Kragujevac Journal of Mathematics
Volume 51(4) (2027), Pages 557–579.

SOLVABILITY OF FOURTH-ORDER COUPLED SYSTEM WITH THREE-POINT BOUNDARY CONDITIONS

ROBERT DE SOUSA[1,2] AND MARCO ANTÓNIO DE SALES MONTEIRO FERNANDES[1]

Abstract. In this paper we deal with a coupled system of fourth-order differential equations with three-point boundary conditions where the nonlinearities depend on the unknown function and its first and second derivatives. More specifically, using Guo-Krasnosel'skii compression-expansion theory on cones, superlinearity/sublinearity conditions near 0 and $+\infty$ and putting some adequate estimates and imposing certain positivity conditions on the Green's function and its first and second derivatives, we guarantee the existence of positive solutions to our problem. The last section contains an example to illustrate the applicability of the theorem.

1. Introduction

Over the decades to the present, researchers around the world have been poring over fourth-order differential equations and their applications.

Fourth-order differential equations are used essentially to study and model elastic beams [9, 13, 14, 17], Abel-Gontscharoff interpolation [1], infinite beam resting on granular foundations [20], soil settlement of compressed beams [10], suspension bridge [7, 19], systems modeling degenerate plates with piers [8], deformation of structures and vibrations of shells and plates [23], beam columns and elasticity and many more applications [24].

In the literature there are several works and techniques on fourth-order boundary value problems and their applications with different conditions (two points, three

Key words and phrases. Fourth-order coupled systems, Green's function, Guo-Krasnosel'skii fixed-point in cones, positive solution.

2020 Mathematics Subject Classification. Primary: 34B15, 34B18, 34B27, 34B60. Secondary: 34L30, 34A34, 45G15.

DOI

Received: February 16, 2024.

Accepted: November 05, 2025.

points, impulsive, mixed, Lidstone type,...) [3,6,11,16,22,26]. However, in this work we essentially focus on three-point boundary conditions.

Citing just a few works in three-points boundary value problems, we have for example, in [2], Bai, by using Guo-Krasnosel'skii fixed point theorem in a cone, obtains some existence results of positive solutions. Ma et al. in [18], study the existence of positive solutions for the fourth-order p-Laplacian three-point boundary value problem through Leggett-Williams fixed point theorem. Applying Schauder fixed point theorem, the upper and lower solution method and the topological degree theory, Çetin and Agarwal in [4], investigate the existence of solutions for fourth-order three-point boundary value problems on a half-line.

On the other hand, systems of fourth-order differential equations appear less in the literature. As a system model, we mention the work of Wang and Yang in [25], where, by using fixed point theory in cones, they obtained the existence of positive solutions with superlinear or sublinear nonlinearities for the system

$$\begin{aligned}
&u^4(t) + \beta_1 u''(t) - \alpha_1 u(t) = f_1(t, u(t), v(t)), \quad t \in (0,1),\\
&v^4(t) + \beta_2 v''(t) - \alpha_2 v(t) = f_2(t, u(t), v(t)), \quad t \in (0,1),\\
&u(0) = u(1) = u''(0) = u''(1) = 0,\\
&v(0) = v(1) = v''(0) = v''(1) = 0,
\end{aligned}$$

where $f_i \in C\Big([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty)\Big)$ and $\beta_i, \alpha_i \in \mathbb{R}$, for $i = 1, 2$, satisfy the conditions $\beta_i < 2\pi^2$, $-\frac{\beta_i^2}{4} \leq \alpha_i$, $\frac{\alpha_i}{\pi^4} + \frac{\beta_i}{\pi^2} < 1$.

In [5], de Sousa and Minhós, apply lower and upper solutions and degree theory to study the fourth-order coupled system composed of the nonlinear fully differential equations

$$\begin{aligned}
u^{(4)}(t) &= f(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t), u'''(t), v'''(t)), \quad t \in (0,1),\\
v^{(4)}(t) &= h(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t), u'''(t), v'''(t)), \quad t \in (0,1),
\end{aligned}$$

where $f, h : [0,1] \times \mathbb{R}^8 \to \mathbb{R}$ are continuous functions, with the Lidstone boundary conditions

$$\begin{aligned}
u(0) = u(1) = A_1, \quad u''(0) = A_2, \quad u''(1) = A_3,\\
v(0) = v(1) = B_1, \quad v''(0) = B_2, \quad v''(1) = B_3,
\end{aligned}$$

and $A_i, B_i \in \mathbb{R}$, for $i = 1, 2, 3$.

In fact, coupled or not, systems of differential equations of the fourth-order with three-point boundary conditions are scarce and little explored in terms of applications. To contribute to filling this gap and motivated by the aforementioned works, we study the nonlinear coupled system of fourth-order three-point boundary value problem, more specifically

$$u^4(t) = f(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t)), \tag{1.1}$$

$$v^4(t) = h(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t)), \tag{1.2}$$

$$u'(0) = u''(0) = u(1) = 0, \tag{1.3}$$

$$\alpha u(0) = -u'''(\eta), \tag{1.4}$$

$$v'(0) = v''(0) = v(1) = 0, \tag{1.5}$$

$$\alpha v(0) = -v'''(\eta), \tag{1.6}$$

where $f, h : [0,1] \times [0,+\infty)^2 \times \mathbb{R}^4 \to [0,+\infty)$ are L^∞-Carathéodory functions satisfying adequate superlinear and sublinear conditions and $0 < \eta < 1$, $1 < \alpha < 6$.

