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SOME q-ANALOGUES OF GRANVILLE AND SUN'S CONGRUENCES

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Abstract. In this paper, we use q-binomial theorem to establish some new q-analogues of Granville and Sun's congruence:

$$\sum_{k=1}^{p-1} \frac{x^k}{k} \equiv \frac{1-x^p-(x-1)^p}{p} \pmod{p},$$

and

$$\sum_{k=1}^{p-1} \frac{x^k}{k^2} \equiv \frac{1}{p}\left(\frac{1+(x-1)^p-x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k-1}{k}\right) \pmod{p},$$

where x is a variable and p is an odd prime.

1. Introduction

In recent years, q-analogues of congruences involving harmonic numbers have been widely studied by many authors. In 2004, Granville [1] showed that for any prime $p \geq 5$,

$$\sum_{k=1}^{p-1} \frac{x^k}{k} \equiv \frac{1-x^p-(x-1)^p}{p} \pmod{p}. \tag{1.1}$$

For any postive integer n, the q-integer is defined as

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1}.$$

It is easy to see that $\lim_{q\to 1}[n]_q = n$.

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Pan [3, (5.1)] showed that Granville [1] confirmed a conjecture of Skula: For any prime $p \geq 5$,

$$\left(\frac{2^{p-1}-1}{p}\right)^2 \equiv -\sum_{j=1}^{p-1} \frac{2^j}{j^2} \pmod p. \tag{1.2}$$

Pan [3, Theorem 5.1] established a q-analogue of (1.2) as follows:

$$\sum_{j=1}^{p-1} \frac{q^j (-q;q)_j}{[j]_q^2} + Q_p(2,q)^2$$
$$\equiv -(p-1) Q_p(2,q)(1-q) - \frac{(7p-5)(p-1)(1-q)^2}{24} \pmod{[p]_q},$$

where $Q_p(2,q) = \frac{(-q;q)_{p-1}-1}{[p]_q}$.

The harmonic numbers are given by

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{and} \quad H_0 = 0.$$

A q-analogues of harmonic numbers H_n is given by

$$H_n(q) = \sum_{k=1}^{n} \frac{q^k}{[k]_q}.$$

The q-Pochhammer symbol is given by

$$(x;q)_n = (1-x)(1-xq)\cdots\left(1-xq^{n-1}\right), \quad (x;q)_0 = 1.$$

The q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear to see that $\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$.

In [4, Theorem 1], Shi and Pan showed that for prime $p \geq 5$,

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q} \equiv \frac{(p-1)(1-q)}{2} + \frac{(p^2-1)(1-q)^2 [p]_q}{24} \pmod{[p]_q^2}.$$

In [5, Lemma 4.1], Sun established an interesting congruence:

$$\sum_{k=1}^{p-1} \frac{x^k}{k^2} \equiv \frac{1}{p}\left(\frac{1+(x-1)^p - x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k - 1}{k}\right) + p\sum_{k=2}^{p-1} \frac{x^k}{k^2} H_{k-1} \pmod{p^2},$$

where x is a real number and p is an odd prime. Thus, it is easy to get the following congruence:

$$\sum_{k=1}^{p-1}\frac{x^k}{k^2}\equiv\frac{1}{p}\left(\frac{1+(x-1)^p-x^p}{p}-\sum_{k=1}^{p-1}\frac{(1-x)^k-1}{k}\right)\pmod p. \tag{1.3}$$

In [6, Theorem 2.1], Tauraso proved that if p is an odd prime, and a,b,d are integers such that $a,d>0, b\geq 0$ and gcd(a, p)=1, then

$$\sum_{k=1}^{p-1}\frac{q^{bk}}{[ak]_q^d}\equiv\frac{(1-q)^d}{p^d}\left((-1)^d p\sum_{s=0}^{d-1}c_s\binom{r_0+sp}{d}-\sum_{s=0}^{d}(-1)^s\binom{d}{s}\binom{sp}{2d}\right)\pmod{[p]_q},$$

where

$$r_0\equiv\frac{-b}{a}\pmod p,\quad r_0\in\{0,1,\ldots,p-1\},$$
$$c_s=\sum_{k=0}^{s}(-1)^{s-k}\binom{r_0+kp+d-1}{d-1}\binom{d}{s-k}.$$

The nth cyclotomic polynomial is given by

$$\Phi_n(q)=\prod_{\substack{1\leq k\leq n\\(n,k)=1}}(q-\zeta^k),$$

where ζ denotes a primitive nth root of unity.

The first aim of the paper is to give two new q-analogues of (1.1).

