \le F_N(z) + \mathcal{P}(\mathcal{X}_N > z, \mathcal{X}_{N_n} \le y)$ and st_N-$\lim\limits_{n\to+\infty} \mathcal{P}(\mathcal{X}_N > z, \mathcal{X}_{N_n} \le y) = 0$. Finally, we have st_N-$\overline{\lim} F_{N_n}(y) \le F_N(z)$. Therefore, for $x < y < z$, we get

$$F_N(x) \le st_N - \underline{\lim} F_{N_n}(y) \le st_N - \overline{\lim} F_{N_n}(y) \le F_N(z).$$

If F is continuous at y, then

$$F_N(y) = \lim_{x\to y^-} F_N(x) \le st_N - \underline{\lim} F_{N_n}(y) \le st_N - \overline{\lim} F_{N_n}(y) \le \lim_{z\to y^+} F_N(z) = F_N(y).$$

Therefore, we have st_N-$\underline{\lim} F_{N_n}(y) = st_N$-$\overline{\lim} F_{N_n}(y) = F_N(y)$. This proves $\mathcal{X}_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$. □

The converse of Theorem 5.1 need not be true as can be seen from the following example.

Example 5.1. Let us consider neutrosophic random variables $\mathcal{X}_N, \mathcal{X}_{N_1}, \mathcal{X}_{N_2}, \dots$ having identical neutrosophic distribution. Let the spectrum of the two dimensional neutrosophic random variable $(\mathcal{X}_{N_n}, \mathcal{X}_N)$ be $(-I_n, -I)$, $(-I_n, 1-I)$, $(1-I_n, -I)$, $(1-I_n, 1-I)$ and

$$\begin{aligned}\mathcal{P}(\mathcal{X}_n = -I_n, \mathcal{X} = -I) &= 0 = \mathcal{P}(\mathcal{X}_n = 1 - I_n, \mathcal{X} = 1 - I), \\ \mathcal{P}(\mathcal{X}_n = -I_n, \mathcal{X} = 1 - I) &= \frac{1}{2} = \mathcal{P}(\mathcal{X}_n = 1 - I_n, \mathcal{X} = -I).\end{aligned}$$

Therefore, st_N-$\lim\limits_{n\to+\infty} F_{N_n}(x) = F_N(x)$ for all $x \in \mathbb{R}$. But, $\mathcal{P}(|\mathcal{X}_{N_n} - \mathcal{X}_N| \ge \frac{1}{2}) = \mathcal{P}(|\mathcal{X}_{N_n} - \mathcal{X}_N| = 1) = \mathcal{P}(\mathcal{X}_n = -I_n, \mathcal{X} = 1 - I) + \mathcal{P}(\mathcal{X}_n = 1 - I_n, \mathcal{X} = -I) = 1$. Therefore, st_N-$\lim\limits_{n\to+\infty} \mathcal{P}(|\mathcal{X}_{N_n} - \mathcal{X}| \ge \frac{1}{2}) \ne 0$. Then, st_N-$\lim\limits_{n\to+\infty} \mathcal{P}(|\mathcal{X}_{N_n} - \mathcal{X}| \ge \varepsilon) \ne 0$. Hence, the result follows.

Theorem 5.2. *If $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ and $\{\mathcal{Y}_{N_n}\}_{n\in\mathbb{N}}$ are sequences of neutrosophic random variables on some probability space with $\mathcal{X}_{N_n} - \mathcal{Y}_{N_n} \xrightarrow{st_{p_N}} 0$ and $\mathcal{Y}_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$, then $\mathcal{X}_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$.*

Proof. Let x and $x+\varepsilon$ be points of continuity of the neutrosophic distribution function F_N corresponding to the neutrosophic random variable $\mathcal{X}_N$, where $\varepsilon > 0$. Thus, $\mathcal{P}(\mathcal{X}_{N_n} \leq x) = \mathcal{P}(\mathcal{Y}_{N_n} \leq x + \mathcal{Y}_{N_n} - \mathcal{X}_{N_n}) = \mathcal{P}(\mathcal{Y}_{N_n} \leq x + \mathcal{Y}_{N_n} - \mathcal{X}_{N_n}; \mathcal{Y}_{N_n} - \mathcal{X}_{N_n} \leq \varepsilon)$ $+\mathcal{P}(\mathcal{Y}_{N_n} \leq x + \mathcal{Y}_{N_n} - \mathcal{X}_{N_n}; \mathcal{Y}_{N_n} - \mathcal{X}_{N_n} > \varepsilon) \leq \mathcal{P}(\mathcal{Y}_{N_n} \leq x + \varepsilon) + \mathcal{P}(\mathcal{Y}_{N_n} - \mathcal{X}_{N_n} > \varepsilon)$. Therefore, this implies $st_N\text{-}\overline{\lim} F_{N_n}(x) \leq F(x+\varepsilon-I)$ and analogously $F(x-\varepsilon-I) \leq st_N\text{-}\underline{\lim} F_{N_n}(x)$. Since ε is arbitrary, so $F(x-I) = F_N(x) = st_N\text{-}\lim F_{N_n}(x)$. ☐

