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ON CERTAIN NEW MODULAR RELATIONS BETWEEN THE QUADRATIC FORMS AND RAMANUJAN'S CONTINUED FRACTIONS OF ORDER SIXTEEN

NASSER ABDO SAEED BULKHALI[1], A. VANITHA[2], AND M. P. CHAUDHARY [3]

Abstract. In this paper, we derive several modular relations between series of the form $\sum_{x,y=-\infty}^{+\infty} q^{a\,x^2+b\,xy+c\,y^2+d\,x+e\,y}$ and Ramanujan's continued fractions of order sixteen by using Ramanujan's theta functions.

1. Introduction

Throughout this paper, we assume that $|q| < 1$ and use the standard product notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a;q)_{+\infty} := \prod_{n=0}^{+\infty}(1-aq^n).$$

For convenience, we sometimes use the multiple q-shifted factorial notation, which is defined as

$$(a_1, a_2, \ldots, a_m; q)_{+\infty} = (a_1;q)_{+\infty}(a_2;q)_{+\infty}\cdots(a_m;q)_{+\infty}.$$

Ramanujan's continued fractions of order sixteen are defined by

$$I_1(q) := \frac{q^{1/2}\,(1-q^3)}{1-q^4} \; + \; \frac{q^4(1-q)\,(1-q^7)}{(1-q^4)\,(1+q^8)} \; + \; \frac{q^4\,(1-q^9)\,(1-q^{15})}{(1-q^4)\,(1+q^{16})} \; + \cdots,$$

and

$$I_2(q) := \frac{q^{3/2}\,(1-q)}{1-q^4} \; + \; \frac{q^4(1-q^3)\,(1-q^5)}{(1-q^4)\,(1+q^8)} \; + \; \frac{q^4\,(1-q^{11})\,(1-q^{13})}{(1-q^4)\,(1+q^{16})} \; + \cdots,$$

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which can be expressed in terms of Ramanujan's theta function as

$$I_1(q) = \frac{q^{1/2} f(-q^3, -q^{13})}{f(-q^5, -q^{11})} \quad \text{and} \quad I_2(q) = \frac{q^{3/2} f(-q, -q^{15})}{f(-q^7, -q^9)}, \tag{1.1}$$

where

$$f(a,b) := \sum_{n=-\infty}^{+\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

In [8], Surekha proved modular relations for $I_1(q)$ and $I_2(q)$, and found $2-, 4-, 8-,$ $16-$dissections for $I_1^{-1}(q)$. She also showed that the sign of the coefficients in the power series expansion of $q^{-1/2} I_1(q)$ and its reciprocal are periodic with period 16. Vanitha [9] found $2-, 4-, 8-, 16-$dissections for the continued fraction $q^{-3/2} I_2(q)$ and its reciprocal. She also gave combinatorial interpretations for the coefficients in the power series expansion of this continued fraction. Park [5] investigated the continued fractions $I_1(q)$ and $I_2(q)$ by adopting the theory of modular functions, and he proved the modularity of $I_1(q)$ and $I_2(q)$ to find the relation to the generator of the field of modular functions on $\Gamma_0(16)$. He also proved that the values $2(I_1^2(q) + 1/I_1^2(q))$ and $2(I_2^2(q) + 1/I_2^2(q))$ are algebraic integers for certain imaginary quadratic quantity of q. In [10], Vanitha et al. proved many new identities related to Ramanujan's continued fraction of order sixteen and the Ramanujan-Göllnitz-Gordon continued fraction. In addition, they have established several new identities for Eisenstein series in connection with Ramanujan's continued fraction of order sixteen.

In his second notebook [6, Chapter 16, Entry 11] and the lost notebook [7], Ramanujan recorded many beautiful q-continued fractions and some of their explicit values. For example, the extremely pivotal entry in Chapter 16 of the second notebook [6], is Entry 11, which we give as follows.

Suppose that either q, a and b are complex numbers with $|q| < 1$ or q, a and b are complex numbers with $a = bq^m$ for some integer m. Then,

$$\begin{aligned} &\frac{(-a;q)_{+\infty}(b;q)_{+\infty} - (a;q)_{+\infty}(-b;q)_{+\infty}}{(-a;q)_{+\infty}(b;q)_{+\infty} + (a;q)_{+\infty}(-b;q)_{+\infty}} \\ &= \frac{a-b}{1-q\,+}\;\frac{(a-bq)(aq-b)}{1-q^3\,+}\;\frac{q(a-bq^2)(aq^2-b)}{1-q^5\,+\cdots}. \end{aligned} \tag{1.2}$$

Setting $q \to q^8$, $(a,b) = (q^7, -q)$ in the right-hand side of (1.2), we obtain

$$A(q^2) := \frac{q(1+q^6)}{1-q^8\,+}\;\frac{q^8(1+q^2)(1+q^{14})}{1-q^{24}\,+}\;\frac{q^{16}(1+q^{10})(1+q^{22})}{1-q^{40}\,+\cdots}.$$

Setting $q \to q^8$, $(a,b) = (q^5, -q^3)$ in the right-hand side of (1.2), we obtain

$$B(q^2) := \frac{q^3(1+q^2)}{1-q^8\,+}\;\frac{q^8(1+q^6)(1+q^{10})}{1-q^{24}\,+}\;\frac{q^{16}(1+q^{14})(1+q^{18})}{1-q^{40}\,+\cdots}.$$

The following lemma can be found in [2, Entry 30(ii) and (iii)].

Lemma 1.1. *We have*

$$f(a,b)+f(-a,-b)=2f(a^3b,ab^3), \tag{1.3}$$

$$f(a,b)-f(-a,-b)=2\,af(b/a,a^5b^3). \tag{1.4}$$

The following identity is an easy consequence of [2, Entry 31] when $n=2$:

$$f(a,b)=f(a^3b,ab^3)+a\ f(b/a,a^5b^3).$$

The Jacobi triple product identity [2, Entry 19] in Ramanujan's notation is

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Using the left-hand side of (1.2) with help of (1.3) and (1.4) we find that, respectively

$$A(q^2)=q\frac{f(q^6,q^{26})}{f(q^{10},q^{22})}\quad\text{and}\quad B(q^2)=q^3\frac{f(q^2,q^{30})}{f(q^{14},q^{18})}.$$

Thus, we have

$$A(q)=q^{1/2}\frac{f(q^3,q^{13})}{f(q^5,q^{11})}\quad\text{and}\quad B(q)=q^{3/2}\frac{f(q,q^{15})}{f(q^7,q^9)}. \tag{1.5}$$

Using (1.1) and (1.5), we find that

$$\begin{aligned}A(-q)&=i\,I_1(q),\quad B(-q)=-i\,I_2(q),\\ I_1(-q)&=i\,A(q)\quad\text{and}\quad I_2(-q)=-i\,B(q),\end{aligned}$$

where $i=\sqrt{-1}$.

