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EXISTENCE OF MULTIPLE SOLUTIONS FOR NONLOCAL ROBIN PROBLEMS WITH DEGENERACY

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ABSTRACT. This paper is devoted to a nonlocal degenerate problem subject to nonhomogeneous Robin boundary conditions. We show that the problem admits multiple positive solutions, with their number being precisely twice the number of zeros of the degenerate term. Our approach is based on a transition from a variational framework to a one-dimensional fixed-point formulation, supplemented by sub- and supersolution techniques and suitable truncations.

1. INTRODUCTION

In the present work, we consider the following nonlocal degenerate problem subject to Robin boundary conditions

$$(\mathrm{P})\qquad \begin{cases} \mathcal{K}\left(\int_\Omega \mathcal{H}(v(z))\, dz\right)(-\Delta_q v(z) + \xi(z)v(z)^{q-1}) = \alpha(z)g(v(z)), & \text{in } \Omega,\\ v > 0, & \text{in } \Omega,\\ |\nabla v|^{q-2}\dfrac{\partial v}{\partial n} + \beta(z)|v|^{q-2}v = \varphi(v), & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded domain; the diffusion factor $\mathcal{K}(\cdot) \in C([0,+\infty);\mathbb{R})$, $\xi(\cdot) \in L^\infty(\Omega)$ is an indefinite potential function, $\mathcal{H}(s) := \sum_{j=1}^m \lambda_j s^{\sigma_j}$ with $\sigma_j > 1$ and $\lambda_j > 0$, the boundary weight term $\beta(\cdot) \in C^{0,\rho}(\partial\Omega)$ $(0 < \rho < 1)$ with $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$, the reaction term $g(\cdot) \in C([0,s_*];\mathbb{R})$ for some $s_* > 0$, the term

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on the boundary $\varphi(\cdot) \in C^{1,\nu}([0, s_*])$ $(0 < \nu < 1)$, the weight function $\alpha(\cdot) \in L^\infty(\Omega)$, with $\underline{\alpha} := \operatorname{essinf}_\Omega \alpha > 0$, n is the outer normal, and

$$-\Delta_q v := -\operatorname{div}(|\nabla v|^{q-2}\nabla v),$$

denotes the classical q-Laplacian operator with $q \in (1, +\infty)$.

The q-Laplacian problem with Robin boundary conditions arises naturally in a variety of nonlinear physical, biological, and engineering contexts, in which both the interior dynamics and the boundary interaction play crucial roles. For instance, in heat transfer through materials with nonlinear conductivity, Robin conditions model convective exchange with an external environment, while in nonlinear electrostatics they represent imperfectly conducting interfaces with capacitive properties. Moreover, similar formulations appear in population dynamics, where the Robin term models semi-permeable boundaries allowing migration driven by both density and flux, we refer to [1, 19, 23, 25–27, 29, 30].

Nonlocal problems of the form (P) with $q = 2$, $\mathcal{H}(u) = u$ and $\xi(u) = 0$ are motivated by biological models describing population diffusion, in which the diffusion coefficient is

$$\nu = -a\left(\int_\Omega u\, dz\right)\nabla u,$$

depending only on the total population [10], regardless of whether individuals are uniformly distributed or accumulated, where $u(z)$ represents the population density at point z. Furthermore, when $N = 1$, such nonlocal models find other applications in the description of transversal vibrations of elastic strings subject to potentially large displacements [8].

Later, a more general type of diffusion was considered, where $\mathcal{H}(u) = u^p$ with $p > 1$, leading to a diffusion coefficient of the form

$$\nu = -a\left(\int_\Omega u^p\, dz\right)\nabla u.$$

In this scenario, large subdomains increase dispersion compared to a uniform distribution of population, since u^p (with $p > 1$) amplifies high-density regions, thereby making their contribution to the velocity integral disproportionately larger.

The study of nonlocal problems of this type has witnessed considerable advances in recent years. Notable contributions include Gasiński et al. [15, 16] on the Laplacian, Candito et al. [5] and Crespo-Blanco et al. [12] on the q-Laplacian, and the recent work [6] on the q-Laplacian.

Papageorgiou et al. [28] considered the Robin problem with an indefinite potential

$$-\Delta u + \xi(z)u = f(z, u), \qquad \text{in } \Omega.$$

Applying variational tools, truncation, and critical groups, they proved the existence of at least three smooth solutions, positive, negative, and nodal, providing precise sign information. Under C^1-regularity of the reaction, a fourth nodal solution is obtained.

This paper is devoted to the study of a nonhomogeneous Robin problem in which the standard L^p-type nonlocal term is generalized to $\int_\Omega \mathcal{H}(v)\, dz$, where $\mathcal{H}(v) = \sum_{j=1}^m \lambda_j v^{\sigma_j}$, and whose diffusion is modulated by a coefficient $\mathcal{K}(\cdot)$ that undergoes multiple sign changes.

Set

$$\gamma_g := \lim_{s\to 0^+} \frac{g(s)}{s^{q-1}}, \quad \gamma_\varphi := \lim_{s\to 0^+} \frac{\varphi(s)}{s^{q-1}},$$

and assume the following hypotheses.

(H_1) $\xi(\cdot) \in L^\infty(\Omega)$ with $\xi_0 := \text{essinf}_\Omega \xi > 0$.

(H_2) $\beta(\cdot) \in C^{0,\rho}(\partial\Omega)$ with $\rho \in (0,1)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

(H_3) $\alpha(\cdot) \in L^\infty(\Omega)$, with $\underline{\alpha} := \text{essinf}_\Omega\, \alpha > 0$.

(H_4) $g(\cdot) \in C([0,s_*];\mathbb{R})$ for some $s_* > 0$, such that $g(s) > 0$ for all $s \in (0,s_*)$, $g(s_*) = 0$, and the map $(0,s_*) \ni s \mapsto \psi(s) := \frac{g(s)}{s^{q-1}}$ is strictly decreasing.

