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ON THE CHARACTERIZATION OF NON-LINEAR MIXED BI-SKEW JORDAN TRIPLE DERIVATIONS ON $*$-ALGEBRAS

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Abstract. In this article, it is shown that a map $\xi : \mathfrak{A} \to \mathfrak{A}$ (not necessarily linear) satisfies $\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C)$ holds for all $A, B, C \in \mathfrak{A}$ if and only if ξ is an additive $*$-derivation where $\mathfrak{A}$ a unital $*$-algebra over the complex fields $\mathbb{C}$. As applications, we apply our main result to some special classes of unital $*$-algebras such as prime $*$-algebras, standard operator algebras, factor von Neumann algebras and von Neumann algebras with no central summands of type I_1.

1. Introduction

Let $\mathfrak{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. A map $\xi : \mathfrak{A} \to \mathfrak{A}$ is called an additive derivation if $\xi(A + B) = \xi(A) + \xi(B)$ and $\xi(AB) = \xi(A)B + A\xi(B)$ holds for all $A, B \in \mathfrak{A}$. Moreover, ξ is said to be an additive $*$-derivation if it is an additive derivation and $\xi(A^*) = \xi(A)^*$ holds for all $A \in \mathfrak{A}$. For $A, B \in \mathfrak{A}$, define the Jordan product and bi-skew Jordan product of A and B by $A \circ B = AB + BA$ and $A \bullet B = AB^* + BA^*$, respectively. A map $\xi : \mathfrak{A} \to \mathfrak{A}$ (not necessarily linear) is said to be nonlinear Jordan derivation (resp. nonlinear Jordan triple derivation) if $\xi(A \circ B) = \xi(A) \circ B + A \circ \xi(B)$ (resp. $\xi((A \circ B) \circ C) = (\xi(A) \circ B) \circ C + (A \circ \xi(B)) \circ C + (A \circ B) \circ \xi(C))$ holds for all $A, B, C \in \mathfrak{A}$. Analogously, a map $\xi : \mathfrak{A} \to \mathfrak{A}$ (not necessarily linear) is called a nonlinear bi-skew Jordan derivation (resp. nonlinear bi-skew Jordan triple derivation) if $\xi(A \bullet B) = \xi(A) \bullet B + A \bullet \xi(B)$ (resp. $\xi((A \bullet B) \bullet C) = (\xi(A) \bullet B) \bullet C + (A \bullet \xi(B)) \bullet C + (A \bullet B) \bullet \xi(C))$ holds for all $A, B, C \in \mathfrak{A}$. In

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the last decade, many mathematicians devoted themselves to the study of mappings involving new products on various kind of rings and algebras. These kind of new products are playing an increasingly important role in some research topics, and their study has attracted the attention of many authors (see [1, 2, 4–8, 12, 13, 15–18]). In 2016, Zhang [17] studied nonlinear skew Jordan derivations on factor von Neumann algebras and proved that every nonlinear skew Jordan derivation on a factor von Neumann algebra is an addtive $*$-derivation. Later, this result has been extended to skew Jordan triple derivation and skew Jordan-type derivations on $*$-algebras in [8, 18], respectively. Khan [5] proved that every multiplicative bi-skew Jordan triple derivations on a prime $*$-algebra is an additive $*$-derivations. This result has been generalized in [2] by Ashraf et al. where they proved that every nonlinear bi-skew Jordan-type derivation on a factor von Nuemann algebra is an additive $*$-derivation. Recently, many researchers paid more attention to the study of Lie (Jordan) mappings involving two different kinds of products at the same time in the nonlinear settings (see, for examples [9–11, 19, 20]). In [20], Zhou et al. proved that every nonlinear mixed Lie triple derivation on prime $*$-algebra is an additive $*$-derivation. Rehman et al. in [11] proved that every nonlinear mixed Jordan triple derivation on a $*$-algebra is an additive $*$-derivation. Similar kinds of problems has been investigated in [9, 10, 19]. Motivated by the above cited works, in this article, we combine the Jordan product and the bi-skew Jordan product in order to get the mixed bi-skew Jordan triple product $(A \circ B) \bullet C$, where $A, B, C \in \mathfrak{A}$. Correspondingly, a map $\xi : \mathfrak{A} \to \mathfrak{A}$ (not necessarily linear) is called a nonlinear mixed bi-skew Jordan triple derivation if

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C)$$

holds for all $A, B, C \in \mathfrak{A}$. The aim of this article is to find the relationship between nonlinear mixed bi-skew Jordan triple derivations and additive $*$-derivation on arbitrary $*$-algebras. More precisely, we show that, under mild assumptions, every nonlinear mixed bi-skew Jordan triple derivation on a unital $*$-algebra is an additive $*$-derivation.

2. The Main Results

The main result of this article states as follows.

Theorem 2.1. *Let $\mathfrak{A}$ be a $*$-algebra with the unity I and containing a nontrivial projection P_1. Write $P_2 = I - P_1$ and assume that $\mathfrak{A}$ satisfies*

$$X\mathfrak{A}P_k = 0 \Rightarrow X = 0 \quad (k = 1, 2), \tag{2.1}$$

where $X \in \mathfrak{A}$. Then a map $\xi : \mathfrak{A} \to \mathfrak{A}$ (not necessarily linear) satisfies

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C), \tag{2.2}$$

for all $A, B, C \in \mathfrak{A}$ if and only if ξ is an additive $*$-derivation.