Recently, Bulkhali et al. [4] derived several beautiful modular relations for Ramanujan's continued fraction of order sixteen by using Ramanujan's theta functions. We need the following lemma to prove our main results.

Lemma 1.2. *We have*

$$\frac{\sqrt{q}f\left(q^{10},q^{22}\right)}{f\left(q^{14},q^{18}\right)}=\frac{\left(A\left(q\right)+B\left(q\right)\right)\left(1+B\left(q^2\right)\right)}{\left(1+A\left(q\right)B\left(q\right)\right)\left(1+A\left(q^2\right)\right)} \tag{1.6}$$

and

$$\frac{q^{-3/2}f\left(q^6,q^{26}\right)}{f\left(q^2,q^{30}\right)}=\frac{A\left(q^2\right)\left(A\left(q\right)+B\left(q\right)\right)\left(1+B\left(q^2\right)\right)}{B\left(q^2\right)\left(1+A\left(q\right)B\left(q\right)\right)\left(1+A\left(q^2\right)\right)}. \tag{1.7}$$

The proof of Lemma 1.2 is given in [4, Theorem 2].

We will frequently use the following lemma in this paper.

Lemma 1.3. *Let* $m=[s/(s-r)]$, $l=m(s-r)-r$, $k=-m(s-r)+s$ *and* $h=mr-m(m-1)(s-r)/2$, $0\le r<s$. *Here* $[x]$ *denote the largest integer less than or equal to* x. *Then,* $f(q^{-r},q^s)=q^{-h}f(q^l,q^k)$.

The proof of Lemma 1.3 is given in [1].

Motivated by these works, in this paper we derive several modular relations between series of the form $\sum_{x,y=-\infty}^{+\infty} q^{a\,x^2+b\,xy+c\,y^2+d\,x+e\,y}$ and Ramanujan's continued fractions of order sixteen.

For convenience, we define the function

$$A(a,b,c,d,e;q) := \sum_{x,y=-\infty}^{+\infty} q^{a\,x^2+b\,xy+c\,y^2+d\,x+e\,y},$$

where a,b,c,d,e are integers with $a,c>0$ and $b^2-4\,ac<0$.

2. Main Results

In this section, we derive several modular relations involving $A(q)$ and $B(q)$.

Theorem 2.1. *Define, for any positive integer n,*

$$S_n := S(q^n) := \frac{\left(A\left(q^n\right)+B\left(q^n\right)\right)\left(1+B\left(q^{2\,n}\right)\right)}{\left(1+A\left(q^n\right)B\left(q^n\right)\right)\left(1+A\left(q^{2\,n}\right)\right)}.$$

Then,

(2.1)
$$\left\{1+B\left(q^2\right)B\left(q^8\right)\right\}+S_1S_4\left\{1+A\left(q^2\right)A\left(q^8\right)\right\}=\frac{A\left(52,4,5,-2,-5;\sqrt[4]{q}\right)}{2\,f\left(q^{56},q^{72}\right)f\left(q^{14},q^{18}\right)},$$

(2.2)
$$\left\{1+B\left(q^2\right)B\left(q^{26}\right)\right\}+S_1S_{13}\left\{1+A\left(q^2\right)A\left(q^{26}\right)\right\}=\frac{A\left(32,8,7,-4,-7;\sqrt{q}\right)}{2\,f\left(q^{14},q^{18}\right)f\left(q^{182},q^{234}\right)},$$

(2.3)
$$\left\{1+B\left(q^2\right)B\left(q^{38}\right)\right\}+S_1S_{19}\left\{1+A\left(q^2\right)A\left(q^{38}\right)\right\}=\frac{A\left(16,4,5,-2,-5;q\right)}{2\,f\left(q^{14},q^{18}\right)f\left(q^{266},q^{342}\right)},$$

(2.4)
$$\left\{1+B\left(q^2\right)B\left(q^{54}\right)\right\}+S_1S_{27}\left\{1+A\left(q^2\right)A\left(q^{54}\right)\right\}=\frac{A\left(19,10,7,-5,-7;q\right)}{2\,f\left(q^{378},q^{486}\right)f\left(q^{14},q^{18}\right)},$$

(2.5)
$$\left\{1+B\left(q^2\right)B\left(q^{62}\right)\right\}+S_1S_{31}\left\{1+A\left(q^2\right)A\left(q^{62}\right)\right\}=\frac{A\left(8,2,4,-1,-4;q^2\right)}{2\,f\left(q^{14},q^{18}\right)f\left(q^{434},q^{558}\right)},$$

(2.6)
$$\left\{1+B\left(q^2\right)B\left(q^{70}\right)\right\}+S_1S_{35}\left\{1+A\left(q^2\right)A\left(q^{70}\right)\right\}=\frac{A\left(21,14,9,-7,-9;q\right)}{2\,f\left(q^{490},q^{630}\right)f\left(q^{14},q^{18}\right)},$$

(2.7)
$$\left\{1+B\left(q^2\right)B\left(q^{78}\right)\right\}+S_1S_{39}\left\{1+A\left(q^2\right)A\left(q^{78}\right)\right\}=\frac{A\left(11,8,5,-4,-5;q^2\right)}{2\,f\left(q^{546},q^{702}\right)f\left(q^{14},q^{18}\right)},$$