Example 5.2. Let us consider the neutrosophic random variables $\mathcal{X}_{N_n}$ and $\mathcal{Y}_{N_n}$ defined on the standard probability space $([0,1], \mathcal{B}, \lambda)$, where λ is the Lebesgue measure. Define

$$\mathcal{X}_{N_n}(\omega) = \omega + \frac{1}{n} \cdot I, \quad \mathcal{Y}_{N_n}(\omega) = \omega, \quad \text{for } \omega \in [0,1],$$

and let the limiting neutrosophic random variable be:

$$\mathcal{X}_N(\omega) = \omega.$$

Here, I represents an indeterminacy component in the neutrosophic framework, such that $\frac{1}{n} \cdot I$ tends to 0 neutrosophically in probability as $n \to +\infty$. We observe that:

$$\mathcal{X}_{N_n} - \mathcal{Y}_{N_n} = \frac{1}{n} \cdot I \xrightarrow{st_{p_N}} 0,$$

and clearly,

$$\mathcal{Y}_{N_n} = \omega \xrightarrow{st_{d_N}} \mathcal{X}_N = \omega.$$

Therefore, by Theorem 5.2, it follows that

$$\mathcal{X}_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N.$$

Thus, we have verified the conditions and conclusion of Theorem 5.2 using a simple, explicit construction of neutrosophic random variables.

Theorem 5.3. *If $\{\mathcal{X}_{N_n}\}_{n\in\mathbb{N}}$ and $\{\mathcal{Y}_{N_n}\}_{n\in\mathbb{N}}$ are sequences of neutrosophic random variables on some probability space and α is a constant, then the following hold.*

(a) *If $X_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$ and $Y_{N_n} \xrightarrow{st_{p_N}} \alpha$, then $X_{N_n} + Y_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N + \alpha$.*
(b) *If $X_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$ and $Y_{N_n} \xrightarrow{st_{p_N}} \alpha$, then $X_{N_n} Y_{N_n} \xrightarrow{st_{d_N}} \alpha\mathcal{X}_N$.*
(c) *If $X_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$ and $Y_{N_n} \xrightarrow{st_{p_N}} \alpha$, then $\frac{X_{N_n}}{Y_{N_n}} \xrightarrow{st_{d_N}} \frac{\mathcal{X}_N}{\alpha}$, where $\alpha \neq 0$.*
(d) *If $X_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$ and $\iota_{N_n} \xrightarrow{st_N} 0$, then $\iota_{N_n} X_{N_n} \xrightarrow{st_{p_N}} 0$, where ι_{N_n} is a sequence of positive constants.*
(e) *If $X_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$, then $X_{N_n} + \alpha \xrightarrow{st_{d_N}} \mathcal{X}_N + \alpha$.*
(f) *If $X_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$, then $\alpha X_{N_n} \xrightarrow{st_{d_N}} \alpha\mathcal{X}_N$, where $\alpha \neq 0$.*
(g) *If $X_{N_n} \xrightarrow{st_{d_N}} \alpha$ if and only if $X_{N_n} \xrightarrow{st_{p_N}} \alpha$.*
(h) *If $X_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$ and $Y_{N_n} \xrightarrow{st_{p_N}} 0$, then $X_{N_n} Y_{N_n} \xrightarrow{st_{p_N}} 0$.*

Proof. Proofs of (a)-(g) are followed easily and hence are omitted.