Theorem 1.1. *For any positive integer n and variable x, we have*

$$\sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q}\equiv\frac{1-(qx)^n-(qx;q)_n}{[n]_q}+\frac{(n-1)(1-q)(x^n+1)}{2}\pmod{\Phi_n(q)}, \tag{1.4}$$

and

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{q^{-k}(x;q^{-1})_k}{[k]_q}&\\ \equiv&\frac{1-(qx)^n-(qx;q)_n}{[n]_q}+\frac{(q-1)(x-1)}{x}\\ &+\frac{x(n-1)(1-q)(x^n+1)-2(1-q)(1-xq^{1-n})(x;q^{-1})_{n-1}}{2x}\pmod{\Phi_n(q)}.\end{aligned} \tag{1.5}$$

Letting $q\to 1$ and $n=p$ in Theorem 1.1, (1.4) and (1.5) will reduce to (1.1). So (1.4) and (1.5) are q-analogues of (1.1).

When $x=-q$ and $n=p$, we will see that Theorem 1.1 is a q-analogue of [3, (5.5)]:

$$\frac{2^{p-1}-1}{p}\equiv-\sum_{j=1}^{p-1}\frac{2^{j-1}}{j}\pmod p.$$

The second aim of the paper is to examine two q-analogues of (1.3).

Theorem 1.2. *For any odd positive integer n and variable x, we have*

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{x^k}{[k]_q^2}\equiv&\left(\frac{(n-1)(1-q)}{2}-\frac{1}{[n]_q}\right)\sum_{k=1}^{n-1}\frac{(q^{-1}x;q^{-1})_k-1}{[k]_q}\\&+\frac{t_0\left(1-(q^{-1}x;q^{-1})_n\right)-(q^{-1}x)^n}{[n]_q^2}\qquad\pmod{\Phi_n(q)},\end{aligned}\tag{1.6}$$

and

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{q^k(x;q)_k}{[k]_q^2}\equiv&\left(\frac{(n-1)(1-q)}{2}-\frac{1}{[n]_q}\right)\sum_{k=1}^{n-1}\frac{x^k-1}{[k]_q}\\&+\frac{t_0(1-x^n)-(x;q)_n}{[n]_q^2}\qquad\pmod{\Phi_n(q)},\end{aligned}\tag{1.7}$$

where $t_0=1-\frac{(n-1)(1-q^n)}{2}+\frac{(n-1)(n-3)(1-q^n)^2}{8}$.

In particular, it is easy to see that (1.6) is a q-analogue of (1.2) when $x=2$, and (1.7) is a q-analogue of (1.2) when $x=-q$.

We will prove Theorems 1.1 and 1.2 in Sections 2 and 4. In addition, we establish some generalizations of Theorems 1.1 and 1.2 in Sections 3 and 5.

2. Proof of Theorem 1.1

Firstly, we need to build some lemmas.

Lemma 2.1. *For any positive integer n and variable x, we have*

$$\sum_{k=1}^{n-1}\frac{x^k-1}{[k]_q}=\sum_{k=1}^{n-1}\frac{(-q)^k(x;q)_k}{[k]_q}q^{\binom{k+1}{2}-nk}\begin{bmatrix}n-1\\k\end{bmatrix}_q.\tag{2.1}$$

Proof. By [7, (1.1)], for any positive integer j and n

$$\sum_{k=j}^{n}q^{k-j}\begin{bmatrix}k\\j\end{bmatrix}_q=\begin{bmatrix}n+1\\j+1\end{bmatrix}_q.\tag{2.2}$$

Letting $j\to j-1$, $n\to n-2$ and $q\to q^{-1}$ in (2.2) gives

$$q^{(j-1)^2+j-1}\sum_{k=j-1}^{n-2}q^{-jk}\begin{bmatrix}k\\j-1\end{bmatrix}_q=q^{j^2-(n-1)j}\begin{bmatrix}n-1\\j\end{bmatrix}_q.$$

By q-binomial theorem: for variable x, positive integer n

$$x^n=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q\prod_{j=0}^{i-1}(x-q^j).\tag{2.3}$$

Letting $n\to k$ and $q\to q^{-1}$ in (2.3), we have

$$x^k-1=\sum_{i=1}^{k}\begin{bmatrix}k\\i\end{bmatrix}_q(-1)^i(x;q)_i\,q^{\binom{i+1}{2}-ki}.\tag{2.4}$$