(H_5) $\mathcal{K}(\cdot) \in C([0,+\infty);\mathbb{R})$ and there exist positive numbers $0 =: s_0 \leqslant s_1 < s_2 \leqslant s_3 < s_4 \leqslant \cdots \leqslant s_{2\ell-1} < s_{2\ell}$ $(\ell \geqslant 1)$ such that $\mathcal{K} > 0$ in (s_{2i-1}, s_{2i}) and $\mathcal{K}(s_{2i-1}) = \mathcal{K}(s_{2i}) = 0$ for any $i \in \mathbb{Z}(1,\ell) := \{1,\dots,\ell\}$. Moreover,

$$\sup_{s\in(s_{2i-1},s_{2i})} \mathcal{K}(s) < \gamma_g \min\left\{\frac{\|\alpha\|_{L^1(\Omega)}}{\mathcal{S}}, \frac{\underline{\alpha}}{\widehat{\lambda}_1(\xi)}\right\}, \quad \mathcal{S} := \|\xi\|_{L^1(\Omega)} + \|\beta\|_{L^1(\partial\Omega)},$$

where $\widehat{\lambda}_1(\xi)$ represents the first eigenvalue of the following problem

$$\begin{cases} -\Delta_q v(z) + \xi(z)|v(z)|^{q-2}v(z) = \lambda|v(z)|^{q-2}v(z), & \text{in } \Omega, \\ |\nabla v|^{q-2}\dfrac{\partial v}{\partial n} + \beta(z)|v|^{q-2}v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

(H_6) $\varphi(\cdot) \in C^{1,\nu}([0,s_*])$, $\nu \in (0,1]$, $\varphi(s_*) = 0$ with s_* as in (H_4) and $\varphi(s) \geqslant 0$ in $(0,s_*)$. Moreover, $(0,s_*) \ni s \mapsto \frac{\varphi(s)}{s^{q-1}}$ is strictly decreasing and $\gamma_\varphi < \mathcal{S}/|\partial\Omega|$ where $\mathcal{S}$ is given in condition (H_5).

(H_7) There is $k \in \mathbb{Z}(1,m)$, such that

$$\min\left\{\lambda_1 s_*^{\sigma_1}\int_\Omega \widehat{v}_1^{\sigma_1}\,dz,\ \lambda_2 s_*^{\sigma_2}\int_\Omega \widehat{v}_1^{\sigma_2}\,dz, \dots, \lambda_m s_*^{\sigma_m}\int_\Omega \widehat{v}_1^{\sigma_m}\,dz\right\} = \lambda_k s_*^{\sigma_k}\int_\Omega \widehat{v}_1^{\sigma_k}\,dz$$

and

$$s_{2\ell} < m\lambda_k s_*^{\sigma_k}\int_\Omega \widehat{v}_1^{\sigma_k}\,dz,$$

where $\widehat{v}_1$ denotes the eigenfunction corresponding to the first eigenvalue $\widehat{\lambda}_1(\xi)$ of problem (1.1), normalized with respect to the L^∞-norm.

(H_8) For any $i \in \mathbb{Z}(1,\ell)$, there exists $\delta \in (s_{2i-1}, s_{2i})$ such that

$$\tau\|\alpha\|_\infty\,|\Omega|\mathcal{M}_g < \mathcal{K}(\delta)\left(\delta^q - \tau\,|\partial\Omega|\mathcal{M}_\varphi\right),$$

where

$$\mathcal{M}_\varphi := \sup_{s\in[0,s_*]} s\varphi(s), \quad \mathcal{M}_g := \sup_{s\in[0,s_*]} sg(s), \quad \tau = \frac{q}{\xi_0}|\Omega|^{q-1}\left(\sum_{i=1}^m \lambda_i s_*^{\sigma_i - 1}\right)^q. \tag{1.2}$$

The main result of this paper is the following theorem.

Theorem 1.1. *Suppose that (H_1)-(H_8) hold. Then, problem (P) has at least ℓ pairs of positive solutions $v_{i,1}, v_{i,2}$, which satisfy*

$$t_1 < \int_\Omega \mathcal{H}(v_{1,1}) < \int_\Omega \mathcal{H}(v_{1,2}) < t_2 < \cdots < t_{2\ell-1} < \int_\Omega \mathcal{H}(v_{\ell,1}) < \int_\Omega \mathcal{H}(v_{\ell,2}) < t_{2\ell}.$$

The main idea of our approach is to reduce problem (P) to a family of auxiliary one-dimensional truncated fixed-point problems, formulated for each $i \in \mathbb{Z}(1,\ell)$ and $\theta \in (s_{2i-1}, s_{2i})$, as follows

$$(P_{i,\theta,g_*}) \qquad \begin{cases} \mathcal{K}(\theta)\left(-\Delta_q v(z) + \xi(z)|v(z)|^{q-2}v(z)\right) = \alpha(z) g_*(v), & \text{in } \Omega, \\ |\nabla v|^{q-2}\dfrac{\partial v}{\partial n} + \beta(z)|v|^{q-2}v = \varphi_*(v), & \text{on } \partial\Omega, \end{cases}$$

where the truncations $g_*(\cdot)$ and $\varphi_*(\cdot)$ are defined by

$$g_*(s) = \begin{cases} g(0), & \text{if } s \le 0, \\ g(s), & \text{if } 0 < s < s_*, \\ 0, & \text{if } s_* \le s, \end{cases} \tag{1.3}$$

and

$$\varphi_*(s) = \begin{cases} \varphi(0), & \text{if } s \le 0, \\ \varphi(s), & \text{if } 0 < s < s_*, \\ 0, & \text{if } s_* \le s. \end{cases} \tag{1.4}$$

We first establish that (P_{i,θ,g_*}) admits a unique positive solution v_θ satisfying $v_\theta \leqslant s_*$. This allows us to introduce the following map

$$\mathcal{Q}_i : (s_{2i-1}, s_{2i}) \to \mathbb{R}, \quad \mathcal{Q}_i(\theta) := \int_\Omega \mathcal{H}(v_\theta)\, dz,$$

defined for every $\theta \in (s_{2i-1}, s_{2i})$. This mapping is crucial to our proof. Indeed, by the first step, for every $\theta \in (s_{2i-1}, s_{2i})$, the function v_θ solves (P_{i,θ,g_*}). Furthermore, if $\theta \in \mathrm{Fix}(\mathcal{Q}_i)$, then the corresponding v_θ is also a solution to (P), where $\mathrm{Fix}(\mathcal{Q}_i) = \{\theta \in (s_{2i-1}, s_{2i}) : \mathcal{Q}_i(\theta) = \theta\}$, the fixed-point set of $\mathcal{Q}_i$. The final step is to establish the existence of at least two fixed points for Q_i.