Proof. Let us choose an arbitrary nontrivial projection P_1 and write $P_2 = I - P_1$. Then, $\mathfrak{A}$ can be written as $\mathfrak{A} = P_1\mathfrak{A}P_1 + P_1\mathfrak{A}P_2 + P_2\mathfrak{A}P_1 + P_2\mathfrak{A}P_2$.

Let $\mathfrak{A}^+ = \{H \in \mathfrak{A} : H^* = H\}$ and $\mathfrak{A}^- = \{S \in \mathfrak{A} : S^* = -S\}$. Further write $\mathfrak{A}^+_{ii} = P_i\mathfrak{A}^+P_i$, $i = 1, 2$ and $\mathfrak{A}^+_{12} = \{P_1HP_2 + P_2HP_1 : H \in \mathfrak{A}^+\}$. Then, any $H \in \mathfrak{A}^+$ can be written as $H = H_{11} + H_{12} + H_{22}$, where $H_{ii} \in \mathfrak{A}_{ii}$, $i = 1, 2$ and $H_{12} \in \mathfrak{A}^+_{12}$. It is easy to verify that if ξ is an additive $*$-derivation, then it satisfies (2.2). Thus, here we only need to prove the necessity part, which will be established by checking the following series of lemmas. Taking $A = B = C = 0$ in (2.2), the following lemma is easy to obtain. □

Lemma 2.1. $\xi(0) = 0$.

Lemma 2.2. *$\xi(H)^* = \xi(H)$ for every $H \in \mathfrak{A}^+$.*

Proof. For any $H \in \mathfrak{A}^+$, we have $H = \Big(H \circ \frac{I}{2}\Big) \bullet \frac{I}{2}$ Thus, we obtain

$$\begin{aligned}\xi(H) =& \xi\bigg(\Big(H \circ \frac{I}{2}\Big) \bullet \frac{I}{2}\bigg)\\ =& \bigg(\Big(\xi(H) \circ \frac{I}{2}\Big) \bullet \frac{I}{2}\bigg) + \bigg(\Big(H \circ \xi\Big(\frac{I}{2}\Big)\Big) \bullet \frac{I}{2}\bigg) + \bigg(\Big(H \circ \frac{I}{2}\Big) \bullet \xi\Big(\frac{I}{2}\Big)\bigg)\\ =& \xi(H)\frac{I}{2} + \frac{I}{2}\xi(H)^* + \bigg(H\xi\Big(\frac{I}{2}\Big) + \xi\Big(\frac{I}{2}\Big)H\bigg)\frac{I}{2} + \frac{I}{2}\bigg(\xi\Big(\frac{I}{2}\Big)^* H + H\xi\Big(\frac{I}{2}\Big)^*\bigg)\\ &+ H\xi\Big(\frac{I}{2}\Big)^* + \xi\Big(\frac{I}{2}\Big)H,\end{aligned}$$

and hence,

$$\begin{aligned}\xi(H)^* =& \xi(H)\frac{I}{2} + \frac{I}{2}\xi(H)^* + \bigg(H\xi\Big(\frac{I}{2}\Big) + \xi\Big(\frac{I}{2}\Big)H\bigg)\frac{I}{2}\\ &+ \frac{I}{2}\bigg(\xi\Big(\frac{I}{2}\Big)^* H + H\xi\Big(\frac{I}{2}\Big)^*\bigg) + H\xi\Big(\frac{I}{2}\Big)^* + \xi\Big(\frac{I}{2}\Big)H.\end{aligned}$$

That is, $\xi(H) = \xi(H)^*$ for all $H \in \mathfrak{A}^+$. □

Lemma 2.3. *For any $H_{11} \in \mathfrak{A}^+_{11}, H_{12} \in \mathfrak{A}^+_{12}$ and $H_{22} \in \mathfrak{A}^+_{22}$, we have*

$$\xi(H_{11} + H_{12}) = \xi(H_{11}) + \xi(H_{12}) \quad \textit{and} \quad \xi(H_{12} + H_{22}) = \xi(H_{12}) + \xi(H_{22}).$$

Proof. Assume that $T = \xi(H_{11} + H_{12}) - \xi(H_{11}) - \xi(H_{12})$. It is easy to observe that $T^* = T$ by Lemma 2.2. Our target is to show that $T = 0$. Using the fact that $(H_{11} \circ P_2) \bullet P_2 = 0$ and making use of Lemma 2.1, we have

$$\begin{aligned}&\xi\Big(((H_{11} + H_{12}) \circ P_2) \bullet P_2\Big)\\ =& \xi((H_{11} \circ P_2) \bullet P_2) + \xi((H_{12} \circ P_2) \bullet P_2)\\ =& \Big((\xi(H_{11}) + \xi(H_{12})) \circ P_2\Big) \bullet P_2\\ &+ ((H_{11} + H_{12}) \circ \xi(P_2)) \bullet P_2 + ((H_{11} + H_{12}) \circ P_2) \bullet \xi(P_2).\end{aligned}$$

