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SOME IDENTITIES IN RINGS AND NEAR-RINGS WITH DERIVATIONS

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Abstract. In the present paper we investigate commutativity in prime rings and 3-prime near-rings admitting a generalized derivation satisfying certain algebraic identities. Some well-known results characterizing commutativity of prime rings and 3-prime near-rings have been generalized.

1. Introduction

In this paper, $\mathcal{N}$ will denote a right near-ring with center $Z(\mathcal{N})$. A near-ring $\mathcal{N}$ is called zero-symmetric if $x0 = 0$ for all $x \in \mathcal{N}$ (recall that right distributivity yields $0x = 0$). A non empty subset U of $\mathcal{N}$ is said to be a semigroup left (resp. right) ideal of $\mathcal{N}$ if $\mathcal{N}U \subseteq U$ (resp. $U\mathcal{N} \subseteq U$) and if U is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $\mathcal{N}$. As usual for all x, y in $\mathcal{N}$, the symbol $[x, y]$ stands for Lie product (commutator) $xy - yx$ and $x \circ y$ stands for Jordan product (anticommutator) $xy + yx$. We note that for a near-ring, $-(x + y) = -y - x$. Recall that $\mathcal{N}$ is 3-prime if for a, b in $\mathcal{N}$, $a\mathcal{N}b = \{0\}$ implies that $a = 0$ or $b = 0$. $\mathcal{N}$ is said to be 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{N}$, then $x = 0$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [20], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. The concept of derivation in rings has been generalized in several ways by various authors. Generalized derivation has been introduced already in rings by M. Brešar [10]. Also the notions of generalized derivation has been introduced in near-rings by Öznur Gölbasi [14]. An additive mapping $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ is called a right generalized derivation with associated derivation d if $\mathcal{F}(xy) = \mathcal{F}(x)y + xd(y)$ for all $x, y \in \mathcal{N}$ and $\mathcal{F}$ is called a left generalized derivation with associated derivation d if

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$\mathcal{F}(xy) = d(x)y + x\mathcal{F}(y)$, for all $x, y \in \mathcal{N}$. $\mathcal{F}$ is called a generalized derivation with associated derivation d if it is both a left as well as a right generalized derivation with associated derivation d. An additive mapping $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ is said to be a left (resp. right) multiplier (or centralizer) if $\mathcal{F}(xy) = \mathcal{F}(x)y$ (resp. $\mathcal{F}(xy) = x\mathcal{F}(y)$) holds for all $x, y \in \mathcal{N}$. $\mathcal{F}$ is said to be a multiplier if it is both left as well as right multiplier. Notice that a right (resp. left) generalized derivation with associated derivation $d = 0$ is a left (resp. right) multiplier. Over the past few years, many authors have investigated commutativity of prime and semi-prime rings admitting suitably constrained derivations [3, 11–13, 16, 18] and [19]. Some comparable results on near-rings have also been derived, see e.g. [1, 2, 4, 7, 9, 15] and [17]. In [11] the authors showed that a prime ring $\mathcal{R}$ must be commutative if it admits a derivation d such that either $d([x, y]) = [x, y]$ for all $x, y \in K$ or $d([x, y]) = -[x, y]$ for all $x, y \in K$, where K is a nonzero ideal of $\mathcal{R}$.

In 2002, Rehman [18] established that if a prime ring of a characteristic not 2 admits a generalized derivation F associated with a nonzero derivation such that $F([x, y]) = [x, y]$ (resp. $F([x, y]) = -[x, y]$) for all x, y in a nonzero square closed Lie ideal U, then $U \subseteq Z(\mathcal{R})$. Quadri, Khan and Rehman [16], without the characteristic assumption on the ring, proved that a prime ring must be commutative if it admits a generalized derivation F, associated with a nonzero derivation, such that $F([x, y]) = [x, y]$ (resp. $F([x, y]) = -[x, y]$) for all x, y in a nonzero ideal I. Motivated by the above results, in the following theorem we explore the commutativity of a prime ring, provided with a generalized derivation F and left multiplier G satisfying the following conditions: $F([x, y]_{\alpha,\beta}) = [x, y]_{u,v}$, $F([x, y]_{\alpha,\beta}) = G([\beta(x), y])$ for all $x, y \in \mathcal{R}$, where α, β, u, v automorphisms of $\mathcal{R}$ and $[x, y]_{\alpha,\beta} = \alpha(x)y - y\beta(x)$.

2. Some Preliminaries

For the proofs of our main theorems, we need the following lemmas. The first lemmas appear in [7] and [20] in the context of left near-rings, and it is easy to see that they hold for right near-rings as well.