The structure of the paper is as follows. In Section 2, we introduce the functional framework and present some preliminary results. In Section 3, we establish key auxiliary results that will be used in the main argument. Section 4 is devoted to the proof of the main theorem, which ensures the existence of ℓ pairs of positive solutions, each corresponding to a positive bump of the nonlocal term.

2. Functional Framework and Preliminaries

In this section, we recall some background on Sobolev spaces. For $q > 1$, the Sobolev space $W^{1,q}(\cdot)$ is characterized by

$$W^{1,q}(\Omega) = \left\{ v \in L^q(\Omega) : D^k v \in L^q(\Omega),\ |k| \leq 1 \right\}.$$

Throughout the paper, we endow the Sobolev space $W^{1,q}(\Omega)$ with the following norm

$$\|v\|_\beta := \left(\mathcal{R}(v) + \int_{\partial\Omega} \beta(z)|v|^q \, d\sigma \right)^{1/q},$$

where

$$\mathcal{R}(v) := \|\nabla v\|_q^q + \int_\Omega \xi(z)|v|^q \, dz.$$

This norm is equivalent to the classical Sobolev norm, (cf. [3,14]), where $\|\cdot\|_q$ stands for the $L^q(\Omega)$-norm.

We remind that the q-Laplacian operator $-\Delta_q : W^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ is continuous, bounded, monotone, and of (S_+)-type (see [7, Example 2.110]).

We present the following proposition for problem (1.1), established by Fragnelli et al. [13] (see also Mugnai et al. [22] and Papageorgiou et al. [24] where special cases of (1.1) are studied).

Proposition 2.1. *If hypotheses* (H_1)-(H_2) *hold, then problem* (1.1) *has a smallest eigenvalue* $\widehat{\lambda}_1 = \widehat{\lambda}_1(\xi) \in \mathbb{R}$ *which is isolated, simple, and the infimum in* (1.1) *is realized on the one-dimensional eigenspace of* λ*; the elements of this eigenspace do not change sign and if* v_1 *denotes the positive,* L^q*-normalized* (*that is,* $\|v_1\|_q = 1$) *eigenfunction, then* v_1 *is nonnegative.*

Definition 2.1. Let $L^q_+(\Omega) = \{v \in L^q(\Omega) :\ v \geqslant 0\}$. We state that $\underline{v} \in W^{1,q}(\Omega)$ is a sub-solution to (P_{i,θ,g_*}) if

$$\mathcal{K}(\theta) \left(\int_\Omega (|\nabla \underline{v}|^{q-2} (\nabla \underline{v}, \nabla \phi)_{\mathbb{R}^N} + \xi(z)|\underline{v}|^{q-2}\underline{v}\phi) \, dz \right) + \mathcal{K}(\theta) \int_{\partial\Omega} \beta(z)|\underline{v}|^{q-2}\underline{v}\phi \, d\sigma$$
$$\leqslant \int_\Omega \alpha(z) g_*(\underline{v})\phi \, dz + \mathcal{K}(\theta) \int_{\partial\Omega} \varphi_*(\underline{v})\phi \, d\sigma, \quad \text{for all } \phi \in W^{1,q}(\Omega) \cap L^q_+(\Omega).$$

We state that $\overline{v} \in W^{1,q}(\Omega)$ is a super-solution to (P_{i,θ,g_*}) if

$$\mathcal{K}(\theta) \left(\int_\Omega (|\nabla \overline{v}|^{q-2} (\nabla \overline{v}, \nabla \phi)_{\mathbb{R}^N} + \xi(z)|\overline{v}|^{q-2}\overline{v}\phi) \, dz \right) + \mathcal{K}(\theta) \int_{\partial\Omega} \beta(z)|\overline{v}|^{q-2}\overline{v}\phi \, d\sigma$$
$$\geqslant \int_\Omega \alpha(z) g_*(\overline{v})\phi \, dz + \mathcal{K}(\theta) \int_{\partial\Omega} \varphi_*(\overline{v})\phi \, d\sigma, \quad \text{for all } \phi \in W^{1,q}(\Omega) \cap L^q_+(\Omega).$$

We say that $v \in W^{1,q}(\Omega)$ is a solution to (P_{i,θ,g_*}) if it is both a sub-solution and a super-solution to (P_{i,θ,g_*}), that is,

(2.1)

$$\mathcal{K}(\theta) \left(\int_\Omega (|\nabla v|^{q-2} (\nabla v, \nabla \phi)_{\mathbb{R}^N} + \xi(z)|v|^{q-2}v\phi) \, dz \right) + \mathcal{K}(\theta) \int_{\partial\Omega} \beta(z)|v|^{q-2}v\phi \, d\sigma$$
$$= \int_\Omega \alpha(z) g_*(v)\phi \, dz + \mathcal{K}(\theta) \int_{\partial\Omega} \varphi_*(v)\phi \, d\sigma, \quad \text{for all } \phi \in W^{1,q}(\Omega).$$

3. Auxiliary Results

The energy functional corresponding to problem (P_{i,θ,g_*}) is given by:

$$\mathcal{J}_{i,\theta}(v) := \frac{1}{q}\mathcal{K}(\theta)\|v\|_\beta^q - \int_\Omega \alpha(z)G_*(v)\,dz - \mathcal{K}(\theta)\int_{\partial\Omega}\Phi_*(v)\,d\sigma,$$

where $G_*(t) := \int_0^t g_*(s)\,ds$ and $\Phi_*(t) := \int_0^t \varphi_*(s)\,ds$. We note that the critical points of $\mathcal{J}_{i,\theta}(v)$ represent weak solutions to (P_{i,θ,g_*}).