On the other hand, we have

$$\xi\Big(((H_{11}+H_{12})\circ P_2)\bullet P_2\Big) =(\xi(H_{11}+H_{12})\circ P_2)\bullet P_2+((H_{11}+H_{12})\circ\xi(P_2))\bullet P_2 + ((H_{11}+H_{12})\circ P_2)\bullet\xi(P_2).$$

On comparing the above two expressions for $\xi(((H_{11}+H_{12})\circ P_2)\bullet P_2)$, we get $((T\circ P_2)\bullet P_2)=0$. Application of the fact that $T^*=T$ leads us to $T_{12}=T_{21}=T_{22}=0$. Invoking the fact $(H_{12}\circ(P_2-P_1))\bullet P_1=0$ and Lemma 2.1, we find that

$$\begin{aligned}&\xi\Big(((H_{11}+H_{12})\circ(P_2-P_1))\bullet P_1\Big)\\ =&\xi\Big((H_{11}\circ(P_2-P_1))\bullet P_1\Big)+\xi\Big((H_{12}\circ(P_2-P_1))\bullet P_1\Big)\\ =&\Big((\xi(H_{11})+\xi(H_{12}))\circ(P_2-P_1)\Big)\bullet P_1+((H_{11}+H_{12})\circ\xi(P_2-P_1))\bullet P_1\\ &+((H_{11}+H_{12})\circ(P_2-P_1))\bullet\xi(P_1).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\xi\Big(((H_{11}+H_{12})\circ(P_2-P_1))\bullet P_1\Big)\\ =&(\xi(H_{11}+H_{12})\circ(P_2-P_1))\bullet P_1+((H_{11}+H_{12})\circ\xi(P_2-P_1))\bullet P_1\\ &+((H_{11}+H_{12})\circ(P_2-P_1))\bullet\xi(P_1).\end{aligned}$$

From the last two expressions for $\xi(((H_{11}+H_{12})\circ(P_2-P_1))\bullet P_1)$, we obtain $((T\circ(P_2-P_1))\bullet P_1)=0$. Using the fact $T^*=T$ and simplifying gives $T_{11}=0$. Hence, $T=0$, that is, $\xi(H_{11}+H_{12})=\xi(H_{11})+\xi(H_{12})$.

Symmetrically, one can prove that $\xi(H_{12}+H_{22})=\xi(H_{12})+\xi(H_{22})$. □

Lemma 2.4. *For any* $H_{11}\in\mathfrak{A}_{11}^+, H_{12}\in\mathfrak{A}_{12}^+$ *and* $H_{22}\in\mathfrak{A}_{22}^+$, *we have*

$$\xi(H_{11}+H_{12}+H_{22})=\xi(H_{11})+\xi(H_{12})+\xi(H_{22}).$$

Proof. Let $T=\xi(H_{11}+H_{12}+H_{22})-\xi(H_{11})-\xi(H_{12})-\xi(H_{22})$. It is easy to observe that $T^*=T$ by Lemma 2.2. We show that $T=0$. Using the fact that $(H_{11}\circ P_2)\bullet P_2=0$ and Lemmas 2.1 and 2.3, we have

$$\begin{aligned}&\xi\Big(((H_{11}+H_{12}+H_{22})\circ P_2)\bullet P_2\Big)\\ =&\xi((H_{11}\circ P_2)\bullet P_2)+\xi(((H_{12}+H_{22})\circ P_2)\bullet P_2)\\ =&\Big((\xi(H_{11})+\xi(H_{12})+\xi(H_{22}))\circ P_2\Big)\bullet P_2+((H_{11}+H_{12}+H_{22})\circ\xi(P_2))\bullet P_2\\ &+((H_{11}+H_{12}+H_{22})\circ P_2)\bullet\xi(P_2).\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}&\xi\Big(((H_{11}+H_{12}+H_{22})\circ P_2)\bullet P_2\Big)\\ =&\Big(\xi(H_{11}+H_{12}+H_{22})\circ P_2\Big)\bullet P_2+((H_{11}+H_{12}+H_{22})\circ\xi(P_2))\bullet P_2\\ &+((H_{11}+H_{12}+H_{22})\circ P_2)\bullet\xi(P_2).\end{aligned}$$

Comparing the above two expressions for $\xi(((H_{11}+H_{12}+H_{22})\circ P_2)\bullet P_2)$, we find that $(T\circ P_2)\bullet P_2=0$, which in turn implies that $T_{12}=T_{21}=T_{22}=0$.