Therefore, we have

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{x^k-1}{[k]_q}&=\sum_{k=1}^{n-1}\frac{1}{[k]_q}\sum_{i=1}^{k}(-1)^i\,(x;q)_i\begin{bmatrix}k\\i\end{bmatrix}_q q^{\binom{i+1}{2}-ik}\\&=\sum_{i=1}^{n-1}(-1)^i\,(x;q)_i\,q^{\binom{i+1}{2}}\sum_{k=i}^{n-1}\frac{q^{-ik}}{[k]_q}\begin{bmatrix}k\\i\end{bmatrix}_q\\&=\sum_{i=1}^{n-1}(-1)^i\,(x;q)_i\,q^{\binom{i+1}{2}-i}\sum_{k=i-1}^{n-2}\frac{q^{-ik}}{[k]_q}\begin{bmatrix}k\\i-1\end{bmatrix}_q\\&=\sum_{i=1}^{n-1}\frac{(-q)^i\,(x;q)_i}{[i]_q}q^{\binom{i+1}{2}-ni}\begin{bmatrix}n-1\\i\end{bmatrix}_q.\end{aligned}$$

□

As Shi and Pan [4, Theorem 1] mentioned, the following q-congruence lemma can be proved by the same method. We replace p with n and $[p]$ with $\Phi_n(q)$ in the proof of [4, Theorem 1].

Lemma 2.2. *For any positive integer n, we have*

$$\sum_{k=1}^{n-1}\frac{1}{[k]_q}\equiv\frac{(n-1)\,(1-q)}{2}+\frac{(n^2-1)\,(1-q)^2\,[n]_q}{24}\pmod{\Phi_n(q)^2}. \tag{2.5}$$

Lemma 2.3. *For any positive integer n and variable x, we have*

$$\sum_{k=1}^{n-1}\frac{x^k}{[k]_q}\equiv\frac{1-x^n-(x;q)_n}{[n]_q}+x^n\frac{(n-1)\,(1-q)}{2}\pmod{\Phi_n(q)}. \tag{2.6}$$

Proof. Note that

$$\begin{bmatrix}n-1\\i\end{bmatrix}_q=(-1)^iq^{-\binom{i+1}{2}}\prod_{j=1}^{i}\left(1-\frac{[n]_q}{[j]_q}\right)\equiv(-1)^i\,q^{-\binom{i+1}{2}}\pmod{\Phi_n(q)}, \tag{2.7}$$

and

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{q^k\,(x;q)_k}{[k]_q}&\equiv\sum_{k=1}^{n-1}\frac{q^{n(n-1)/2-nk+k}(x;q)_k}{[k]_q}\\&\equiv-\frac{1}{[n]_q}\sum_{k=1}^{n-1}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n}{2}-nk+\binom{k}{2}}(x;q)_k\\&=-\frac{1}{[n]_q}\sum_{k=1}^{n-1}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}(x;q)_k\\&=-\frac{1}{[n]_q}\left(q^{\binom{n}{2}}x^n-q^{\binom{n}{2}}-(x;q)_n\right)\pmod{\Phi_n(q)}.\end{aligned} \tag{2.8}$$

With the help of (2.1) and (2.7), we have

$$\sum_{k=1}^{n-1}\frac{x^k}{[k]_q}\equiv\sum_{k=1}^{n-1}\frac{q^k\,(x;q)_k}{[k]_q}+\sum_{k=1}^{n-1}\frac{1}{[k]_q}\pmod{\Phi_n(q)}. \tag{2.9}$$

Since

$$\begin{aligned} q^{tn} &= 1-(1-q^{tn}) \\ &= 1-(1-q^n)\left(1+q^n+q^{2n}+\cdots+q^{(t-1)n}\right) \\ &\equiv 1-t\,(1-q^n) \qquad \pmod{\Phi_n(q)^2}, \end{aligned}$$

we have

$$q^{\binom{n}{2}} \equiv 1-\frac{(n-1)\,(1-q^n)}{2} \qquad \pmod{\Phi_n(q)^2}. \tag{2.10}$$

Combining (2.8), (2.9) and (2.10), we get Lemma 2.3. □

Lemma 2.4. *For any positive integer n and variable x, we have*

$$\sum_{k=1}^{n-1}\frac{(x;q^{-1})_k-1}{[k]_q} = \sum_{k=1}^{n-1}\frac{(-qx)^k}{[k]_q}q^{\binom{k+1}{2}-nk}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \tag{2.11}$$

Proof. The q-binomial theorem [2, (3.3.6)]: For variable x, positive integer n,

$$(x;q)_n = \sum_{i=0}^{n}(-1)^i q^{\binom{i}{2}}\begin{bmatrix} n \\ i \end{bmatrix}_q x^i, \tag{2.12}$$

letting $q \to q^{-1}$ in (2.12), we have

$$(x;q^{-1})_n = \sum_{i=0}^{n}(-1)^i q^{\binom{i+1}{2}-ni}\begin{bmatrix} n \\ i \end{bmatrix}_q x^i. \tag{2.13}$$