Lemma 3.1. *Suppose that hypotheses (H_1)-(H_6) hold. Then, for each $i \in \mathbb{Z}(1,\ell)$ and $\theta \in (s_{2i-1}, s_{2i})$ fixed, problem (P_{i,θ,g_*}) admits a unique solution v_θ such that $0 < v_\theta \leqslant s_*$. Moreover, $v_\theta \in C^{1,\lambda}(\overline{\Omega})$ for some $\lambda \in (0,1)$.*

Proof. From (H_4) and (H_6), the functional $\mathcal{J}_{i,\theta}(v)$ is well defined, coercive and lower semi-continuous. Invoking Tonelli-Weierstrass' theorem (cf. [11]), there exists a global minimum $v_\theta \in W^{1,q}(\Omega)$ for $\mathcal{J}_{i,\theta}(v)$ which is a solution to (P_{i,θ,g_*}).

Now, we prove that $0 < v_\theta \leqslant s_*$. Let us consider (2.1) with the test function $\phi_1 = -(v_\theta)^-$, we have

$$\mathcal{K}(\theta)\|v_\theta^-\|_\beta^p = \int_\Omega \alpha(z)g_*(v_\theta)(-(v_\theta)^-)\,dz + \mathcal{K}(\theta)\int_{\partial\Omega}\varphi_*(v_\theta)(-(v_\theta)^-)\,d\sigma \leqslant 0.$$

As a result, $\|v_\theta^-\|_\beta = 0$ and thus $v_\theta \geqslant 0$. Invoking Proposition 2.1, we can apply the strong maximum principle in [18, Theorem 6.4.6], which yields $v_\theta > 0$.

By the same argument, we test (2.1) with $\phi_2 = (v_\theta - s_*)^+$, we get

$$\begin{aligned}\mathcal{K}(\theta)\|(v_\theta - s_*)^+\|_\beta^q =& -\int_\Omega \alpha(z)g_*(v_\theta)(v_\theta - s_*)^+\,dz - \mathcal{K}(\theta)\int_{\partial\Omega}\varphi_*(v_\theta)(v_\theta - s_*)^+\,d\sigma \\ \leqslant& 0.\end{aligned}$$

Therefore, $v_\theta \leqslant s_*$.

The uniqueness follows by an argument analogous to that of [12, Lemma 2.1].

Given that $v_\theta \in L^\infty(\Omega)$ and $\varphi \in C^{1,\nu}([0,s_*])$ by (H_6), [20, Theorem 2] ensures that $v_\theta \in C^{1,\lambda}(\overline{\Omega})$ for some $\lambda \in (0,1)$. ☐

We now establish that, for any $i \in \mathbb{Z}(1,\ell)$, $\mathcal{Q}_i$ is continuous and admits at least two fixed points.

Proposition 3.1. *Assume that (H_1), (H_3)-(H_6) hold. Then, the mapping*

$$\mathcal{Q}_i : (s_{2i-1}, s_{2i}) \to \mathbb{R}, \quad \mathcal{Q}_i(\theta) := \int_\Omega \mathcal{H}(v_\theta)\,dz,$$

is continuous for each $i \in \mathbb{Z}(1,\ell)$, where v_θ is obtained from Lemma 3.1.

Proof. Let $i \in \mathbb{Z}(1,\ell)$ be fixed. Consider a sequence $(\theta_n)_{n\geqslant 1} \subset (s_{2i-1}, s_{2i})$ converging to some $\theta_* \in (s_{2i-1}, s_{2i})$, and denote by v_{θ_n} the corresponding minimizers of the energy functional associated with θ_n. Let $\psi_1 = \frac{1}{|\partial\Omega|}$ and $v = t\psi_1$, t small enough. A direct computation yields

$$\frac{\mathcal{J}_{i,\theta_n}(t\psi_1)}{t^q} = \frac{\mathcal{K}(\theta_n)\mathcal{S}}{q|\partial\Omega)|^q} - \int_\Omega \alpha(z)U_t(z)\,dz - \mathcal{K}(\theta_n)\int_{\partial\Omega} V_t(z)\,d\sigma,$$

where

$$U_t(z) := \frac{G_{s_*}(t\psi_1)}{(t\psi_1)^q}\psi_1^q, \quad V_t(z) := \frac{\Phi_{s_*}(t\psi_1)}{(t\psi_1)^q}\psi_1^q.$$

Applying Höpital's rule together with the dominated convergence theorem, we obtain

$$\int_\Omega \alpha(z)U_t(z)\,dz \to \frac{\gamma_g\|\alpha\|_{L^1(\Omega)}}{q|\partial\Omega|^q}, \quad \int_{\partial\Omega} V_t(z)\,d\sigma \to \frac{\gamma_\varphi}{q|\partial\Omega|^{q-1}},$$

as $t \to 0^+$. Putting the limits together leads to

$$\lim_{t\to 0^+}\frac{\mathcal{J}_{i,\theta_n}(t\psi_1)}{t^q} = \frac{1}{q|\partial\Omega|^q}\left(\mathcal{K}(\theta_n)\mathcal{S} - \gamma_g\|\alpha\|_{L^1(\Omega)} - \mathcal{K}(\theta_n)\gamma_\varphi|\partial\Omega|\right). \tag{3.1}$$