Invoking the fact $(H_{22} \circ P_1) \bullet P_1 = 0$ and using Lemmas 2.1 and 2.3, we find that

$$\begin{aligned}
&\xi\Big(((H_{11} + H_{12} + H_{22}) \circ P_1) \bullet P_1\Big) \\
=&\xi\Big(((H_{11} + H_{12}) \circ P_2) \bullet P_1\Big) + \xi((H_{22} \circ P_1) \bullet P_1) \\
=&\Big((\xi(H_{11}) + \xi(H_{12}) + \xi(H_{22})) \circ P_1\Big) \bullet P_1 + ((H_{11} + H_{12} + H_{22}) \circ \xi(P_2)) \bullet P_1 \\
&+ ((H_{11} + H_{12} + H_{22}) \circ P_1) \bullet \xi(P_1).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\xi\Big(((H_{11} + H_{12} + H_{22}) \circ P_1) \bullet P_1\Big) \\
=&(\xi(H_{11} + H_{12} + H_{22}) \circ P_1) \bullet P_1 + ((H_{11} + H_{12} + H_{22}) \circ \xi(P_1)) \bullet P_1 \\
&+ ((H_{11} + H_{12} + H_{22}) \circ P_1) \bullet \xi(P_1).
\end{aligned}$$

Comparing the above two expressions for $\xi(((H_{11} + H_{12} + H_{22}) \circ P_1) \bullet P_1)$, we obtain that $(T \circ P_1) \bullet P_1 = 0$ which further implies that $T_{11} = 0$. Hence, $T = 0$, that is,

$$\xi(H_{11} + H_{12} + H_{22}) = \xi(H_{11}) + \xi(H_{12}) + \xi(H_{22}).$$

□

Lemma 2.5. *For any $H_{12}, H_{12}^{'} \in \mathfrak{A}_{12}^{+}$, we have $\xi(H_{12} + H_{12}^{'}) = \xi(H_{12}) + \xi(H_{12}^{'})$.*

Proof. For any $X_{12}, Y_{12} \in \mathfrak{A}_{12}$, let $H_{12} = X_{12} + X_{12}^*$ and $H_{12}^{'} = Y_{12} + Y_{12}^*$. Then, it is easy to calculate that

$$\begin{aligned}
&((P_1 + H_{12}) \circ (P_2 + H_{12}^{'})) \bullet \frac{I}{2} \\
=&\Big((P_1 + X_{12} + X_{12}^*) \circ (P_2 + Y_{12} + Y_{12}^*)\Big) \bullet \frac{I}{2} \\
=&(X_{12} + X_{12}^*) + (Y_{12} + Y_{12}^*) + (X_{12}Y_{12}^* + Y_{12}X_{12}^*) + (X_{12}^*Y_{12} + Y_{12}^*X_{12}) \\
=&H_{12} + H_{12}^{'} + H_{11} + H_{22},
\end{aligned}$$

where $H_{11} = X_{12}Y_{12}^* + Y_{12}X_{12}^* \in \mathfrak{A}_{11}^{+}$ and $H_{22} = X_{12}^*Y_{12} + Y_{12}^*X_{12} \in \mathfrak{A}_{22}^{+}$. Therefore, using Lemmas 2.3 and 2.4, we have

$$\begin{aligned}
&\xi(H_{12} + H_{12}^{'}) + \xi(H_{11}) + \xi(H_{22}^{'}) \\
=&\xi(H_{12} + H_{12}^{'} + H_{11} + H_{22}^{'}) \\
=&\xi\left(((P_1 + X_{12} + X_{12}^*) \circ (P_2 + Y_{12} + Y_{12}^*)) \bullet \frac{I}{2}\right) \\
=&\left(((\xi(P_1) + \xi(X_{12} + X_{12}^*)) \circ (P_2 + Y_{12} + Y_{12}^*)) \bullet \frac{I}{2}\right) \\
&+ \left(((P_1 + X_{12} + X_{12}^*) \circ ((\xi(P_2) + \xi(Y_{12} + Y_{12}^*))) \bullet \frac{I}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(((P_1+X_{12}+X_{12}^*)\circ(P_2+Y_{12}+Y_{12}^*))\bullet\xi\left(\frac{I}{2}\right)\right)\\
=&\xi\left((P_1\circ P_2)\bullet\frac{I}{2}\right)+\xi\left((P_1\circ(Y_{12}+Y_{12}^*))\bullet\frac{I}{2}\right)\\
&+\xi\left(((X_{12}+X_{12}^*)\circ P_2)\bullet\frac{I}{2}\right)+\xi\left(((X_{12}+X_{12}^*)\circ(Y_{12}+Y_{12}^*))\bullet\frac{I}{2}\right)\\
=&\xi(H_{12})+\xi(H_{12}^{'})+\xi(H_{11})+\xi(H_{22}^{'}).
\end{aligned}
$$

Hence, $\xi(H_{12}+H_{12}^{'})=\xi(H_{12})+\xi(H_{12}^{'})$ for any $H_{12}\in\mathfrak{A}_{12}^+$ and $H_{12}^{'}\in\mathfrak{A}_{12}^+$. □

Lemma 2.6. *For any $H_{ii},H_{ii}^{'}\in\mathfrak{A}_{ii}^+$ for $i=1,2$, we have*

$$\xi(H_{11}+H_{11}^{'})=\xi(H_{11})+\xi(H_{11}^{'})\quad\textit{and}\quad\xi(H_{22}+H_{22}^{'})=\xi(H_{22})+\xi(H_{22}^{'}).$$

Proof. Let $T=\xi(H_{11}+H_{11}^{'})-\xi(H_{11})-\xi(H_{11}^{'})$. It is easy to observe that $T^*=T$ by Lemma 2.2. We show that $T=0$.