(2.8)
$$\left\{1+B\left(q^{2}\right)B\left(q^{126}\right)\right\}+S_1S_{63}\left\{1+A\left(q^{2}\right)A\left(q^{126}\right)\right\}=\frac{A\left(14,14,8,-7,-8;q^{2}\right)}{2f\left(q^{882},q^{1134}\right)f\left(q^{14},q^{18}\right)},$$
(2.9)
$$\left\{1+B\left(q^{6}\right)B\left(q^{8}\right)\right\}+S_3S_{4}\left\{1+A\left(q^{6}\right)A\left(q^{8}\right)\right\}=\frac{A\left(112,8,7,-4,-7;\sqrt[4]{q}\right)}{2\,f\left(q^{56},q^{72}\right)f\left(q^{42},q^{54}\right)},$$
(2.10)
$$\left\{1+B\left(q^{6}\right)B\left(q^{14}\right)\right\}+S_3S_{7}\left\{1+A\left(q^{6}\right)A\left(q^{14}\right)\right\}=\frac{A\left(68,4,5,-2,-5;\sqrt{q}\right)}{2\,f\left(q^{42},q^{54}\right)f\left(q^{98},q^{126}\right)},$$
(2.11)
$$\left\{1+B\left(q^{6}\right)B\left(q^{50}\right)\right\}+S_3S_{25}\left\{1+A\left(q^{6}\right)A\left(q^{50}\right)\right\}=\frac{A\left(43,2,7,-1,-7;q\right)}{2\,f\left(q^{350},q^{450}\right)f\left(q^{42},q^{54}\right)},$$
(2.12)
$$\left\{1+B\left(q^{6}\right)B\left(q^{58}\right)\right\}+S_3S_{29}\left\{1+A\left(q^{6}\right)A\left(q^{58}\right)\right\}=\frac{A\left(22,2,4,-1,-4;q^{2}\right)}{2\,f\left(q^{406},q^{522}\right)f\left(q^{42},q^{54}\right)},$$
(2.13)
$$\left\{1+B\left(q^{6}\right)B\left(q^{62}\right)\right\}+S_3S_{31}\left\{1+A\left(q^{6}\right)A\left(q^{62}\right)\right\}=\frac{A\left(89,10,17,-5,-17;\sqrt{q}\right)}{2\,f\left(q^{434},q^{558}\right)f\left(q^{42},q^{54}\right)},$$
(2.14)
$$\left\{1+B\left(q^{6}\right)B\left(q^{74}\right)\right\}+S_3S_{37}\left\{1+A\left(q^{6}\right)A\left(q^{74}\right)\right\}=\frac{A\left(23,4,5,-2,-5;q^{2}\right)}{2\,f\left(q^{518},q^{666}\right)f\left(q^{42},q^{54}\right)},$$
(2.15)
$$\left\{1+B\left(q^{10}\right)B\left(q^{4}\right)\right\}+S_5S_{2}\left\{1+A\left(q^{10}\right)A\left(q^{4}\right)\right\}=\frac{A\left(95,10,7,-5,-7;\sqrt[4]{q}\right)}{2\,f\left(q^{70},q^{90}\right)f\left(q^{28},q^{36}\right)},$$
(2.16)
$$\left\{1+B\left(q^{10}\right)B\left(q^{46}\right)\right\}+S_5S_{23}\left\{1+A\left(q^{10}\right)A\left(q^{46}\right)\right\}=\frac{A\left(68,8,7,-4,-7;q\right)}{2\,f\left(q^{322},q^{414}\right)f\left(q^{70},q^{90}\right)},$$
(2.17)
$$\left\{1+B\left(q^{10}\right)B\left(q^{54}\right)\right\}+S_5S_{27}\left\{1+A\left(q^{10}\right)A\left(q^{54}\right)\right\},=\frac{A\left(34,2,4,-1,-4;q^{2}\right)}{2\,f\left(q^{70},q^{90}\right)f\left(q^{378},q^{486}\right)},$$
(2.18)
$$\left\{1+B\left(q^{10}\right)B\left(q^{94}\right)\right\}+S_5S_{47}\left\{1+A\left(q^{10}\right)A\left(q^{94}\right)\right\}=\frac{A\left(73,6,13,-3,-13;q\right)}{2\,f\left(q^{658},q^{846}\right)f\left(q^{70},q^{90}\right)},$$
(2.19)
$$\left\{1+B\left(q^{10}\right)B\left(q^{118}\right)\right\}+S_5S_{59}\left\{1+A\left(q^{10}\right)A\left(q^{118}\right)\right\}=\frac{A\left(40,10,8,-5,-8;q^{2}\right)}{2\,f\left(q^{70},q^{90}\right)f\left(q^{826},q^{1062}\right)},$$

(2.20)
$$\left\{1+B\left(q^{14}\right)B\left(q^{26}\right)\right\}+S_7S_{13}\left\{1+A\left(q^{14}\right)A\left(q^{26}\right)\right\}=\frac{A\left(76,8,5,-4,-5;q\right)}{2\,f\left(q^{98},q^{126}\right)f\left(q^{182},q^{234}\right)},$$
(2.21)
$$\left\{1+B\left(q^{14}\right)B\left(q^{50}\right)\right\}+S_7S_{25}\left\{1+A\left(q^{14}\right)A\left(q^{50}\right)\right\}=\frac{A\left(44,2,4,-1,-4;q^2\right)}{2\,f\left(q^{350},q^{450}\right)f\left(q^{98},q^{126}\right)},$$
(2.22)
$$\left\{1+B\left(q^{14}\right)B\left(q^{114}\right)\right\}+S_7S_{57}\left\{1+A\left(q^{14}\right)A\left(q^{114}\right)\right\}=\frac{A\left(56,14,8,-7,-8;q^2\right)}{2\,f\left(q^{98},q^{126}\right)f\left(q^{798},q^{1026}\right)},$$
(2.23)
$$\left\{1+B\left(q^{22}\right)B\left(q^{34}\right)\right\}+S_{11}S_{17}\left\{1+A\left(q^{22}\right)A\left(q^{34}\right)\right\}=\frac{A\left(107,2,7,-1,-7;q\right)}{2\,f\left(q^{154},q^{198}\right)f\left(q^{238},q^{306}\right)},$$
(2.24)
$$\left\{1+B\left(q^{22}\right)B\left(q^{42}\right)\right\}+S_{11}S_{21}\left\{1+A\left(q^{22}\right)A\left(q^{42}\right)\right\}=\frac{A\left(58,2,4,-1,-4;q^2\right)}{2\,f\left(q^{294},q^{378}\right)f\left(q^{154},q^{198}\right)},$$
(2.25)
$$\left\{1+B\left(q^{26}\right)B\left(q^{38}\right)\right\}+S_{13}S_{19}\left\{1+A\left(q^{26}\right)A\left(q^{38}\right)\right\}=\frac{A\left(62,2,4,-1,-4;q^2\right)}{2\,f\left(q^{182},q^{234}\right)f\left(q^{266},q^{342}\right)},$$
(2.26)
$$\left\{1+B\left(q^{42}\right)B\left(q^{62}\right)\right\}+S_{31}S_{21}\left\{1+A\left(q^{42}\right)A\left(q^{62}\right)\right\}+\frac{A\left(201,6,13,-3,-13;q\right)}{2\,f\left(q^{294},q^{378}\right)f\left(q^{434},q^{558}\right)}.$$