Meanwhile, by hypotheses (H_5)-(H_6) we have

$$\sup_{s\in(s_{2i-1},s_{2i})}\mathcal{K}(s) < \gamma_g\frac{\|\alpha\|_{L^1(\Omega)}}{\mathcal{S}-\gamma_\varphi|\partial\Omega|},$$

where $\mathcal{S}$ is given in condition (H_5). Combining the last estimate with (3.1), we conclude that, for $t > 0$ sufficiently small, $d_{\theta_n} \leqslant \mathcal{J}_{i,\theta_n}(t\psi_1) < 0$ where

$$d_{\theta_n} := \mathcal{J}_{i,\theta_n}(v_{\theta_n}) = \inf_{v\in W^{1,q}(\Omega)}\mathcal{J}_{i,\theta_n}(v).$$

This yields

$$\|v_{\theta_n}\|_\beta^q \leqslant \frac{q}{\mathcal{K}(\theta_n)}\left(\|\alpha\|_\infty\mathcal{M}_g\,|\Omega| + \mathcal{K}(\theta_n)\mathcal{M}_\varphi|\partial\Omega|\right). \tag{3.2}$$

Therefore, $(v_{\theta_n})_{n\geqslant 1} \subset W^{1,q}(\Omega)$ is a bounded sequence. Thus, by [4, Theorem 4.9],

$$\begin{aligned} &v_{\theta_n} \to v_*, \quad \text{in } L^1(\Omega),\\ &v_{\theta_n}(z) \to v_*(z), \quad \text{a.e. in } \Omega, \end{aligned} \tag{3.3}$$

for some $v_* \in W^{1,q}(\Omega)$. By testing (2.1) with $\phi = v_{\theta_n} - v_*$, then taking $\limsup_{n\to\infty}$ and exploiting the boundedness of g_* and φ_*, together with the convergence $\theta_n \to \theta_*$ and (3.3), we deduce

$$\mathcal{K}(\theta_*)\limsup_{n\to+\infty}\langle -\Delta_q v_{\theta_n}, v_n - v\rangle \leqslant 0.$$

Applying the (S_+)-property of the q-Laplacian operator, we conclude that $v_{\theta_n} \to v_*$ in $W^{1,q}(\Omega)$. To prove that v_* is nontrivial, we introduce the constant

$$\bar{\mathcal{K}} := \sup_{i\in\mathbb{Z}(1,\ell)}\ \sup_{s\in(s_{2i-1},s_{2i})}\mathcal{K}(s).$$

Thanks to (H_4), the map $\psi(\cdot)$ is strictly decreasing and thus admits an inverse $\psi^{-1} : (0, \gamma_g) \to (0, s_*)$. Hence, by (H_5), for any $\epsilon \in \left(0, \gamma_g - \frac{1}{\underline{\alpha}}\hat{\lambda}_1(\xi)\bar{\mathcal{K}}\right)$, it makes sense to consider the function

$$y_{\theta_n} := \psi^{-1}\left(\frac{1}{\underline{\alpha}}\widehat{\lambda}_1(\xi)\mathcal{K}(\theta_n) + \epsilon\right)\widehat{v}_1. \tag{3.4}$$

We note that $y_{\theta_n} \in W^{1,q}(\Omega)$ and it holds that $y_{\theta_n} \leqslant s_*$ in Ω. Moreover, $\psi(\cdot)$ is strictly decreasing. Therefore

$$qG_*(y_{\theta_n}) \geqslant q\frac{g_*(y_{\theta_n})}{y_{\theta_n}^{q-1}}\int_0^{y_{\theta_n}} t^{q-1}\,dt = g(y_{\theta_n})y_{\theta_n}. \tag{3.5}$$

On the other hand, exploiting the positivity of $\varphi(\cdot)$ and $\mathcal{K}(\theta_n)$, it follows that

$$\frac{\mathcal{J}_{i,\theta_n}(y_{\theta_n})}{\|y_{\theta_n}\|_\infty^q} \leqslant \frac{1}{q}\mathcal{K}(\theta_n)\|\widehat{v}_1\|_\beta^q - \int_\Omega \frac{\alpha(z)G_*(y_{\theta_n})}{\|y_{\theta_n}\|_\infty^q}\,dz. \tag{3.6}$$

Using (3.5), the definition of y_{θ_n}, the fact that $\psi(\cdot)$ is strictly decreasing and $\|\widehat{v}_1\|_\infty = 1$, we deduce that

$$\frac{G_*(y_{\theta_n})}{\|y_{\theta_n}\|_\infty^q} \geqslant \frac{g(y_{\theta_n})y_{\theta_n}}{q\|y_{\theta_n}\|_\infty^q} \geqslant \frac{g(y_{\theta_n})}{qy_{\theta_n}^{q-1}}\widehat{v}_1^q \geqslant \frac{g(\|y_{\theta_n}\|_\infty)}{\|y_{\theta_n}\|_\infty^{q-1}}\widehat{v}_1^q = \psi(\|y_{\theta_n}\|_\infty)\widehat{v}_1^q. \tag{3.7}$$

Combining the last inequality with (3.7), and substituting these relations into (3.6), together with Proposition 2.1, we infer

$$d_{\theta_n} \leqslant \mathcal{J}_{i,\theta_n}(y_{\theta_n}) \leqslant -\frac{\varepsilon}{q}\|y_{\theta_n}\|_\infty^q \int_\Omega \alpha(z)\widehat{v}_1^q\,dz.$$

Passing to the limit, it follows that

$$\mathcal{J}_{i,\theta_*}(v_*) \leqslant -\frac{\varepsilon}{q}\psi^{-1}\left(\frac{1}{\underline{\alpha}}\widehat{\lambda}_1(\xi)\mathcal{K}(\theta_*) + \epsilon\right)^q \int_\Omega \alpha(z)\widehat{v}_1^q\,dz < 0.$$

Consequently, v_* is nontrivial. This completes the proof. □

4. Proof of Main Result

This section is devoted to the proof of the main theorem, where we establish both existence and multiplicity of positive solutions to (P).