Using the fact that $(H_{11}\circ P_2)\bullet P_2=(H_{11}^{'}\circ P_2)\bullet P_2=0$ and Lemma 2.1, we obtain

$$
\begin{aligned}
&\xi\Big(((H_{11}+H_{11}^{'})\circ P_2)\bullet P_2\Big)\\
=&\xi((H_{11}\circ P_2)\bullet P_2)+\xi((H_{11}^{'}\circ P_2)\bullet P_2)\\
=&((\xi(H_{11})+\xi(H_{11}^{'}))\circ P_2)\bullet P_2+((H_{11}+H_{11}^{'})\circ\xi(P_2))\bullet P_2\\
&+((H_{11}+H_{11}^{'})\circ P_2)\bullet\xi(P_2).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\xi(((H_{11}+H_{11}^{'})\circ P_2)\bullet P_2)=&(\xi((H_{11}+H_{11}^{'})\circ P_2)\bullet P_2+((H_{11}+H_{11}^{'})\circ\xi(P_2))\bullet P_2\\
&+((H_{11}+H_{11}^{'})\circ P_2)\bullet\xi(P_2).
\end{aligned}
$$

Comparing the above two expressions for $\xi(((H_{11}+H_{11}^{'})\circ P_2)\bullet P_2)$, we find that $(T\circ P_2)\bullet P_2=0$, which in turn gives $T_{12}=T_{21}=T_{22}=0$.

Next, we show that $T_{11}=0$. For this, assume that $H_{12}=X_{12}+X_{12}^*$ for some $X_{12}\in\mathfrak{A}_{12}$, then $H_{12}\in\mathfrak{A}_{12}^+$ and it is easy to observe that

$$\left((H_{11}\circ H_{12})\bullet\frac{I}{2}\right),\left((H_{11}^{'}\circ H_{12})\bullet\frac{I}{2}\right)\in\mathfrak{A}_{12}^+.$$

Thus, using Lemma 2.5, we find that

$$
\begin{aligned}
&\xi\left(((H_{11}+H_{11}^{'})\circ H_{12})\bullet\frac{I}{2}\right)\\
=&\xi\left((H_{11}\circ H_{12})\bullet\frac{I}{2}\right)+\xi\left((H_{11}^{'}\circ H_{12})\bullet\frac{I}{2}\right)\\
=&((\xi(H_{11})+\xi(H_{11}^{'}))\circ H_{12})\bullet\frac{I}{2}+((H_{11}+H_{11}^{'})\circ\xi(H_{12}))\bullet\frac{I}{2}
\end{aligned}
$$

$$+ ((H_{11} + H'_{11}) \circ H_{12}) \bullet \xi\left(\frac{I}{2}\right).$$

On the other hand, we have

$$\xi\left(((H_{11} + H'_{11}) \circ H_{12}) \bullet \frac{I}{2}\right) =(\xi(H_{11} + H'_{11}) \circ H_{12}) \bullet \frac{I}{2} + ((H_{11} + H'_{11}) \circ \xi(H_{12})) \bullet \frac{I}{2}$$
$$+ ((H_{11} + H'_{11}) \circ H_{12}) \bullet \xi\left(\frac{I}{2}\right).$$

From the last two expressions for $\xi\Big(((H_{11} + H'_{11}) \circ H_{12}) \bullet \frac{I}{2}\Big)$, we get $(T \circ H_{12}) \bullet \frac{I}{2} = 0$ and using the fact that $T^* = T$, we get $TX_{12} + TX^*_{12} + X_{12}T + X^*_{12}T = 0$. Multiplying it by P_1 and P_2 from left and right, we obtain $T_{11}XP_2 + P_1XT_{22} = 0$. Since $T_{22} = 0$, we obtain $T_{11}XP_2 = 0$. Application of the condition (2.1) yields $T_{11} = 0$. Hence, $T = 0$, that is, $\xi(H_{11} + H'_{11}) = \xi(H_{11}) + \xi(H'_{11})$. Symmetrically, one can prove that $\xi(H_{22} + H'_{22}) = \xi(H_{22}) + \xi(H'_{22})$. □

Using Lemmas 2.4, 2.5 and 2.6, the following lemma is easy to obtain.

Lemma 2.7. *ξ is additive on $\mathfrak{A}^+$.*

Lemma 2.8. $\xi(I) = 0$.