Using (2.2) and (2.13), we get

$$\begin{aligned} \sum_{k=1}^{n-1}\frac{(x;q^{-1})_k-1}{[k]_q} &= \sum_{k=1}^{n-1}\frac{1}{[k]_q}\sum_{i=1}^{k}(-x)^i\begin{bmatrix} k \\ i \end{bmatrix}_q q^{\binom{i+1}{2}-ik} \\ &= \sum_{i=1}^{n-1}(-x)^i q^{\binom{i+1}{2}}\sum_{k=i}^{n-1}\frac{q^{-ik}}{[k]_q}\begin{bmatrix} k \\ i \end{bmatrix}_q \\ &= \sum_{i=1}^{n-1}(-x)^i\, q^{\binom{i+1}{2}-i}\sum_{k=i-1}^{n-2}\frac{q^{-ik}}{[k]_q}\begin{bmatrix} k \\ i-1 \end{bmatrix}_q \\ &= \sum_{i=1}^{n-1}\frac{(-qx)^i}{[i]_q}q^{\binom{i+1}{2}-ni}\begin{bmatrix} n-1 \\ i \end{bmatrix}_q. \end{aligned}$$

□

Next we will proof (1.4).

By (2.11) we have

$$\sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q} = \sum_{k=1}^{n-1}\frac{(-qx)^k}{[k]_q}q^{\binom{k+1}{2}-nk}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \sum_{k=1}^{n-1}\frac{1}{[k]_q}.$$

Using (2.7), we get

$$\sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q} \equiv \sum_{k=1}^{n-1}\frac{(xq)^k}{[k]_q} + \sum_{k=1}^{n-1}\frac{1}{[k]_q} \pmod{\Phi_n(q)},$$

where we have used the fact that $q^{kn} \equiv 1 \pmod{\Phi_n(q)}$. Finally, use (2.5) and (2.6), letting $x \to qx$ in (2.6), we have

$$\sum_{k=1}^{n-1} \frac{(x;q^{-1})_k}{[k]_q} \equiv \frac{1-(qx)^n-(qx;q)_n}{[n]_q} + \frac{(n-1)(1-q)(x^n+1)}{2} \pmod{\Phi_n(q)}.$$

Next we will give the proof of (1.5).

Let

$$M(k,x,q) = \frac{(x;q^{-1})_k}{[k]_q}.$$

For $k > 1$, it is easy to check that

$$(1-q^k)M(k,x,q) = \left(1-q^{k-1}\right)\left(1-xq^{1-k}\right)M(k-1,x,q). \tag{2.14}$$

Summing both sides of (2.14) over k from 1 to $n-1$, we have

$$\sum_{k=1}^{n-1}\left(1-q^k\right)M(k,x,q) = \sum_{k=1}^{n-1}(1-q^{k-1})\left(1-xq^{1-k}\right)M(k-1,x,q).$$

Then

$$\sum_{k=0}^{n-1}\left(1-q^k\right)M(k,x,q) = \sum_{k=0}^{n-2}\left(1-q^k\right)(1-xq^{-k})M(k,x,q).$$

After simplifying, we get

$$\sum_{k=1}^{n-1}\frac{q^{-k}(x;q^{-1})_k}{[k]_q} = \sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q} - \frac{(1-q)(1-xq^{1-n})(x;q^{-1})_{n-1}-(x-1)(q-1)}{x}. \tag{2.15}$$

Finally, combining (1.4), we get (1.5).

So we complete the proof of Theorem 1.1.

3. Corollary 1

Corollary 3.1. *For any positive integer n and variable x, we have*

$$\begin{aligned}\sum_{1\le k\le j\le n-1} q^{-j}\frac{(x;q^{-1})_k}{[k]_q} &\equiv q^{1-n}\left(1-(xq;q)_n\right)-q\left(x^n-1\right)\\ &+\frac{2q\left(1-xq^{1-n}\right)(x;q^{-1})_{n-1}-x\left(q^{1-n}-q\right)\left(x^n+1\right)(n-1)-2q}{2x} \pmod{\Phi_n(q)}.\end{aligned} \tag{3.1}$$

When $q \to 1$, $x = 2$ and $n = p$, the corollary reduce to the following congruence:

$$\sum_{k=1}^{n-1}\sum_{j=1}^{k}\frac{(-1)^j}{j} \equiv -2q_p(2) \pmod{p},$$

where $q_p(2) = \frac{2^{p-1}-1}{p}$.