Lemma 4.1. *Assume that (H_1)-(H_6) are satisfied. Then,*

$$v_\theta \geqslant \chi_\theta := \psi^{-1}\left(\frac{1}{\underline{\alpha}}\widehat{\lambda}_1(\xi)\mathcal{K}(\theta)\right)\widehat{v}_1, \quad \textit{for all } i \in \mathbb{Z}(1,\ell), \theta \in (s_{2i-1}, s_{2i}). \tag{4.1}$$

Proof. Fix $i \in \mathbb{Z}(1,\ell)$ and $\theta \in (s_{2i-1}, s_{2i})$. We consider the following truncated problem

$$(P_{i,\theta,\tilde{g}}) \qquad \begin{cases} \mathcal{K}(\theta)\left(-\Delta_q v + \xi(z)|v|^{q-2}v\right) = \alpha(z)\widetilde{g}(z,v), & \text{in } \Omega, \\ |\nabla v|^{q-2}\dfrac{\partial v}{\partial n} + \beta(z)|v|^{q-2}v = \widetilde{\varphi}(z,v), & \text{on } \partial\Omega. \end{cases}$$

where

$$\widetilde{g}(z,s) = \begin{cases} g_*(s), & \text{if } s > \chi_\theta(z), \\ g_*(\chi_\theta(z)), & \text{if } s \leq \chi_\theta(z), \end{cases}$$

and

$$\widetilde{\varphi}(z,s) = \begin{cases} \varphi_*(s), & \text{if } s > \chi_\theta(z), \\ \varphi_*(\chi_\theta(z)), & \text{if } s \leq \chi_\theta(z). \end{cases}$$

By proceeding as in Lemma 3.1, we get a unique solution $\widetilde{v}_\theta$ of $(P_{i,\theta,\tilde{g}})$ fulfilling $0 < \widetilde{v}_\theta \leqslant s_*$.

Given that $\psi(\cdot)$ is strictly monotone, hypothesis (H_5) holds, and $\|\widehat{v}_1\|_\infty = 1$, it necessarily follows that

$$\frac{\widehat{\lambda}_1(\xi)\mathcal{K}(\theta)}{\underline{\alpha}} = \frac{g\left(\|\chi_\theta\|_\infty\right)}{\|\chi_\theta\|_\infty^{q-1}} \leqslant \frac{g(\chi_\theta)}{\chi_\theta^{q-1}}. \tag{4.2}$$

By (H_2), (H_4), (H_6), and Proposition 2.1, we obtain, for each $\phi \in W^{1,q}(\Omega) \cap L^q_+(\Omega)$,

$$\mathcal{K}(\theta)\langle T(\chi_\theta), \phi\rangle = C(\theta) \int_\Omega \widehat{v}_1^{q-1}\phi\, dz = \widehat{\lambda}_1(\xi)\mathcal{K}(\theta) \int_\Omega \chi_\theta^{q-1}\phi\, dz, \tag{4.3}$$

where

$$C(\theta) := \widehat{\lambda}_1(\xi)\mathcal{K}(\theta)\left(\psi^{-1}\left(\frac{1}{\underline{\alpha}}\widehat{\lambda}_1(\xi)\mathcal{K}(\theta)\right)\right)^{q-1},$$

and $T : W^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ denotes the monotone operator defined by

$$\langle T(v), \phi\rangle := \int_\Omega \left(|\nabla v|^{q-2}(\nabla v, \nabla \phi)_{\mathbb{R}^N} + \xi(z)|v|^{q-2}v\,\phi\right)\, dz + \int_{\partial\Omega} \beta(z)\,|v|^{q-2}v\,\phi\, d\sigma.$$

By (4.2) and since $\chi_\theta \leqslant s_*$, we deduce that, for all $\phi \in W^{1,q}(\Omega) \cap L^q_+(\Omega)$,

$$\widehat{\lambda}_1(\xi)\mathcal{K}(\theta) \int_\Omega \chi_\theta^{q-1}\phi\, dz \leqslant \int_\Omega \alpha(z)g(\chi_\theta) \tag{4.4}$$

From (H_6), it follows that $\varphi_*(\chi_\theta) = \varphi(\chi_\theta) \geqslant 0$ on $\partial\Omega$, which together with (4.3)–(4.4), yields that

$$\mathcal{K}(\theta)\langle T(\chi_\theta), \phi\rangle \leqslant \int_\Omega \alpha(z)g_*(\chi_\theta)\phi\, dz + \mathcal{K}(\theta)\int_{\partial\Omega} \varphi_*(\chi_\theta)\phi\, d\sigma,$$

for each $\phi \in W^{1,q}(\Omega) \cap L^q_+(\Omega)$. Consequently, χ_θ is a sub-solution to (P_{i,θ,g_*}).

Testing $(P_{i,\theta,\widetilde{g}})$ with $\phi = (\chi_\theta - \widetilde{v}_\theta)^+$ and using the fact that χ_θ is a sub-solution to (P_{i,θ,g_*}), we have

$$\begin{aligned}
&\mathcal{K}(\theta)\langle T(\widetilde{v}_\theta), (\chi_\theta - \widetilde{v}_\theta)^+\rangle \\
=& \int_\Omega \alpha(z)\widetilde{g}(z,\widetilde{v}_\theta)(\chi_\theta - \widetilde{v}_\theta)^+\, dz + \mathcal{K}(\theta)\int_{\partial\Omega} \widetilde{\varphi}(z,\widetilde{v}_\theta)(\chi_\theta - \widetilde{v}_\theta)^+\, d\sigma \\
=& \int_\Omega \alpha(z)g_*(\chi_\theta)(\chi_\theta - \widetilde{v}_\theta)^+\, dz + \mathcal{K}(\theta)\int_{\partial\Omega} \varphi_*(\chi_\theta)(\chi_\theta - \widetilde{v}_\theta)^+\, d\sigma \\
\geqslant& \mathcal{K}(\theta)\langle T(\chi_\theta), (\chi_\theta - \widetilde{v}_\theta)^+\rangle.
\end{aligned}$$

Since T is strictly monotone on $W^{1,q}(\Omega)$, we deduce

$$\chi_\theta \leqslant \widetilde{v}_\theta, \quad \text{in } \Omega. \tag{4.5}$$