Proof. The identity (2.1) can be written in the form
$$\frac{A\left(52,4,5,-2,-5;\sqrt[4]{q}\right)}{2\,f\left(q^{56},q^{72}\right)f\left(q^{14},q^{18}\right)}=\left\{1+B\left(q^2\right)B\left(q^8\right)\right\}+\frac{\left(A\left(q\right)+B\left(q\right)\right)\left(1+B\left(q^2\right)\right)}{\left(1+A\left(q\right)B\left(q\right)\right)\left(1+A\left(q^2\right)\right)}$$
$$\times\frac{\left(A\left(q^4\right)+B\left(q^4\right)\right)\left(1+B\left(q^8\right)\right)}{\left(1+A\left(q^4\right)B\left(q^4\right)\right)\left(1+A\left(q^8\right)\right)}\left\{1+A\left(q^2\right)A\left(q^8\right)\right\},$$
Employing (1.6) in the above identity, with changing $q\to q^4$ and then multiply both sides of the resulting identity into $f\left(q^{56},q^{72}\right)f\left(q^{14},q^{18}\right)$, after some simplifications, we obtain
$$\begin{aligned}\frac{1}{2}A\left(52,4,5,-2,-5;\sqrt[4]{q}\right)=&f\left(q^{56},q^{72}\right)f\left(q^{14},q^{18}\right)+q^{15}f\left(q^{120},q^{8}\right)f\left(q^{30},q^{2}\right)\\&+q^{5/2}f\left(q^{88},q^{40}\right)f\left(q^{22},q^{10}\right)\\&+q^{15/2}f\left(q^{104},q^{24}\right)f\left(q^{26},q^{6}\right).\end{aligned}\tag{2.27}$$
Thus to prove (2.1), it suffices to establish (2.27). Now, we start with the following series:
$$A\left(52,4,5,-2,-5;q\right)=\sum_{x,y=-\infty}^{+\infty}q^{52\,x^2+4\,xy+5\,y^2-2\,x-5\,y}.\tag{2.28}$$

In the series (2.28), we make the change of indices by setting

$$2\,x+y=8\,X+t \quad \text{and} \quad -6\,x+y=8\,Y+u, \tag{2.29}$$

where the values of t and u are selected from the complete set of residues modulo 8, which can be chosen as $\{0,\pm1,\pm2,\pm3,4\}$. Then, solving the equations in (2.29) with respect to x and y, we obtain

$$x=X-Y+\frac{t-u}{8} \quad \text{and} \quad y=6\,X+2\,Y+\frac{6\,t+2\,u}{8}. \tag{2.30}$$

Since $x,\ y,\ X$ and Y are all integers, thus the equations in (2.30) yields that $u=t$ and hence $x=X-Y$ and $y=6\,X+2\,Y+t$, where $-3\le t\le 4$.

Thus, there is one-to-one correspondence between the set

$$\{(x,y):-\infty<x,y<+\infty,x,y\in\mathbb{Z}\},$$

and the set

$$\{(X,Y,t):-\infty<X,Y<+\infty,-3\le t\le 4,X,Y\in\mathbb{Z}\}.$$

Thus, using the above arguments, the equality (2.28) can be written as

$$\begin{aligned}
A\,(52,4,5,-2,-5;q)=&\sum_{t=-3}^{4}q^{5\,t^2-5\,t}\left(\sum_{X=-\infty}^{+\infty}q^{256\,X^2+64\,Xt-32\,X}\right)\\
&\times\left(\sum_{Y=-\infty}^{+\infty}q^{64\,Y^2+16\,Yt-8\,Y}\right)\\
=&\sum_{t=-3}^{4}q^{5\,t^2-5\,t}f\left(q^{224+64\,t},q^{288-64\,t}\right)f\left(q^{56+16\,t},q^{72-16\,t}\right)\\
=&q^{60}f\left(q^{32},q^{480}\right)f\left(q^{8},q^{120}\right)+q^{30}f\left(q^{96},q^{416}\right)f\left(q^{24},q^{104}\right)\\
&+q^{10}f\left(q^{160},q^{352}\right)f\left(q^{40},q^{88}\right)+f\left(q^{224},q^{288}\right)f\left(q^{56},q^{72}\right)\\
&+f\left(q^{288},q^{224}\right)f\left(q^{72},q^{56}\right)+q^{10}f\left(q^{352},q^{160}\right)f\left(q^{88},q^{40}\right)\\
&+q^{30}f\left(q^{416},q^{96}\right)f\left(q^{104},q^{24}\right)+q^{60}f\left(q^{480},q^{32}\right)f\left(q^{120},q^{8}\right),
\end{aligned}$$

which is the same as (2.27), after changing $q\to q^{1/4}$, this completes the proof of (2.1). The proofs of (2.2)–(2.26) follow in a similar way. So, we omit the details. □