(h) For any $\delta > 0$, take $\pm\beta$ in set of points of continuity of the neutrosophic distribution function $\mathcal{F}_N$ of $\mathcal{X}_N$ such that $F_N(\beta) - F_N(-\beta) \geq 1 - \beta$. Any $\varepsilon > 0$, $\mathcal{P}(|\mathcal{X}_{N_n}\mathcal{Y}_{N_n}| \geq \varepsilon) = \mathcal{P}(|\mathcal{X}_{N_n}\mathcal{Y}_{N_n}| \geq \varepsilon, |\mathcal{Y}_{N_n}| < \varepsilon/\beta) + \mathcal{P}(|\mathcal{X}_{N_n}\mathcal{Y}_{N_n}| \geq \varepsilon, |\mathcal{Y}_{N_n}| \geq \varepsilon/\beta) \leq \mathcal{P}(|\mathcal{X}_{N_n}| > \beta) + \mathcal{P}(|\mathcal{Y}_{N_n}| \geq \varepsilon/\beta)$. Thus, $st_N\text{-}\overline{\lim}\mathcal{P}(|\mathcal{X}_{N_n}\mathcal{Y}_{N_n}| \geq \varepsilon) < \delta$. Therefore, this implies $X_{N_n}Y_{N_n} \xrightarrow{st_{p_N}} 0$. □

Example 5.3. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the standard probability space $([0,1], \mathcal{B}, \lambda)$, and let I represent the neutrosophic indeterminacy component. Define the sequences:

$$\mathcal{X}_{N_n}(\omega) = \omega + \frac{1}{n}I, \quad \mathcal{Y}_{N_n}(\omega) = \frac{1}{n} + \frac{1}{n^2}I, \quad \text{for } \omega \in [0,1].$$

Let $\alpha = 1$ and the limiting neutrosophic random variable:

$$\mathcal{X}_N(\omega) = \omega.$$

(a) Since $\mathcal{X}_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N$ and $\mathcal{Y}_{N_n} \xrightarrow{st_{p_N}} 1$, then

$$\mathcal{X}_{N_n} + \mathcal{Y}_{N_n} \xrightarrow{st_{d_N}} \mathcal{X}_N + 1 = \omega + 1.$$

(b) Using the same sequences, the product:

$$\mathcal{X}_{N_n} \cdot \mathcal{Y}_{N_n} \xrightarrow{st_{d_N}} \omega \cdot 1 = \omega.$$

(c) For the quotient, since $\mathcal{Y}_{N_n} \xrightarrow{st_{p_N}} 1 \neq 0$, we get

$$\frac{\mathcal{X}_{N_n}}{\mathcal{Y}_{N_n}} \xrightarrow{st_{d_N}} \frac{\omega}{1} = \omega.$$

(d) Let $\iota_{N_n} = \frac{1}{n}$ (a sequence of positive constants with $\iota_{N_n} \xrightarrow{st_N} 0$). Then,

$$\iota_{N_n} \cdot \mathcal{X}_{N_n} \xrightarrow{st_{p_N}} 0.$$

(e) Adding a constant to the sequence

$$\mathcal{X}_{N_n} + \alpha \xrightarrow{st_{d_N}} \mathcal{X}_N + 1 = \omega + 1.$$

(f) Multiplying the sequence by a constant $\alpha = 2$

$$2 \cdot \mathcal{X}_{N_n} \xrightarrow{st_{d_N}} 2 \cdot \omega = 2\omega.$$

(g) Let $\mathcal{X}_{N_n} = 1 + \frac{1}{n}I$ so that

$$\mathcal{X}_{N_n} \xrightarrow{st_{d_N}} 1 \quad \text{if and only if} \quad \mathcal{X}_{N_n} \xrightarrow{st_{p_N}} 1.$$

(h) Since $\mathcal{X}_{N_n} \xrightarrow{st_{d_N}} \omega$ and $\mathcal{Y}_{N_n} \xrightarrow{st_{p_N}} 0$, then

$$\mathcal{X}_{N_n} \cdot \mathcal{Y}_{N_n} \xrightarrow{st_{p_N}} 0.$$

6. Conclusion

In this paper, we have defined the notion of statistical convergence of a sequence of neutrosophic random variables. Also, we proved and showed some of their basic properties and relationships. For further results, we suggest studding λ-statistical convergence, f-statistical convergence of order α and many others based on sequences of neutrosophic random variables and to find some relations or equivalences between the notions which were defined in this paper. Additionally, we can imply that the results presented in this paper, are more general than the classical theory presented by [1, 2]. Indeed, this supports that Smarandache said [3, 4] that statistical Neutrosophic theory is more general than classical statistical theory.

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