Lemma 2.1. *Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. Let d be a nonzero derivation on $\mathcal{N}$.*

(i) *If $x, y \in \mathcal{N}$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.*
(ii) *If $x \in \mathcal{N}$ and $xU = \{0\}$ or $Ux = \{0\}$, then $x = 0$.*
(iii) *If $z \in Z(\mathcal{N})$, then $d(z) \in Z(\mathcal{N})$.*

Lemma 2.2. *Let d be an arbitrary derivation of a near-ring $\mathcal{N}$. Then $\mathcal{N}$ satisfies the following partial distributive laws:*

(i) *$z(xd(y) + d(x)y) = zxd(y) + zd(x)y$ for all $x, y, z \in \mathcal{N}$;*
(ii) *$z(d(x)y + xd(y)) = zd(x)y + zxd(y)$ for all $x, y, z \in \mathcal{N}$.*

Lemma 2.3. ([5, Theorem 2.1]). *Let $\mathcal{N}$ be a 3-prime near-ring, U a nonzero semigroup left ideal or semigroup right ideal. If $\mathcal{N}$ admits a nonzero derivation d such that $d(U) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

3. Some Results Involving Prime Rings

Theorem 3.1. *Let $\mathcal{R}$ be a prime ring, I a nonzero ideal of $\mathcal{R}$ and α, β, u, v automorphisms of $\mathcal{R}$ such that $\beta(I) = I$. If F is a generalized derivation of $\mathcal{R}$ associated with a derivation d and G is a left multiplier of $\mathcal{R}$ which satisfy one of the following conditions:*

(i) *$F([x,y]_{\alpha,\beta}) = [x,y]_{u,v}$ for all $x, y \in I$;*
(ii) *$F([x,y]_{\alpha,\beta}) = G([\beta(x), y])$ for all $x, y \in I$,*

then $\mathcal{R}$ is commutative.

Proof. (i) Suppose that

$$F([x,y]_{\alpha,\beta}) = [x,y]_{u,v}, \quad \text{for all } x, y \in I. \tag{3.1}$$

Replacing y by $y\beta(x)$ in (3.1), and using the fact that $[x, y\beta(x)]_{\alpha,\beta} = [x,y]_{\alpha,\beta}\beta(x)$ and $[x, y\beta(x)]_{u,v} = [x,y]_{u,v}\beta(x) + y[v(x), \beta(x)]$ for all $x, y \in I$, we arrive at

$$F([x,y]_{\alpha,\beta})\beta(x) + [x,y]_{\alpha,\beta}d(\beta(x)) = [x,y]_{u,v}\beta(x) + y[v(x), \beta(x)], \quad \text{for all } x, y \in I. \tag{3.2}$$

Using (3.1), (3.2) implies that

$$[x,y]_{\alpha,\beta}d(\beta(x)) = y[v(x), \beta(x)], \quad \text{for all } x, y \in I. \tag{3.3}$$

Substituting ry instead of y in (3.3) where $r \in \mathcal{R}$, we arrive at

$$[\alpha(x), r]Id(\beta(x)) = \{0\}, \quad \text{for all } x \in I, r \in \mathcal{R}.$$

By Lemma 2.1 (i), we get $[\alpha(x), r] = 0$ or $d(\beta(x)) = 0$ for all $x \in I$, $r \in \mathcal{R}$ which gives $\alpha(x) \in Z(\mathcal{R})$ or $d(\beta(x)) = 0$ for all $x \in I$. Since α and β are automorphisms of $\mathcal{R}$, we get $x \in Z(\mathcal{R})$ or $d(\beta(x)) = 0$ for all $x \in I$. Using Lemma 2.1 (iii), we obtain $d(\beta(I)) \subseteq Z(\mathcal{R})$ i.e, $d(I) \subseteq Z(\mathcal{R})$ which forces that $\mathcal{R}$ is commutative by Lemma 2.3.
(ii) Assume that

$$F([x,y]_{\alpha,\beta}) = G([\beta(x), y]), \quad \text{for all } x, y \in I. \tag{3.4}$$

Putting $y\beta(x)$ instead of y in (3.4), we get

$$F([x,y]_{\alpha,\beta})\beta(x) + [x,y]_{\alpha,\beta}d(\beta(x)) = G([\beta(x), y])\beta(x), \quad \text{for all } x, y \in I.$$

Using (3.4), we obtain $[x,y]_{\alpha,\beta}d(\beta(x)) = 0$ for all $x, y \in I$, which implies that

$$\alpha(x)yd(\beta(x)) = y\beta(x)d(\beta(x)), \quad \text{for all } x, y \in I. \tag{3.5}$$

Taking ry in place of y in (3.5) where $r \in \mathcal{R}$ and using it again, we conclude that

$$[\alpha(x), r]Id(\beta(x)) = \{0\}, \quad \text{for all } x \in I, r \in \mathcal{R}.$$

By Lemma 2.1 (i), we get $\alpha(x) \in Z(\mathcal{R})$ or $d(\beta(x)) = 0$ for all $x \in \mathcal{R}$ and using the same techniques as used above, we conclude that $\mathcal{R}$ is commutative. □

For $\alpha = \beta = u = v = id_{\mathcal{R}}$, we get the following result.