Proof. It follows from $\frac{I}{2} = \Big(\frac{I}{2} \circ \frac{I}{2}\Big) \bullet \frac{I}{2}$ that

$$\xi\left(\frac{I}{2}\right) =\xi\left(\left(\frac{I}{2} \circ \frac{I}{2}\right) \bullet \frac{I}{2}\right)$$
$$=\left(\xi\left(\frac{I}{2}\right) \circ \frac{I}{2}\right) \bullet \frac{I}{2} + \left(\frac{I}{2} \circ \xi\left(\frac{I}{2}\right)\right) \bullet \frac{I}{2} + \left(\frac{I}{2} \circ \frac{I}{2}\right) \bullet \xi\left(\frac{I}{2}\right).$$

Simplifying with the help of Lemma 2.2, we obtain

$$\xi\left(\frac{I}{2}\right) =\xi\left(\frac{I}{2}\right) + \xi\left(\frac{I}{2}\right) + \xi\left(\frac{I}{2}\right).$$

Hence, $\xi\Big(\frac{I}{2}\Big) = 0$. Since ξ is additive on $\mathfrak{A}^+$, we get $\xi(I) = \xi\Big(\frac{I}{2}\Big) + \xi\Big(\frac{I}{2}\Big) = 0$. □

Lemma 2.9. *For any $S \in \mathfrak{A}^-$, we have $\xi(S)^* = -\xi(S)$.*

Proof. Since $0 = \Big((S \circ I) \bullet \frac{I}{2}\Big)$ for any $S \in \mathfrak{A}^-$, using the facts that $\xi(I) = \xi(\frac{I}{2}) = 0$, we obtain

$$0 = \xi(0) = \xi\left((S \circ I) \bullet \frac{I}{2}\right) = \left((\xi(S) \circ I) \bullet \frac{I}{2}\right),$$

which implies that $\xi(S)^* = -\xi(S)$ for any $S \in \mathfrak{A}^-$. □

Lemma 2.10. $\xi(iI) = 0$.

Proof. We have $\xi(iI)^* = -\xi(iI)$, by Lemma 2.9. Since $(iI \circ iI) \bullet I = -4I$, invoking Lemmas 2.7 and 2.8, we obtain that

$$0 = -4\xi(I) = \xi(-4I) = \xi((iI \circ iI) \bullet I) = (\xi(iI) \circ iI) \bullet I + (iI \circ \xi(iI)) \bullet I = \xi(iI).$$

□

Lemma 2.11. *For any $H \in \mathfrak{A}^+$, we have $\xi(iH) = i\xi(H)$.*

Proof. Observe that $(iH \circ iI) \bullet I = -4H$ for any $H \in \mathfrak{A}^+$. Using Lemmas 2.7, 2.8, 2.9 and 2.10, we find that

$$-4\xi(H) = \xi(-4H) = \xi((iH \circ iI) \bullet I) = (\xi(iH) \circ iI) \bullet I = 4i\xi(iH),$$

which implies that $\xi(iH) = i\xi(H)$. □

Lemma 2.12. *For any $A \in \mathfrak{A}$, we have $\xi(A^*) = \xi(A)^*$.*

Proof. Let $A = H_1 + iH_2 \in \mathfrak{A}$ for some $H_1, H_2 \in \mathfrak{A}^+$. Since $4H_1 = ((H_1 + iH_2) \circ I) \bullet I$, and making use of Lemmas 2.7 and 2.8, we obtain

$$\begin{aligned} 4\xi(H_1) =& \xi(4H_1) = \xi(((H_1 + iH_2) \circ I) \bullet I) \\ =& (\xi(H_1 + iH_2) \circ I) \bullet I = 2\{\xi(H_1 + iH_2) + \xi(H_1 + iH_2)^*\}. \end{aligned}$$

Thus, we have

$$2\xi(H_1) = \xi(H_1 + iH_2) + \xi(H_1 + iH_2)^*. \tag{2.3}$$

On the other hand, we have $4H_2 = ((H_1 + iH_2) \circ I) \bullet iI$. Invoking Lemmas 2.7, 2.8 and 2.10, we obtain

$$\begin{aligned} 4\xi(H_1) =& \xi(4H_1) = \xi(((H_1 + iH_2) \circ I) \bullet iI) \\ =& (\xi(H_1 + iH_2) \circ I) \bullet iI = 2\Big(\xi(H_1 + iH_2) - \xi(H_1 + iH_2)^*\Big). \end{aligned}$$

Therefore, we have

$$2i\xi(H_2) = \xi(H_1 + iH_2) - \xi(H_1 + iH_2)^*. \tag{2.4}$$

The addition of (2.2) and (2.3) yields

$$\xi(H_1 + iH_2) =\xi(H_1) + i\xi(H_2). \tag{2.5}$$

Since ξ is additive on $\mathfrak{A}^+$ and $\xi(0) = 0$, we have $\xi(-H) = -\xi(H)$ for any $H \in \mathfrak{A}^+$. By using (2.4) and Lemmas 2.2 and 2.7, we have

$$\begin{aligned} \xi(A)^* =& \xi(H_1 + iH_2)^* = (\xi(H_1) + i\xi(H_2))^* = \xi(H_1) - i\xi(H_2) \\ =& \xi(H_1 - iH_2) = \xi(A^*), \end{aligned}$$

for all $A \in \mathfrak{A}$. □

Lemma 2.13. *ξ is additive on $\mathfrak{A}$.*

Proof. Let us consider any two arbitrary elements $A_1 = H_1 + iH_2, A_2 = H_1^{'} + iH_2^{'} \in \mathfrak{A}$ for some $H_1, H_2, H_1', H_2' \in \mathfrak{A}^+$. Using (2.4) and Lemma 2.7, we find that

$$\begin{aligned}\xi(A_1 + A_2) =&\xi((H_1 + iH_2) + (H_1^{'} + iH_2^{'})) = \xi((H_1 + H_1^{'}) + i(H_2 + H_2^{'}))\\ =&\xi(H_1 + H_1^{'}) + i\xi(H_2 + H_2^{'}) = \xi(H_1) + \xi(H_1') + i\xi(H_2) + i\xi(H_2^{'})\\ =&\xi(H_1) + i\xi(H_2) + \xi(H_1') + i\xi(H_2^{'}) = \xi(H_1 + iH_2) + \xi(H_1' + iH_2')\\ =&\xi(A_1) + \xi(A_2).\end{aligned}$$