Proof. Note that

$$\begin{aligned}\sum_{1\le k\le j\le n-1} q^{-j}\frac{(x;q^{-1})_k}{[k]_q} &= \sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q}\sum_{j=k}^{n-1}q^{-j}\\ &= \sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q}\cdot\frac{q^{-k}-q^{-n}}{1-q^{-1}}\\ &= \frac{q}{q-1}\sum_{k=1}^{n-1}\frac{q^{-k}(x;q^{-1})_k}{[k]_q} - \frac{q}{(q-1)q^n}\sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q}.\end{aligned}$$

Then combining Theorem 1.1, we get (3.1). □

Corollary 3.2. *For any positive integer n, and variable x, we have*

$$\begin{aligned}\sum_{1\le k\le j\le n-1} q^{j-k}\frac{(x;q^{-1})_k}{[k]_q} \equiv& \frac{2q^n\left((x;q^{-1})_{n-1}-1\right)+x(x^n+1)(n-1)(1-q^n)}{2x}\\ &-(xq;q)_n - q(x;q^{-1})_{n-1} - q^n\left(x^n-1\right)+1 \pmod{\Phi_n(q)^2}.\end{aligned}$$

When $q\to 1$, $x=2$ and $n=p$, the corollary reduce to the following congruence:

$$\sum_{k=1}^{n-1}\sum_{j=1}^{k}\frac{(-1)^j}{j} \equiv -2q_p(2)+q_p(2)^2p \pmod{p^2}.$$

Proof.

$$\begin{aligned}\sum_{1\le k\le j\le n-1} q^{j-k}\frac{(x;q^{-1})_k}{[k]_q} &= \sum_{k=1}^{n-1}\frac{q^{-k}(x;q^{-1})_k}{[k]_q}\sum_{j=k}^{n-1}q^{j}\\ &= \sum_{k=1}^{n-1}\frac{q^{-k}(x;q^{-1})_k}{[k]_q}\cdot\frac{q^k-q^n}{1-q}\\ &= \frac{1}{1-q}\sum_{k=1}^{n-1}\frac{(x;q^{-1})_k}{[k]_q} - \frac{q^n}{1-q}\sum_{k=1}^{n-1}\frac{q^{-k}(x;q^{-1})_k}{[k]_q}.\end{aligned}$$

Combining (1.4) and (2.15), we get Corollary 3.2, where we have used the fact that $q^n\equiv 1 \pmod{\Phi_n(q)}$, so the result can modulo $\Phi_n(q)^2$. □

4. Proof Of Theorem 1.2

In order to prove Theorem 1.2, we need the following lemma.

Lemma 4.1. *For any positive integer n, variable x, we have*

$$\sum_{k=1}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\frac{x^k}{[k]_q} = q^{\binom{n}{2}}\sum_{k=1}^{n}\frac{(x;q^{-1})_k-1}{[k]_q} \tag{4.1}$$

and

$$\sum_{k=1}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\frac{(x;q)_k}{[k]_q} = q^{\binom{n}{2}}\sum_{k=1}^{n}\frac{x^k-1}{[k]_q}. \tag{4.2}$$

Proof. The q-binomial coefficients satisfy the following recurrence relation:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

It is easy to see that

$$q^{\binom{n-k}{2}} = q^{\binom{n}{2}+\binom{k+1}{2}-nk}. \tag{4.3}$$

So, we have

$$\begin{aligned} &\sum_{k=1}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{x^k}{[k]_q} \\ =&\sum_{k=1}^{n}(-1)^k \left(q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right) q^{\binom{n-k}{2}} \frac{x^k}{[k]_q} \\ =&q^{n-1}\sum_{k=1}^{n-1}(-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{\binom{n-1-k}{2}} \frac{x^k}{[k]_q} + \frac{1}{[n]_q}\sum_{k=1}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} x^k. \end{aligned} \tag{4.4}$$

By induction and using (2.13), (4.3) and (4.4), we have

$$\begin{aligned} \sum_{k=1}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{x^k}{[k]_q} =&q^{n-1}\sum_{k=1}^{n-1}(-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{\binom{n-1-k}{2}} \frac{x^k}{[k]_q} + \frac{q^{\binom{n}{2}}}{[n]_q}((x;q^{-1})_n - 1) \\ =&q^{\binom{n}{2}} \sum_{k=1}^{n} \frac{(x;q^{-1})_k - 1}{[k]_q}. \end{aligned}$$

Using (2.4), and (4.3), we have

$$q^{\binom{n}{2}} x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (-1)^k (x;q)_k . \tag{4.5}$$

So, we have

$$\begin{aligned} &\sum_{k=1}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(x;q)_k}{[k]_q} \\ =&\sum_{k=1}^{n}(-1)^k \left(q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right) q^{\binom{n-k}{2}} \frac{(x;q)_k}{[k]_q} \\ =&q^{n-1}\sum_{k=1}^{n-1}(-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{\binom{n-1-k}{2}} \frac{(x;q)_k}{[k]_q} + \frac{1}{[n]_q}\sum_{k=1}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (x;q)_k. \end{aligned} \tag{4.6}$$

By induction and using (4.5) and (4.6), we have

$$\begin{aligned} \sum_{k=1}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(x;q)_k}{[k]_q} &= q^{n-1}\sum_{k=1}^{n-1}(-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{\binom{n-1-k}{2}} \frac{x^k}{[k]_q} + \frac{q^{\binom{n}{2}}}{[n]_q}(x^n - 1) \\ &= q^{\binom{n}{2}} \sum_{k=1}^{n} \frac{x^k - 1}{[k]_q}. \end{aligned}$$

□

Next we will proof Theorem 1.2.