Theorem 2.2. *For S_n as defined in Theorem 2.1, and*

$$T_n:=T(q^n):=\frac{A\,(q^{2\,n})\,(A\,(q^n)+B\,(q^n))\,(1+B\,(q^{2\,n}))}{B\,(q^{2\,n})\,(1+A\,(q^n)\,B\,(q^n))\,(1+A\,(q^{2\,n}))},$$

we have

$$\frac{A\left(53,6,5,-53,-3;\sqrt[4]{q}\right)}{2\,f\,(q^2,q^{30})\,f\,(q^{72},q^{56})}=\left\{1+\frac{B\,(q^8)}{B\,(q^2)}\right\}+S_4T_1\left\{1+\frac{A\,(q^8)}{A\,(q^2)}\right\}, \tag{2.31}$$

$$\frac{A\left(27,2,3,-27,-1;\sqrt{q}\right)}{2\,f\left(q^{2},q^{30}\right)f\left(q^{90},q^{70}\right)}=\left\{1+\frac{B\left(q^{10}\right)}{B\left(q^{2}\right)}\right\}+S_{5}T_{1}\left\{1+\frac{A\left(q^{10}\right)}{A\left(q^{2}\right)}\right\}, \tag{2.32}$$

$$\frac{A\left(55,2,7,-55,-1;\sqrt[4]{q}\right)}{2\,f\left(q^{2},q^{30}\right)f\left(q^{108},q^{84}\right)}=\left\{1+\frac{B\left(q^{12}\right)}{B\left(q^{2}\right)}\right\}+S_{6}T_{1}\left\{1+\frac{A\left(q^{12}\right)}{A\left(q^{2}\right)}\right\}, \tag{2.33}$$

$$\frac{A\left(57,2,9,-57,-1;\sqrt[4]{q}\right)}{2\,f\left(q^{112},q^{144}\right)f\left(q^{30},q^{2}\right)}=\left\{1+\frac{B\left(q^{16}\right)}{B\left(q^{2}\right)}\right\}+S_{8}T_{1}\left\{1+\frac{A\left(q^{16}\right)}{A\left(q^{2}\right)}\right\}, \tag{2.34}$$

$$\frac{A\left(15,2,3,-15,-1;q\right)}{2\,f\left(q^{154},q^{198}\right)f\left(q^{30},q^{2}\right)}=\left\{1+\frac{B\left(q^{22}\right)}{B\left(q^{2}\right)}\right\}+S_{11}T_{1}\left\{1+\frac{A\left(q^{22}\right)}{A\left(q^{2}\right)}\right\}, \tag{2.35}$$

$$\frac{A\left(31,6,7,-31,-3;\sqrt{q}\right)}{2\,f\left(q^{182},q^{234}\right)f\left(q^{30},q^{2}\right)}=\left\{1+\frac{B\left(q^{26}\right)}{B\left(q^{2}\right)}\right\}+S_{13}T_{1}\left\{1+\frac{A\left(q^{26}\right)}{A\left(q^{2}\right)}\right\}, \tag{2.36}$$

$$\frac{A\left(9,4,3,-9,-2;q^{2}\right)}{2\,f\left(q^{322},q^{414}\right)f\left(q^{30},q^{2}\right)}=\left\{1+\frac{B\left(q^{46}\right)}{B\left(q^{2}\right)}\right\}+S_{23}T_{1}\left\{1+\frac{A\left(q^{46}\right)}{A\left(q^{2}\right)}\right\}, \tag{2.37}$$

$$\frac{A\left(79,10,7,-79,-5;\sqrt{q}\right)}{2\,f\left(q^{6},q^{90}\right)f\left(q^{198},q^{154}\right)}=\left\{1+\frac{B\left(q^{22}\right)}{B\left(q^{6}\right)}\right\}+S_{11}T_{3}\left\{1+\frac{A\left(q^{22}\right)}{A\left(q^{6}\right)}\right\}, \tag{2.38}$$

$$\frac{A\left(41,2,5,-41,-1;q\right)}{2\,f\left(q^{6},q^{90}\right)f\left(q^{306},q^{238}\right)}=\left\{1+\frac{B\left(q^{34}\right)}{B\left(q^{6}\right)}\right\}+S_{17}T_{3}\left\{1+\frac{A\left(q^{34}\right)}{A\left(q^{6}\right)}\right\}, \tag{2.39}$$

$$\frac{A\left(83,2,11,-83,-1;\sqrt{q}\right)}{2\,f\left(q^{6},q^{90}\right)f\left(q^{342},q^{266}\right)}=\left\{1+\frac{B\left(q^{38}\right)}{B\left(q^{6}\right)}\right\}+S_{19}T_{3}\left\{1+\frac{A\left(q^{38}\right)}{A\left(q^{6}\right)}\right\}, \tag{2.40}$$

$$\frac{A\left(85,2,13,-85,-1;\sqrt{q}\right)}{2\,f\left(q^{322},q^{414}\right)f\left(q^{90},q^{6}\right)}=\left\{1+\frac{B\left(q^{46}\right)}{B\left(q^{6}\right)}\right\}+S_{23}T_{3}\left\{1+\frac{A\left(q^{46}\right)}{A\left(q^{6}\right)}\right\}, \tag{2.41}$$

$$\frac{A\left(43,2,7,-43,-1;q\right)}{2\,f\left(q^{350},q^{450}\right)f\left(q^{90},q^{6}\right)}=\left\{1+\frac{B\left(q^{50}\right)}{B\left(q^{6}\right)}\right\}+S_{25}T_{3}\left\{1+\frac{A\left(q^{50}\right)}{A\left(q^{6}\right)}\right\}, \tag{2.42}$$

$$\frac{A\left(22,2,4,-22,-1;q^{2}\right)}{2\,f\left(q^{406},q^{522}\right)f\left(q^{90},q^{6}\right)}=\left\{1+\frac{B\left(q^{58}\right)}{B\left(q^{6}\right)}\right\}+S_{29}T_{3}\left\{1+\frac{A\left(q^{58}\right)}{A\left(q^{6}\right)}\right\}, \tag{2.43}$$

$$\frac{A\left(34,2,4,-34,-1;q^{2}\right)}{2\,f\left(q^{10},q^{150}\right)f\left(q^{486},q^{378}\right)}=\left\{1+\frac{B\left(q^{54}\right)}{B\left(q^{10}\right)}\right\}+S_{27}T_{5}\left\{1+\frac{A\left(q^{54}\right)}{A\left(q^{10}\right)}\right\}, \tag{2.44}$$