□

Lemma 2.14. *For any $A \in \mathfrak{A}$, we have $\xi(iA) = i\xi(A)$.*

Proof. Let $A = H_1 + iH_2 \in \mathfrak{A}$ for some $H_1, H_2 \in \mathfrak{A}^+$. Invoking Lemmas 2.11 and 2.13, we obtain

$$\begin{aligned}\xi(iA) =&\xi(iH_1 - H_2) = \xi(iH_1) - \xi(H_2) = i\xi(H_1) - \xi(H_2)\\ =&i(\xi(H_1) + i\xi(H_2)) = i(\xi(H_1 + iH_2)) = i\xi(A).\end{aligned}$$

□

Lemma 2.15. *ξ is an additive $*$-derivation on $\mathfrak{A}$.*

Proof. In view of Lemmas 2.12 and 2.13, it is sufficient to show that $\xi(A_1A_2) = \xi(A_1)A_2 + A_1\xi(A_2)$ for all $A_1, A_2 \in \mathfrak{A}$. First, we show that

$$\xi(HH^{'}) = \xi(H)H^{'} + H\xi(H^{'}), \quad \text{for all } H, H' \in \mathfrak{A}^+.$$

Using Lemmas 2.2, 2.7 and 2.8, we have

$$\begin{aligned}2\xi(HH^{'} + H^{'}H) =&\xi(2(HH^{'} + H^{'}H)) = \xi((H \circ H') \bullet I)\\ =&(\xi(H) \circ H^{'}) \bullet I + (H \circ \xi(H^{'})) \bullet I\\ =&2\Big(\xi(H)H^{'} + H^{'}\xi(H) + H\xi(H^{'}) + \xi(H^{'})H\Big).\end{aligned}$$

Thus, we have

$$\xi(HH^{'}) + \xi(H^{'}H) =\xi(H)H^{'} + H^{'}\xi(H) + H\xi(H^{'}) + \xi(H^{'})H. \tag{2.6}$$

Also, in view of Lemma 2.11, we have

$$\begin{aligned}2i\xi(H^{'}H - HH^{'}) =&\xi(2i(H^{'}H - HH^{'})) = \xi((H \circ I) \bullet iH^{'})\\ =&(\xi(H) \circ I) \bullet iH^{'} + (H \circ I) \bullet \xi(iH^{'})\\ =&2i\Big(\xi(H)H^{'} - H^{'}\xi(H) + H\xi(H^{'}) - \xi(H^{'})H\Big).\end{aligned}$$

Consequently, we get

$$\xi(HH^{'}) - \xi(H^{'}H) =\xi(H)H^{'} - H^{'}\xi(H) + H\xi(H^{'}) - \xi(H^{'})H. \tag{2.7}$$

The addition of (2.5) and (2.6) yields

$$\xi(HH^{'}) = \xi(H)H^{'} + H\xi(H^{'}), \quad \text{for all } H, H^{'} \in \mathfrak{A}^+. \tag{2.8}$$

Now, let $A_1 = H_1 + iH_2$ and $A_2 = H_1^{'} + iH_2^{'}$ be any two arbitrary elements of $\mathfrak{A}$, where $H_1, H_2, H_1^{'}, H_2^{'} \in \mathfrak{A}^+$. In view of (2.7) and Lemmas 2.13 and 2.14, we find that

$$\begin{aligned}\xi(A_1A_2) =& \xi\Big((H_1H_1^{'} - H_2H_2^{'}) + i(H_1H_2^{'} + H_2H_1^{'})\Big)\\ =& \xi(H_1H_1^{'} - H_2H_2^{'}) + i\xi(H_1H_2^{'} + H_2H_1^{'})\\ =& \xi(H_1H_1^{'}) - \xi(H_2H_2^{'}) + i\xi(H_1H_2^{'}) + i\xi(H_2H_1^{'})\\ =& \xi(H_1)H_1^{'} + H_1\xi(H_1^{'}) - \xi(H_2)H_2^{'} - H_2\xi(H_2^{'}) + i\xi(H_1)H_2^{'} + iH_1\xi(H_2^{'})\\ &+ i\xi(H_2)H_1^{'} + iH_2\xi(H_1^{'})\end{aligned}$$

and

$$\begin{aligned}\xi(A_1)A_2 + A_1\xi(A_2) =& \xi(H_1 + iH_2)(H_1^{'} + iH_2^{'}) + (H_1 + iH_2)\xi(H_1^{'} + iH_2^{'})\\ =& (\xi(H_1) + i\xi(H_2))(H_1^{'} + iH_2^{'}) + (H_1 + iH_2)(\xi(H_1^{'}) + i\xi(H_2^{'}))\\ =& \xi(H_1)H_1^{'} + H_1\xi(H_1^{'}) - \xi(H_2)H_2^{'} - H_2\xi(H_2^{'}) + i\xi(H_1)H_2^{'}\\ &+ iH_1\xi(H_2^{'}) + i\xi(H_2)H_1^{'} + iH_2\xi(H_1^{'}).\end{aligned}$$