For any odd positive integer n, variable x, we have

(4.7)
$$\begin{aligned}\sum_{i=1}^{n-1}\frac{(xq)^i}{[i]_q^2} &\equiv \sum_{i=1}^{n-1}\frac{q^{n(n-1)/2-ni}(xq)^i}{[i]_q^2}\\ &\equiv -\frac{1}{[n]_q}\sum_{i=1}^{n-1}(-1)^i\begin{bmatrix}n\\ i\end{bmatrix}_q q^{\binom{n-i}{2}}\frac{x^i}{[i]_q}\\ &= -\frac{q^{\binom{n}{2}}}{[n]_q}\sum_{i=1}^{n}\frac{(x;q^{-1})_i-1}{[i]_q}-\frac{x^n}{[n]_q^2}\\ &= -\frac{q^{\binom{n}{2}}}{[n]_q}\left(\sum_{i=1}^{n-1}\frac{(x;q^{-1})_i-1}{[i]_q}+\frac{(x;q^{-1})_n-1}{[n]_q}\right)-\frac{x^n}{[n]_q^2}\\ &= -\frac{q^{\binom{n}{2}}}{[n]_q}\sum_{i=1}^{n-1}\frac{(x;q^{-1})_i-1}{[i]_q}+\frac{q^{\binom{n}{2}}\left(1-(x;q^{-1})_n\right)-x^n}{[n]_q^2} \qquad (\mathrm{mod}\ \Phi_n(q)).\end{aligned}$$

In the first step we need n to be an odd positive integer, otherwise it doesn't work, where in the third step we used (4.1).

Obviously, we have,

$$\begin{aligned}q^{tn} &= 1-(1-q^n)\left(1+q^n+q^{2n}+\cdots+q^{(t-1)n}\right)\\ &\equiv 1-(1-q^n)\left(t-\frac{t(t-1)(1-q^n)}{2}\right) \qquad (\mathrm{mod}\ \Phi_n(q)^3).\end{aligned}$$

Note that

$$q^{\binom{n}{2}} \equiv 1-\frac{(n-1)(1-q^n)}{2}+\frac{(n-1)(n-3)(1-q^n)^2}{8} \qquad (\mathrm{mod}\ \Phi_n(q)^3), \tag{4.8}$$

where we let $t_0 = 1-\frac{(n-1)(1-q^n)}{2}+\frac{(n-1)(n-3)(1-q^n)^2}{8}$. Then, letting $x \to q^{-1}x$ in (4.7) and combining (2.10), (4.7) and (4.8), we get(1.6).

Furthermore,

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{q^i(x;q)_i}{[i]_q^2} &\equiv \sum_{i=1}^{n-1}\frac{q^{n(n-1)/2-ni}(x;q)_i}{[i]_q^2}\\ &\equiv -\frac{1}{[n]_q}\sum_{i=1}^{n-1}(-1)^i\begin{bmatrix}n\\ i\end{bmatrix}_q q^{\binom{n-i}{2}}\frac{(x;q)_i}{[i]_q}\\ &= -\frac{q^{\binom{n}{2}}}{[n]_q}\sum_{i=1}^{n}\frac{x^i-1}{[i]_q}-\frac{(x;q)_n}{[n]_q^2}\\ &= -\frac{q^{\binom{n}{2}}}{[n]_q}\sum_{i=1}^{n-1}\frac{x^i-1}{[i]_q}+\frac{q^{\binom{n}{2}}(1-x^n)-(x;q)_n}{[n]_q^2} \quad (\mathrm{mod}\ \Phi_n(q)),\end{aligned} \tag{4.9}$$

where in the third step we have used (4.2). Combining (2.10), (4.8) and (4.9), we get (1.7).