Corollary 3.1. ([16, Theorem 2.1]). *Let $\mathcal{R}$ be a prime ring and I a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y] = [x, y]$ for all $x, y \in I$, then $\mathcal{R}$ is commutative.*

For $\alpha = \beta = u = id_{\mathcal{R}}$ and $v = -id_{\mathcal{R}}$, we get the following result.

Corollary 3.2. ([16, Theorem 2.2]). *Let $\mathcal{R}$ be a prime ring and I a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y] + [x, y] = 0$ for all $x, y \in I$, then $\mathcal{R}$ is commutative.*

4. Some Results Involving 3-Prime Near-Rings

In this section, we will present a very important result that generalizes several theorems that are well known in the literature. More precisely, we will show that a 2-torsion prime near-ring $\mathcal{N}$ is a commutative ring if and only if $\mathcal{N}$ admits a derivation d and a left multiplier G such that $G([x, y]) = [d(x), y] - [x, d(y)]$ for all $x, y \in U$.

Theorem 4.1. *Let $\mathcal{N}$ be a 2-torsion free prime near-ring and U a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d and left multiplier G, then the following assertions are equivalents:*

(i) *$G([x, y]) = [d(x), y] - [x, d(y)]$ for all $x, y \in U$;*
(ii) *$\mathcal{N}$ is a commutative ring.*

Proof. It is easy to notice that (ii) implies (i).

(i)⇒(ii) Suppose that

$$G([x, y]) = [d(x), y] - [x, d(y)], \quad \text{for all } x, y \in U. \tag{4.1}$$

Replacing x by xy in (4.1) and using the fact that $[xy, y] = [x, y]y$, we obtain

$$[d(xy), y] - [xy, d(y)] = G([x, y])y, \quad \text{for all } x, y \in U.$$

Which implies that

$$[d(xy), y] - [xy, d(y)] = ([d(x), y] - [x, d(y)])y, \quad \text{for all } x, y \in U.$$

Using Lemma 2.2 and by developing the last expression, we arrive at

$$d(x)y^2 + xd(y)y - yxd(y) - yd(x)y + d(y)xy - xyd(y) = d(x)y^2 - yd(x)y + d(y)xy - xd(y)y.$$

For $x = y$, the equation (4.1) and 2-torsion freeness we give easily $d(y)y = yd(y)$ for all $y \in U$. In this case, by a simplification of last equation, we find that

$$xd(y)y = yxd(y), \quad \text{for all } x, y \in U. \tag{4.2}$$

Substituting tx in place of x, where $t \in \mathcal{N}$ in (4.2) and using it again, we arrive at

$$[y, t]Ud(y) = \{0\}, \quad \text{for all } y \in U, t \in \mathcal{N}.$$

Using Lemma 2.1 (i), we obtain

$$y \in Z(\mathcal{N}) \ \text{ or } \ d(y) = 0, \quad \text{for all } y \in U. \tag{4.3}$$

If there exists $y_0 \in Z(\mathcal{N}) \cap U$, then by (4.1), we get $xd(y_0) = d(y_0)x$ for all $x \in U$, in this case, (4.3) gives $xd(y) = d(y)x$ for all $x, y \in U$. Replace x by tx, where $t \in \mathcal{N}$, we get $[d(y), t]x = 0$ for all $x, y \in U$, $t \in \mathcal{N}$ which implies that $[d(y), t]U = \{0\}$ for all $y \in U$, $t \in \mathcal{N}$. Since $U \neq \{0\}$, by Lemma 2.1 (ii), we obtain $d(U) \subseteq Z(\mathcal{N})$ and Lemma 2.3 assures that $\mathcal{N}$ is a commutative ring. $\square$

If we replace G by the null application or the identical application $id_{\mathcal{N}}$, we get the following results.

Corollary 4.1. ([8, Theorem 2.1]). *Let $\mathcal{N}$ be a 2-torsion free prime near-ring. If $\mathcal{N}$ admits a derivation d such that $[d(x), y] = [x, d(y)]$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Corollary 4.2. *Let $\mathcal{N}$ be a 2-torsion free prime near-ring and U a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d, then the following assertions are equivalent:*
- (i) $[x, y] = [d(x), y] - [x, d(y)]$ *for all* $x, y \in U$;
- (ii) $[d(x), y] = [x, d(y)]$ *for all* $x, y \in U$;
- (iii) $\mathcal{N}$ *is a commutative ring.*

When $d = 0$, we have the following result.

Corollary 4.3. *Let $\mathcal{N}$ be a 2-torsion free prime near-ring and U a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left multiplier G, then the following assertions are equivalent:*
- (i) $G([x, y]) = 0$ *for all* $x, y \in U$;
- (ii) $\mathcal{N}$ *is a commutative ring.*

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