Hence, for all $\phi \in W^{1,q}(\Omega)$, it follows that

$$\mathcal{K}(\theta)\langle T(\widetilde{v}_\theta), \phi\rangle = \int_\Omega \alpha(z) g_*(\widetilde{v}_\theta)\phi \, dz + \mathcal{K}(\theta)\int_{\partial\Omega} \varphi_*(\widetilde{v}_\theta)\phi \, d\sigma,$$

that is, $\widetilde{v}_\theta$ is a positive solution to (P_{i,θ,g_*}) such that $\widetilde{v}_\theta \leqslant s_*$. As a consequence of the uniqueness result in Lemma 3.1, we obtain that

$$\widetilde{v}_\theta = v_\theta, \quad \text{in } \Omega. \tag{4.6}$$

Inserting (4.6) into (4.5) yields (4.1). This completes the proof. □

Proposition 4.1. *Suppose that (H_1), (H_3)-(H_8) hold. Then, the map Q_i admits at least two fixed points $\theta_{i,1}, \theta_{i,2} \in (s_{2i-1}, s_{2i})$ such that $\theta_{i,1} < \theta_{i,2}$, for any $i \in \mathbb{Z}(1,\ell)$.*

Proof. Fix $i \in \mathbb{Z}(1,\ell)$. By (4.1), for every $\theta \in (s_{2i-1}, s_{2i})$, we have

$$\mathcal{Q}_i(\theta) = \int_\Omega \mathcal{H}(v_\theta)\, dz = \sum_{i=1}^m \lambda_i \int_\Omega v_\theta^{\sigma_i}\, dz \geqslant \sum_{i=1}^m \lambda_i \left(\psi^{-1}\left(\frac{1}{\underline{\rho}}\widehat{\lambda}_1 \mathcal{K}(\theta)\right)\right)^{\sigma_i} \int_\Omega \widehat{v}_1^{\sigma_i}\, dz,$$

which along with (H_4), yields

$$\liminf_{\theta\to s_{2i-1}^+} \mathcal{Q}_i(\theta) \geqslant \sum_{i=1}^m \lambda_i s_*^{\sigma_i}\int_\Omega \widehat{v}_1^{\sigma_i}\, dz \quad \text{and} \quad \liminf_{\theta\to s_{2i}^-} \mathcal{Q}_i(\theta) \geqslant \sum_{i=1}^m \lambda_i s_*^{\sigma_i}\int_\Omega \widehat{v}_1^{\sigma_i}\, dz.$$

Consequently, condition (H_7) implies

$$\liminf_{\theta\to s_{2i-1}^+} \mathcal{Q}_i(\theta) > s_{2i-1} \quad \text{and} \quad \liminf_{\theta\to s_{2i}^-} \mathcal{Q}_i(\theta) > s_{2i}.$$

We shall prove the existence of some $\theta_0 \in (s_{2i-1}, s_{2i})$ such that

$$Q_i(\theta_0) < \theta_0.$$

To this end, fix $\theta \in (s_{2i-1}, s_{2i})$ and consider the following auxiliary problem:

$$\begin{cases} -\Delta_q v + \xi(z)v^{q-1} = h(v_\theta), & \text{in } \Omega, \\ |\nabla v|^{q-2}\frac{\partial v}{\partial n} + \beta(z)|v|^{q-2}v = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

where $h(v_\theta) := \sum_{j=1}^m \lambda_j v_\theta^{\sigma_j - 1}$ and v_θ refers to the unique solution of (P_{i,θ,g_*}).

Applying the Minty-Browder's Theorem [4, Theorem 5.16], problem (4.7) admits a unique positive solution u_θ.

Testing (4.7) with v_θ yields

$$\mathcal{Q}_i(\theta) = \int_\Omega \mathcal{H}(v_\theta)\, dz = \langle T(u_\theta), v_\theta\rangle. \tag{4.8}$$

Applying Hölder's inequality to each term on the right-hand side of (4.8), we infer that

$$\mathcal{Q}_i(\theta) \leqslant \|u_\theta\|_\beta^{q-1}\|v_\theta\|_\beta. \tag{4.9}$$

On the other hand, testing (4.7) with u_θ and applying Hölder's inequality, we obtain

$$\|u_\theta\|_\beta^{q-1} = \int_\Omega \frac{h(v_\theta)u_\theta}{\|u_\theta\|_\beta} \leqslant \frac{\|u_\theta\|_q}{\|u_\theta\|_\beta}\sum_{j=1}^m \lambda_j \|v_\theta^{\sigma_j-1}\|_{q'} \leqslant \frac{|\Omega|^{\frac{1}{q'}}}{\xi_0^{\frac{1}{q}}}\sum_{j=1}^m \lambda_j s_*^{\sigma_i-1}. \tag{4.10}$$

Inserting (3.2) and (4.10) into (4.9), we arrive at

$$\mathcal{Q}_i(\theta) \leqslant \left(\frac{C_q(\theta)}{\xi_0}|\Omega|^{q-1}\right)^{1/q}\left(\sum_{j=1}^m \lambda_j s_*^{\sigma_j-1}\right),$$

where

$$C_q(\theta) = \frac{q}{\mathcal{K}(\theta)}\left(|\Omega|\|\alpha\|_\infty \mathcal{M}_g + \mathcal{K}(\theta)|\partial\Omega|\mathcal{M}_\varphi\right).$$

Therefore, from (H_8), there exists at least one $\theta_0 \in (s_{2i-1}, s_{2i})$ such that $\mathcal{Q}_i(\theta_0) < \theta_0$. This completes the proof. □

Proof of Theorem 1.1. Proposition 4.1 ensures that the map $\mathcal{Q}_i$ has at least two fixed points. Hence, problem (P) admits at least two positive solutions for each $i \in \mathbb{Z}(1,\ell)$, which satisfy

$$t_1 < \int_\Omega \mathcal{H}(v_{1,1}) < \int_\Omega \mathcal{H}(v_{1,2}) < t_2 < \cdots < t_{2\ell-1} < \int_\Omega \mathcal{H}(v_{\ell,1}) < \int_\Omega \mathcal{H}(v_{\ell,2}) < t_{2\ell}.$$

□

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References

[1] G. D'Agui, A. Sciammetta, E. Tornatore and P. Winkert, *Parametric Robin double phase problems with critical growth on the boundary*, Discrete Contin. Dyn. Syst. Ser. S **16**(6) (2023), 1286–1299.