5. Corollary 2

Corollary 5.1. *For any odd positive integer n and variable x, we have*

(5.1)
$$\sum_{j=1}^{n-1}\frac{x^j}{[j]_q}\sum_{k=1}^{j}\frac{x^k}{[k]_q}\equiv\frac{1}{2}\left(\frac{1-x^n-(x;q)_n}{[n]_q}+\frac{x^n(n-1)(1-q)}{2}\right)^2 +\frac{t_0\left(1-(q^{-1}x^2;q^{-1})_n\right)-(q^{-1}x^2)^n}{2[n]_q^2} -\left(\frac{1}{2[n]_q}-\frac{(n-1)(1-q)}{4}\right)\sum_{k=1}^{n-1}\frac{(q^{-1}x^2;q^{-1})_k-1}{[k]_q}\pmod{\Phi_n(q)}.$$

Proof. Note that

$$\sum_{j=1}^{n-1}\frac{x^j}{[j]_q}\sum_{k=1}^{j}\frac{x^k}{[k]_q}=\sum_{k=1}^{n-1}\frac{x^k}{[k]_q}\sum_{j=k}^{n-1}\frac{x^j}{[j]_q} =\sum_{k=1}^{n-1}\frac{x^k}{[k]_q}\left(\sum_{j=1}^{n-1}\frac{x^j}{[j]_q}-\sum_{j=1}^{k}\frac{x^j}{[j]_q}+\frac{x^k}{[k]_q}\right) =\sum_{k=1}^{n-1}\frac{x^k}{[k]_q}\sum_{j=1}^{n-1}\frac{x^j}{[j]_q}-\sum_{k=1}^{n-1}\frac{x^k}{[k]_q}\sum_{j=1}^{k}\frac{x^j}{[j]_q}+\sum_{k=1}^{n-1}\frac{x^{2k}}{[k]_q^2}.$$

Then,

$$\sum_{j=1}^{n-1}\frac{x^j}{[j]_q}\sum_{k=1}^{j}\frac{x^k}{[k]_q}=\frac{1}{2}\left(\sum_{k=1}^{n-1}\frac{x^k}{[k]_q}\right)^2+\frac{1}{2}\sum_{k=1}^{n-1}\frac{x^{2k}}{[k]_q^2}. \tag{5.2}$$

By (1.6), letting $x\to x^2$, we get,

$$\sum_{k=1}^{n-1}\frac{x^{2k}}{[k]_q^2}\equiv\left(\frac{(n-1)(1-q)}{2}-\frac{1}{[n]_q}\right)\sum_{k=1}^{n-1}\frac{(q^{-1}x^2;q^{-1})_k-1}{[k]_q} +\frac{q^{\binom{n}{2}}\left(1-(q^{-1}x^2;q^{-1})_n\right)-(q^{-1}x^2)^n}{[n]_q^2}\pmod{\Phi_n(q)}. \tag{5.3}$$

Then combining (2.6), (2.10), (4.8), (5.2) and (5.3), we get Corollary 5.1. □

For example, let $x=-1$. We see that for any odd positive integer n,

$$\sum_{k=1}^{n-1}\frac{(-1)^k}{[k]_q}\sum_{j=1}^{k}\frac{(-1)^j}{[j]_q}\equiv 2Q_n^2(2,q)+(n-1)(1-q) +\frac{(n^2-1)(1-q)^2}{12}\pmod{\Phi_n(q)}, \tag{5.4}$$

where $Q_n(2,q)=\frac{(-q;q)_{n-1}-1}{[n]_q}$. The case $n=p\geq 5$ in (5.4) is Theorem 1.4 in [8].

We replace x with -1 in (5.2), and obtain

$$\sum_{j=1}^{n-1}\frac{(-1)^j}{[j]_q}\sum_{k=1}^{j}\frac{(-1)^k}{[k]_q}=\frac{1}{2}\left(\sum_{k=1}^{n-1}\frac{(-1)^k}{[k]_q}\right)^2+\frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{[k]_q^2}. \tag{5.5}$$

As Shi and Pan [4, (5)] mentioned, the following q-congruence can be proved by the same method. We replace p with n and $[p]$ with $\Phi_n(q)$ in the proof of [4, (5)]:

$$\sum_{k=1}^{n-1}\frac{1}{[k]_q^2}\equiv-\frac{(n-1)(n-5)(1-q)^2}{12}\pmod{\Phi_n(q)}. \tag{5.6}$$

Meanwhile, as B. He [8, (1.7)] mentioned, We replace p with n and $[p]$ with $\Phi_n(q)$ in the proof of [8, (1.7)]:

$$\sum_{k=1}^{n-1}\frac{(-1)^k}{[k]_q}\equiv-2Q_n(2,q)-\frac{(n-1)(1-q)}{2}\pmod{\Phi_n(q)}. \tag{5.7}$$

Finally, combining (5.5), (5.6) and (5.7), we get (5.4). Meanwhile, let $x=-1$ in (5.1), we also can simplify to get (5.4), so we omit this part. We replace n with p and $\Phi_n(q)$ with $[p]_q$ in (5.4). This consequence will be [8, Theorem 1.4].