$$\frac{A\left(137,6,17,-137,-3;\sqrt{q}\right)}{2\,f\left(q^{10},q^{150}\right)f\left(q^{522},q^{406}\right)}=\left\{1+\frac{B\left(q^{58}\right)}{B\left(q^{10}\right)}\right\}+S_{29}T_{5}\left\{1+\frac{A\left(q^{58}\right)}{A\left(q^{10}\right)}\right\}, \tag{2.45}$$

$$\frac{A\left(73,6,13,-73,-3;q\right)}{2\,f\left(q^{658},q^{846}\right)f\left(q^{150},q^{10}\right)}=\left\{1+\frac{B\left(q^{94}\right)}{B\left(q^{10}\right)}\right\}+S_{47}T_{5}\left\{1+\frac{A\left(q^{94}\right)}{A\left(q^{10}\right)}\right\}, \tag{2.46}$$

$$\frac{A\left(38,6,8,-38,-3;q^{2}\right)}{2\,f\left(q^{826},q^{1062}\right)f\left(q^{150},q^{10}\right)}=\left\{1+\frac{B\left(q^{118}\right)}{B\left(q^{10}\right)}\right\}+S_{59}T_{5}\left\{1+\frac{A\left(q^{118}\right)}{A\left(q^{10}\right)}\right\}, \tag{2.47}$$

$$\frac{A\left(51,4,9,-51,-2;q^{2}\right)}{2\,f\left(q^{910},q^{1170}\right)f\left(q^{210},q^{14}\right)}=\left\{1+\frac{B\left(q^{130}\right)}{B\left(q^{14}\right)}\right\}+S_{65}T_{7}\left\{1+\frac{A\left(q^{130}\right)}{A\left(q^{14}\right)}\right\}. \tag{2.48}$$

Proof. Using the definitions of S_n and T_n, we may rewrite the identity (2.31) in the form

$$\begin{aligned}\frac{A\left(53,6,5,-53,-3;\sqrt[4]{q}\right)}{2\,f\left(q^{2},q^{30}\right)f\left(q^{72},q^{56}\right)}=&\left\{1+\frac{B\left(q^{8}\right)}{B\left(q^{2}\right)}\right\}+\frac{\left(A\left(q^{4}\right)+B\left(q^{4}\right)\right)\left(1+B\left(q^{8}\right)\right)}{\left(1+A\left(q^{4}\right)B\left(q^{4}\right)\right)\left(1+A\left(q^{8}\right)\right)}\\&\times\frac{A\left(q^{2}\right)\left(A\left(q\right)+B\left(q\right)\right)\left(1+B\left(q^{2}\right)\right)}{B\left(q^{2}\right)\left(1+A\left(q\right)B\left(q\right)\right)\left(1+A\left(q^{2}\right)\right)}\left\{1+\frac{A\left(q^{8}\right)}{A\left(q^{2}\right)}\right\}.\end{aligned}$$

Using (1.6) and (1.7) in the above identity, with changing $q\to q^{4}$ in (1.6), and then multiply both sides of the resulting identity into $f\left(q^{2},q^{30}\right)f\left(q^{72},q^{56}\right)$, after some simplifications, we deduce that

$$\begin{aligned}\frac{1}{2}A\left(53,6,5,-53,-3;\sqrt[4]{q}\right)=&f\left(q^{2},q^{30}\right)f\left(q^{72},q^{56}\right)+q^{9}f\left(q^{14},q^{18}\right)f\left(q^{120},q^{8}\right)\\&+\sqrt{q}f\left(q^{6},q^{26}\right)f\left(q^{88},q^{40}\right)\\&+q^{7/2}f\left(q^{10},q^{22}\right)f\left(q^{104},q^{24}\right).\end{aligned} \tag{2.49}$$

Thus, to prove (2.31), it suffices to establish (2.49). Now, we start with the following series:

$$A\left(53,6,5,-53,-3;q\right)=\sum_{x,y=-\infty}^{+\infty}q^{53\,x^{2}+6\,xy+5\,y^{2}-53\,x-3\,y}. \tag{2.50}$$

In the series (2.50), we make the change of indices by setting

$$7\,x+y=8\,X+t\quad\text{and}\quad-x+y=8\,Y+u, \tag{2.51}$$

here the values of t and u are selected from the complete set of residues modulo 8, which can be chosen as $\{0,\pm1,\pm2,\pm3,4\}$. Then, solving the equations in (2.51) with respect to x and y, we obtain

$$x=X-Y+\frac{(t-u)}{8}\quad\text{and}\quad y=X+7\,Y+\frac{t+7\,u}{8}. \tag{2.52}$$

Since $x,\ y,\ X$ and Y are all integers, thus the equations in (2.52) yields that $u=t$ and hence $x=X-Y$ and $y=X+7\,Y+t$, where $-3\le t\le 4$.

Thus, there is one-to-one correspondence between the set

$$\left\{(x,y):-\infty<x,y<+\infty,x,y\in\mathbb{Z}\right\},$$

and the set

$$\left\{(X,Y,t):-\infty<X,Y<+\infty,-3\le t\le 4,X,Y\in\mathbb{Z}\right\}.$$

Thus, using the above arguments, the equality (2.50) can be written as

$$A\left(53,6,5,-53,-3;q\right)=\sum_{t=-3}^{4}q^{5\,t^{2}-3\,t}\left(\sum_{X=-\infty}^{+\infty}q^{64\,X^{2}+16\,Xt-56\,X}\right)$$

$$
\begin{aligned}
&\times\left(\sum_{Y=-\infty}^{+\infty} q^{256\,Y^2+64\,Yt+32\,Y}\right)\\
=&\sum_{t=-3}^{4} q^{5\,t^2-3\,t} f\left(q^{8+16\,t},q^{120-16\,t}\right) f\left(q^{288+64\,t},q^{224-64\,t}\right)\\
=&q^{54} f\left(q^{-40},q^{168}\right) f\left(q^{96},q^{416}\right)+q^{26} f\left(q^{-24},q^{152}\right) f\left(q^{160},q^{352}\right)\\
&+q^{8} f\left(q^{-8},q^{136}\right) f\left(q^{224},q^{288}\right)+f\left(q^{8},q^{120}\right) f\left(q^{288},q^{224}\right)\\
&+q^{2} f\left(q^{24},q^{104}\right) f\left(q^{352},q^{160}\right)+q^{14} f\left(q^{40},q^{88}\right) f\left(q^{416},q^{96}\right)\\
&+q^{36} f\left(q^{56},q^{72}\right) f\left(q^{480},q^{32}\right)+q^{68} f\left(q^{72},q^{56}\right) f\left(q^{544},q^{-32}\right),
\end{aligned}
$$

which is the same as (2.49), after changing $q \to q^{1/4}$, this completes the proof of (2.31). The proofs of (2.32)–(2.48) follow in a similar way. So, we omit the details. □