[2] W. Allegretto and Y. X. Huang, *A Picone's identity for the p-Laplacian and applications*, Nonlinear Anal. **32**(7) (1998), 819–830.

[3] G. Bonanno and A. Sciammetta, *Existence and multiplicity results to Neumann problems for elliptic equations involving the p-Laplacian*, J. Math. Anal. Appl. **390**(1) (2012), 59–67.

[4] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[5] P. Candito, L. Gasiíski, R. Livrea and J. R. Santos Júnior, *Multiplicity of positive solutions for a degenerate nonlocal problem with p-Laplacian*, Adv. Nonlinear Anal. **11**(1) (2022), 357–368.

[6] P. Candito, G. Failla and R. Livrea, *Pairs of positive solutions for a carrier $p(x)$-Laplacian type equation*, Mathematics **12**(16) (2024), Article ID 2441.

[7] S. Carl, V. K. Le and D. Motreanu, *Nonsmooth Variational Problems and their Inequalities. Comparison Principles and Applications*, Springer, New York, 2007.

[8] G. F. Carrier, *On the nonlinear vibration problem of the elastic string*, Quart. Appl. Math. **3** (1945), 151–165.

[9] E. Casas and L. A. Fernández, *A Green's formula for quasilinear elliptic operators*, J. Math. Anal. Appl. **142**(1) (1989), 62–73.

[10] M. Chipot and J. F. Rodrigues, *On a class of nonlocal nonlinear elliptic problems*, ESAIM Math. Model. Numer. Anal. **26**(3) (1992), 447–467.

[11] D. G. Costa, *An Invitation To Variational Methods in Differential Equations*, Birkhäuser, Basel, 2007.

[12] A. Crespo-Blanco, G. Failla and B. Vassallo, *Multiple solutions for a nonlocal Neumann p-Laplacian problem*, Nonlinear Anal, Real World Applic. **87** (2026) Paper ID 104449, 10 pages.

[13] G. Fragnelli, D. Mugnai and N. S. Papageorgiou, *The Brezis-Oswald result for quasilinear Robin problems*, Adv. Nonlinear Anal. **16**(3) (2016), 603–622.

[14] L. Gasiński and N. S. Papageorgiou, *Positive solutions for the Robin p-Laplacian problem with competing nonlinearities*, Adv. Calc. Var. **12**(1) (2019), 31–56.

[15] L. Gasiński and J. R. Santos Júnior, *Multiplicity of positive solutions for an equation with degenerate nonlocal diffusion*, Comput. Math. Appl. **78**(1) (2019), 136–143.

[16] L. Gasiński and J. R. Santos Júnior, *Nonexistence and multiplicity of positive solutions for an equation with degenerate nonlocal diffusion*, Bull. Lond. Math. Soc. **52**(3) (2020), 489–497.

[17] L. Gasiński and N. S. Papageorgiou, *Existence and uniqueness of positive solutions for the Neumann p-Laplacian*, Positivity **17**(2) (2013), 309–332.

[18] L. Gasiński and N. S. Papageorgiou, *Nonlinear Analysis*, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[19] S. Leonardi and N. S. Papageorgiou, *Existence and multiplicity of positive solutions for parametric nonlinear nonhomogeneous singular Robin problems*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **114**(2) (2020), Article ID 100.

[20] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12**(11) (1988), 1203–1219.

[21] Z. Liu, D. Motreanu and S. Zeng, *Multiple solutions for a Kirchhoff-type problem with vanishing nonlocal term and fractional p-Laplacian*, Front. Math. **18**(5) (2023), 1067–1082.

[22] D. Mugnai and N. S. Papageorgiou, *Resonant nonlinear Neumann problems with indefinite weight*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **11**(5) (2012), 729–788.

[23] N. S. Papageorgiou and A. Scapellato, *Concave-convex problems for the Robin p-Laplacian plus an indefinite potential*, Mathematics **8**(3) (2020), Article ID 421.

[24] N. S. Papageorgiou and V. D. Rădulescu, *Multiplicity of solutions for resonant Neumann problems with an indefinite and unbounded potential*, Trans. Amer. Math. Soc. **367**(12) (2015), 8723–8756.

[25] N. S. Papageorgiou and V. D. Rădulescu, *Multiple solutions with precise sign for nonlinear parametric Robin problems*, J. Differential Equations **256**(7) (2014), 2449–2479.

[26] N. S. Papageorgiou and V. D. Rădulescu, *Nonlinear parametric Robin problems with combined nonlinearities*, Adv. Nonlinear Stud. **15**(3) (2015), 715–748.

[27] N. S. Papageorgiou and V. D. Rădulescu and D. D. Repovš, *Nonlinear, nonhomogeneous Robin problems with indefinite potential and general reaction*, Appl. Math. Optim. **81**(3) (2020), 823–857.

[28] N. S. Papageorgiou, J. Zhang and W. Zhang, *Multiple solutions with sign information for Robin equations with indefinite potential*, Bull. Math. Sci. **15**(1) (2025), 23 pages.

[29] J. C. Sabina de Lis and S. Segura de León, *Higher Robin eigenvalues for the p-Laplacian operator as p approaches* 1, Calc. Var. Partial Differential Equations **63**(7) (2024), Article ID 169.

[30] C. Vetro, *Perturbed eigenvalue problems for the Robin p-Laplacian plus an indefinite potential*, Anal. Math. Phys. **10**(4) (2020), Article ID 69.

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