Corollary 5.2. *For any odd positive integer n and variable x,*

$$\begin{aligned}\sum_{1\leq k\leq j\leq n-1}\frac{q^jx^k}{[k]_q^2}\equiv&\frac{t_0\left(1-(q^{-1}x;q^{-1})_n\right)-(q^{-1}x)^n}{[n]_q}-\sum_{k=1}^{n-1}\frac{x^k}{[k]_q}\\&-\frac{2-(n-1)(1-q^n)}{2}\sum_{k=1}^{n-1}\frac{(q^{-1}x;q^{-1})_k-1}{[k]_q}\pmod{\Phi_n(q)^2}.\end{aligned} \tag{5.8}$$

Proof. Note that

$$\begin{aligned}\sum_{1\leq k\leq j\leq n-1}\frac{q^jx^k}{[k]_q^2}&=\sum_{k=1}^{n-1}\frac{x^k}{[k]_q^2}\sum_{j=k}^{n-1}q^j\\&=\sum_{k=1}^{n-1}\frac{x^k}{[k]_q^2}\left(\frac{1-q^n}{1-q}-\frac{1-q^k}{1-q}\right)\\&=\frac{1-q^n}{1-q}\sum_{k=1}^{n-1}\frac{x^k}{[k]_q^2}-\sum_{k=1}^{n-1}\frac{x^k}{[k]_q}.\end{aligned} \tag{5.9}$$

Finally, combining (1.6) and (5.9), we get the desired result. □

For example, when $x=1$, $x=q^k$, $x=(-1)^k$ in (5.8), we will get

(5.10)
$$\sum_{j=1}^{n-1} q^j \sum_{k=1}^{j} \frac{1}{[k]_q^2} \equiv -\frac{(n-1)(1-q)}{2} - \frac{(n-1)(n-3)(1-q)^2[n]_q}{8} \pmod{\Phi_n(q)^2},$$

(5.11)
$$\sum_{j=1}^{n-1} q^j \sum_{k=1}^{j} \frac{q^k}{[k]_q^2} \equiv \frac{(n-1)(1-q)}{2} - \frac{(n^2-1)(1-q)^2[n]_q}{8} \pmod{\Phi_n(q)^2},$$

and

(5.12)
$$\begin{aligned}&\sum_{j=1}^{n-1} q^j \sum_{k=1}^{j} \frac{(-1)^k}{[k]_q^2}\\ \equiv&2Q_n(2,n) + \frac{(n-1)(1-q)}{2}\\ &-\left(Q_n(2,q)^2 + 3Q_n(2,q)(1-q) + \frac{(n+7)(n-1)(1-q)^2}{12}\right)[n]_q \pmod{\Phi_n(q)^2},\end{aligned}$$

where n is an odd positive integer.

As B. He [8, Theorem 1.2] mentioned, we replace p with n and $[p]$ with $\Phi_n(q)$ in the proof of [8, Theorem 1.2]:

(5.13)
$$\begin{aligned}\sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q} \equiv& -2Q_n(2,q) - \frac{(n-1)(1-q)}{2}\\ &+\left(Q_n(2,q)^2 + Q_n(2,q)(1-q) + \frac{(n^2-1)(1-q)}{12}\right)[n]_q \pmod{\Phi_n(q)^2}\end{aligned}$$

and

$$\sum_{k=1}^{n-1} \frac{(-1)^k}{[k]_q^2} \equiv -2Q_n(2,q)(1-q) + \frac{(1-n)(1-q)^2}{2} \pmod{\Phi_n(q)}, \tag{5.14}$$

Furthermore, note that

$$\sum_{k=1}^{n-1} \frac{1}{[k]_q} = \sum_{k=1}^{n-1} \frac{1-q^k+q^k}{[k]_q} = (1-q)(n-1) + \sum_{k=1}^{n-1} \frac{q^k}{[k]_q} \tag{5.15}$$

and

$$\sum_{k=1}^{n-1} \frac{1}{[k]_q^2} = (1-q)\sum_{k=1}^{n-1} \frac{1}{[k]_q} + \sum_{k=1}^{n-1} \frac{q^k}{[k]_q^2}. \tag{5.16}$$

Then combining (2.6), (5.6),(5.9), (5.13), (5.14), (5.15) and (5.16), it is not difficult for us to get (5.10), (5.11) and (5.12). Meanwhile, letting $x=1$, $x=q$ and $x=-1$ in (5.8), we also can simplify to get (5.10), (5.11) and (5.12), so we omit this part. We replace n with p and $\Phi_n(q)$ with $[p]_q$ in (5.10), (5.11) and (5.12). These consequences will be [8, (1.3)], [8, (1.4)] and [8, Theorem 1.3].

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