Theorem 2.3. *For S_n and T_n as defined in Theorem 2.1 and Theorem 2.2, respectively, we have*

$$\frac{A\left(43,2,3,-43,-1;\sqrt[4]{q}\right)}{2\,f\left(q^{20},q^{44}\right)f\left(q^{26},q^{6}\right)}=\left\{1+\frac{A\left(q^{4}\right)}{A\left(q^{2}\right)}\right\}+\frac{1}{S_2T_1}\left\{1+\frac{B\left(q^{4}\right)}{B\left(q^{2}\right)}\right\}, \tag{2.53}$$

$$\frac{A\left(77,2,5,-77,-1;\sqrt[4]{q}\right)}{2\,f\left(q^{12},q^{52}\right)f\left(q^{66},q^{30}\right)}=\left\{1+\frac{A\left(q^{6}\right)}{A\left(q^{4}\right)}\right\}+\frac{1}{S_3T_2}\left\{1+\frac{B\left(q^{6}\right)}{B\left(q^{4}\right)}\right\}, \tag{2.54}$$

$$\frac{A\left(95,10,7,-95,-5;\sqrt[4]{q}\right)}{2\,f\left(q^{50},q^{110}\right)f\left(q^{52},q^{12}\right)}=\left\{1+\frac{A\left(q^{10}\right)}{A\left(q^{4}\right)}\right\}+\frac{1}{S_5T_2}\left\{1+\frac{B\left(q^{10}\right)}{B\left(q^{4}\right)}\right\}, \tag{2.55}$$

$$\frac{A\left(111,6,7,-111,-3;\sqrt[4]{q}\right)}{2\,f\left(q^{18},q^{78}\right)f\left(q^{88},q^{40}\right)}=\left\{1+\frac{A\left(q^{8}\right)}{A\left(q^{6}\right)}\right\}+\frac{1}{S_4T_3}\left\{1+\frac{B\left(q^{8}\right)}{B\left(q^{6}\right)}\right\}, \tag{2.56}$$

$$\frac{A\left(69,6,5,-69,-3;\sqrt{q}\right)}{2\,f\left(q^{70},q^{154}\right)f\left(q^{78},q^{18}\right)}=\left\{1+\frac{A\left(q^{14}\right)}{A\left(q^{6}\right)}\right\}+\frac{1}{S_7T_3}\left\{1+\frac{B\left(q^{14}\right)}{B\left(q^{6}\right)}\right\}, \tag{2.57}$$

$$\frac{A\left(103,2,7,-103,-1;\sqrt{q}\right)}{2\,f\left(q^{90},q^{198}\right)f\left(q^{130},q^{30}\right)}=\left\{1+\frac{A\left(q^{18}\right)}{A\left(q^{10}\right)}\right\}+\frac{1}{S_9T_5}\left\{1+\frac{B\left(q^{18}\right)}{B\left(q^{10}\right)}\right\}, \tag{2.58}$$

$$\frac{A\left(28,2,2,-28,-1;q^{2}\right)}{2\,f\left(q^{110},q^{242}\right)f\left(q^{130},q^{30}\right)}=\left\{1+\frac{A\left(q^{22}\right)}{A\left(q^{10}\right)}\right\}+\frac{1}{S_{11}T_5}\left\{1+\frac{B\left(q^{22}\right)}{B\left(q^{10}\right)}\right\}, \tag{2.59}$$

$$\frac{A\left(32,2,2,-32,-1;q^{2}\right)}{2\,f\left(q^{42},q^{182}\right)f\left(q^{198},q^{90}\right)}=\left\{1+\frac{A\left(q^{18}\right)}{A\left(q^{14}\right)}\right\}+\frac{1}{S_9T_7}\left\{1+\frac{B\left(q^{18}\right)}{B\left(q^{14}\right)}\right\}, \tag{2.60}$$

$$\frac{A\left(73,2,5,-73,-1;q\right)}{2\,f\left(q^{130},q^{286}\right)f\left(q^{182},q^{42}\right)}=\left\{1+\frac{A\left(q^{26}\right)}{A\left(q^{14}\right)}\right\}+\frac{1}{S_{13}T_7}\left\{1+\frac{B\left(q^{26}\right)}{B\left(q^{14}\right)}\right\}, \tag{2.61}$$

$$\frac{A\left(41,4,3,-41,-2;q^{2}\right)}{2\,f\left(q^{170},q^{374}\right)f\left(q^{182},q^{42}\right)}=\left\{1+\frac{A\left(q^{34}\right)}{A\left(q^{14}\right)}\right\}+\frac{1}{S_{17}T_7}\left\{1+\frac{B\left(q^{34}\right)}{B\left(q^{14}\right)}\right\}, \tag{2.62}$$

$$\frac{A\left(189,6,13,-189,-3;\sqrt{q}\right)}{2\,f\left(q^{170},q^{374}\right)f\left(q^{234},q^{54}\right)}=\left\{1+\frac{A\left(q^{34}\right)}{A\left(q^{18}\right)}\right\}+\frac{1}{S_{17}T_9}\left\{1+\frac{B\left(q^{34}\right)}{B\left(q^{18}\right)}\right\},\tag{2.63}$$

$$\frac{A\left(223,2,15,-223,-1;\sqrt{q}\right)}{2\,f\left(q^{190},q^{418}\right)f\left(q^{286},q^{66}\right)}=\left\{1+\frac{A\left(q^{38}\right)}{A\left(q^{22}\right)}\right\}+\frac{1}{S_{19}T_{11}}\left\{1+\frac{B\left(q^{38}\right)}{B\left(q^{22}\right)}\right\},\tag{2.64}$$

$$\frac{A\left(62,2,4,-62,-1;q^2\right)}{2\,f\left(q^{78},q^{338}\right)f\left(q^{418},q^{190}\right)}=\left\{1+\frac{A\left(q^{38}\right)}{A\left(q^{26}\right)}\right\}+\frac{1}{S_{19}T_{13}}\left\{1+\frac{B\left(q^{38}\right)}{B\left(q^{26}\right)}\right\}.\tag{2.65}$$