Therefore, we have

$$\xi(A_1A_2) = \xi(A_1)A_2 + A_1\xi(A_2), \quad \text{for all } A_1, A_2 \in \mathfrak{A}.$$

Hence, the proof of Theorem 2.1 is complete. □

3. Applications

In this section, we apply Theorem 2.1 to certain special classes of $*$-algebras, namely prime $*$-algebras, standard operator algebras, factor von Neumann algebras and von Neumann algebras with no central summands of type I_1. Recall that an algebra $\mathfrak{A}$ is prime if for any $A, B \in \mathfrak{A}$, $A\mathfrak{A}B = \{0\}$ implies that either $A = 0$ or $B = 0$. It is easy to verify that every prime $*$-algebra satisfies (2.1). Therefore, as a direct consequence of Theorem 2.1, we have the following result.

Corollary 3.1. *Let $\mathfrak{A}$ be a unital prime $*$-algebra containing a nontrivial projection. Then a map $\xi : \mathfrak{A} \to \mathfrak{A}$ satisfies*

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C),$$

for all $A, B, C \in \mathfrak{A}$ if and only if ξ is an additive $*$-derivation.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ denote the subalgebra of all bounded finite rank operators. A subalgebra $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a standard operator algebra if it contains $\mathcal{F}(\mathcal{H})$. Now we have the following result.

Corollary 3.2. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathfrak{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity operator I. Suppose that $\mathfrak{A}$ is* *closed under the adjoint operation. Then, a map* $\xi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ *satisfies*

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C),$$

for all $A, B, C \in \mathfrak{A}$ *if and only if* ξ *is a linear* $*$*-derivation. Moreover, there exists an operator* $T \in \mathcal{B}(\mathcal{H})$ *satisfying* $T + T^* = 0$ *such that* $\xi(A) = AT - TA$ *for all* $A \in \mathfrak{A}$*, that is,* ξ *is inner.*

Proof. Since $\mathfrak{A}$ is a unital prime $*$-algebra containing nontrivial projections, then by Corollary 3.1, we see that ξ is an additive $*$-derivation. It follows from [14] that ξ is an linear inner derivation, that is, there exists an operator $S \in \mathcal{B}(\mathcal{H})$ such that $\xi(A) = AS - SA$ for all $A \in \mathfrak{A}$. Using the fact that $\xi(A^*) = \xi(A)^*$, we have

$$A^*S - SA^* = \xi(A^*) = \xi(A)^* = S^*A^* - A^*S^*,$$

for any $A \in \mathfrak{A}$, This leads to $A^*(S + S^*) = (S + S^*)A^*$. Hence, $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Letting $T = S - \frac{1}{2}\lambda I$, one can check that $T + T^* = 0$ and $\xi(A) = AT - TA$ for all $A \in \mathfrak{A}$. □

A von Neumann algebra $\mathfrak{A}$ is a weakly closed, self adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. A von Neumann algebra $\mathfrak{A}$ is a factor von Neumann algebra if its center contains only the scalar operators. It is well known that a factor von Neumann algebra is prime, thus, it always satisfies (2.1). Hence, as an immediate consequence of Corollary 3.1, we get the following.

Corollary 3.3. *Let* $\mathfrak{A}$ *be a factor von Neumann algebra with* $\dim(\mathfrak{A}) \geq 2$. *Then, a map* $\xi : \mathfrak{A} \to \mathfrak{A}$ *satisfies*

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C),$$

for all $A, B, C \in \mathfrak{A}$ *if and only if* ξ *is an additive* $*$*-derivation.*

Further, it is well-known that every von Neumann algebra with no central summands of type I_1 satisfies (2.1) (see [3, 6] for details). Therefore, applying Theorem 2.1, we have the following result.

Corollary 3.4. *Let* $\mathfrak{A}$ *be a von Neumann algebra having no central summands of type* I_1*. Then a map* $\xi : \mathfrak{A} \to \mathfrak{A}$ *satisfies*

$$\xi((A \circ B) \bullet C) = (\xi(A) \circ B) \bullet C + (A \circ \xi(B)) \bullet C + (A \circ B) \bullet \xi(C),$$

for all $A, B, C \in \mathfrak{A}$ *if and only if* ξ *is an additive* $*$*-derivation.*

4. Conclusion

In this paper, we have studied the relationship between nonlinear mixed bi-skew Jordan triple derivations and additive *-derivations on arbitrary *-algebras. In fact, it is shown that, under certain assumptions, every nonlinear mixed bi-skew Jordan triple derivation on a unital *-algebra is an additive *-derivation.

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