Proof. Using the definitions of S_n and T_n, we may rewrite the identity (2.53) in the form

$$\begin{aligned}\frac{A\left(43,2,3,-43,-1;\sqrt[4]{q}\right)}{2\,f\left(q^{20},q^{44}\right)f\left(q^{26},q^{6}\right)}=&\left\{1+\frac{A\left(q^{4}\right)}{A\left(q^{2}\right)}\right\}+\frac{\left(1+A\left(q^2\right)B\left(q^2\right)\right)\left(1+A\left(q^4\right)\right)}{\left(A\left(q^2\right)+B\left(q^2\right)\right)\left(1+B\left(q^4\right)\right)}\\&\times\frac{B\left(q^2\right)\left(1+A\left(q\right)B\left(q\right)\right)\left(1+A\left(q^2\right)\right)}{A\left(q^2\right)\left(A\left(q\right)+B\left(q\right)\right)\left(1+B\left(q^2\right)\right)}\left\{1+\frac{B\left(q^{4}\right)}{B\left(q^{2}\right)}\right\}.\end{aligned}$$

Using (1.6) and (1.7) in the above identity, with changing $q\to q^2$ in (1.6), and then multiply both sides of the resulting identity into $f\left(q^{20},q^{44}\right)f\left(q^{26},q^{6}\right)$, after some simplifications, we find that

$$\begin{aligned}\frac{1}{2}A\left(43,2,3,-43,-1;\sqrt[4]{q}\right)=&f\left(q^{20},q^{44}\right)f\left(q^{26},q^{6}\right)+qf\left(q^{12},q^{52}\right)f\left(q^{22},q^{10}\right)\\&+q^{1/2}f\left(q^{28},q^{36}\right)f\left(q^{30},q^{2}\right)\\&+q^{7/2}f\left(q^{4},q^{60}\right)f\left(q^{18},q^{14}\right).\end{aligned}\tag{2.66}$$

Thus, to prove (2.53), it suffices to establish (2.66). Now, we start with the following series:

$$M(q):=A\left(43,2,3,-43,-1;q\right)=\sum_{x,y=-\infty}^{+\infty}q^{43\,x^2+2\,xy+3\,y^2-43\,x-y}.\tag{2.67}$$

In the series (2.67), we make the change of indices by setting

$$3x+y=8X+t\quad\text{and}\quad-5x+y=8Y+u,\tag{2.68}$$

where the values of t and u are selected from the complete set of residues modulo 8, which can be chosen as $\{0,\pm1,\pm2,\pm3,4\}$. Then, solving the equations in (2.68) with respect to x and y, we obtain

$$x=X-Y+\frac{t-u}{8}\quad\text{and}\quad y=5\,X+3\,Y+\frac{5\,t+3\,u}{8}.\tag{2.69}$$

Since x,y,X and Y are all integers, thus the equations in (2.69) yields that $u=t$ and hence $x=X-Y$ and $y=5\,X+3\,Y+t$, where $-3\leq t\leq 4$. Thus, there is one-to-one correspondence between the set

$$\left\{(x,y):-\infty<x,y<+\infty,x,y\in\mathbb{Z}\right\},$$

and the set

$$\{(X,Y,t) : -\infty < X, Y < +\infty, -3 \leq t \leq 4, X, Y \in \mathbb{Z}\}.$$

Thus, using the above arguments, (2.67) can be written as

$$\begin{aligned} M(q) = & \sum_{t=-3}^{4} q^{3t^2-t} \left(\sum_{X=-\infty}^{+\infty} q^{128X^2+32Xt-48X} \right) \\ & \times \left(\sum_{Y=-\infty}^{+\infty} q^{64Y^2+16Yt+40Y} \right) \\ = & \sum_{t=-3}^{4} q^{3t^2-t} f\left(q^{80+32t}, q^{176-32t}\right) f\left(q^{104+16t}, q^{24-16t}\right) \\ = & q^{30} f\left(q^{-16}, q^{272}\right) f\left(q^{56}, q^{72}\right) + q^{14} f\left(q^{16}, q^{240}\right) f\left(q^{72}, q^{56}\right) \\ & + q^{4} f\left(q^{48}, q^{208}\right) f\left(q^{88}, q^{40}\right) + f\left(q^{80}, q^{176}\right) f\left(q^{104}, q^{24}\right) \\ & + q^{2} f\left(q^{112}, q^{144}\right) f\left(q^{120}, q^{8}\right) + q^{10} f\left(q^{144}, q^{112}\right) f\left(q^{136}, q^{-8}\right) \\ & + q^{24} f\left(q^{176}, q^{80}\right) f\left(q^{152}, q^{-24}\right) + q^{44} f\left(q^{208}, q^{48}\right) f\left(q^{168}, q^{-40}\right), \end{aligned}$$

which is the same as (2.66), after changing $q \to q^{1/4}$, this completes the proof of (2.53). The proofs of (2.54)–(2.65) follow in a similar way. So, we omit the details. ☐

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[1]Department of Mathematics,
Hodeidah University,
Faculty of Education-Zabid, Hodeidah, Yemen
Email address: nassbull@hotmail.com
ORCID iD: https://orcid.org/0000-0001-8097-0100

[2]Department of Mathematics,
The National Institute of Engineering,
Mysuru-570 008, India
Email address: a.vanitha4@gmail.com
ORCID iD: https://orcid.org/0009-0008-5208-0790

[3]Department of Mathematical Sciences,
International Scientific Research and Welfare Organization,
New Delhi, India
Email address: dr.m.p.chaudhary@gmail.com
ORCID iD: https://orcid.org/0000-0002-7329-2732