Note that our system, in addition to being fourth-order and nonlinear in the unknown functions and their derivatives (first and second), has non-separated variables and the boundary conditions are in three points. These characteristics make research and above all applications in this type of system scarce and very interesting.

To obtain our results, that is, the existence of positive solutions, we base our study on the work of Minhós and Robert in [21], we use Guo–Krasnosel'skii compression-expansion theorem on cones, we impose the functions f and h conditions of superlinearity/sublinearity, in addition we guarantee that the Green's function associated to the linear problem and its first and second derivative are nonnegative with certain conditions and verify some adequate estimates.

The paper is organized in the following way. In Section 2 we present some auxiliary results and definitions. Section 3 contains an existence result for the existence of positive solutions of the coupled system (1.1)–(1.6). In the last section, an example is used to show the applicability of the main result.

2. Preliminary

Lemma 2.1. *The pair of functions* $(u(t), v(t)) \in \left(C^4[0,1], (0,+\infty)\right)^2$ *is a solution of problem* (1.1)–(1.6) *if and only if it is a solution of the following system of integral equations*

$$u(t) = \int_0^1 G(t,s) f\left(s, v(s), u(s), v'(s), u'(s), v''(s), u''(s)\right) ds, \tag{2.1}$$

$$v(t) = \int_0^1 G(t,s) h(s, u(s), v(s), u'(s), v'(s), u''(s), v''(s)) ds, \tag{2.2}$$

where $G(t,s)$ *is the Green's function associated to problem* (1.1)–(1.6)*, defined by*

$$G(t,s) = \frac{1}{6(\alpha-6)} \begin{cases} (\alpha t^3 - 1)(s-1)^3, & 0 \le t \le \eta \le s \le 1, \\ (\alpha-6)(t-s)^3 + (\alpha t^3 - 1)(s-1)^3 \\ \quad + 6(t^3-1), & 0 \le s \le \eta \le t \le 1, \\ (\alpha-6)(t-s)^3 + (\alpha t^3 - 1)(s-1)^3, & 0 \le \eta \le s \le t \le 1, \\ (\alpha t^3 - 1)(s-1)^3 + 6(t^3-1), & 0 \le t \le s \le \eta < 1. \end{cases} \tag{2.3}$$

Proof. Firstly, we solve the associated homogeneous system of (1.1)–(1.6) using the method of variation of parameters for constant coefficients.

Let $u_h(t) = At^3 + Bt^2 + Ct + D$. Computing the Wronskian we get

$$W = \begin{vmatrix} t^3 & t^2 & t & 1 \\ 3t^2 & 2t & 1 & 0 \\ 6t & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{vmatrix} = 12, \quad W_1 = \begin{vmatrix} 0 & t^2 & t & 1 \\ 0 & 2t & 1 & 0 \\ 0 & 2 & 0 & 0 \\ f(t) & 0 & 0 & 0 \end{vmatrix} = 2f(t),$$

$$W_2 = \begin{vmatrix} t^3 & 0 & t & 1 \\ 3t^2 & 0 & 1 & 0 \\ 6t & 0 & 0 & 0 \\ 6 & f(t) & 0 & 0 \end{vmatrix} = -6f(t), \quad W_3 = \begin{vmatrix} t^3 & t^2 & 0 & 1 \\ 3t^2 & 2t & 0 & 0 \\ 6t & 2 & 0 & 0 \\ 6 & 0 & f(t) & 0 \end{vmatrix} = 6t^2 f(t),$$

$$W_4 = \begin{vmatrix} t^3 & t^2 & t & 0 \\ 3t^2 & 2t & 1 & 0 \\ 6t & 2 & 0 & 0 \\ 6 & 0 & 0 & f(t) \end{vmatrix} = -2t^3 f(t).$$

Then, we have:

$$u_1' = \frac{W_1}{W} = \frac{1}{6}f(t), \qquad u_2' = \frac{W_2}{W} = -\frac{1}{2}tf(t),$$
$$u_3' = \frac{W_3}{W} = \frac{1}{2}t^2 f(t), \quad u_4' = \frac{W_4}{W} = -\frac{1}{6}t^3 f(t).$$

The solution and its derivatives are expressed as

$$u(t) = At^3 + Bt^2 + Ct + D + \frac{1}{6}\int_0^t (t^3 - s^3) f(s)ds + \frac{1}{2}\int_0^t (ts^2 - t^2 s) f(s)ds, \tag{2.4}$$

$$u'(t) = 3At^2 + 2Bt + C + \frac{1}{2}t^2 \int_0^t f(s)ds - t\int_0^t sf(s)ds + \frac{1}{2}\int_0^t s^2 f(s)ds, \tag{2.5}$$

$$u''(t) = 6At + 2B + t\int_0^t f(s)ds - \int_0^t sf(s)ds, \tag{2.6}$$

$$u'''(t) = 6A + \int_0^t f(s)ds. \tag{2.7}$$

Applying the boundary conditions on (2.4)–(2.7) we get

$$A = -D - \frac{1}{6}\int_0^1 (1 - s^3) f(s)ds - \frac{1}{2}\int_0^1 (s^2 - s) f(s)ds,$$
$$D = \frac{1}{6-\alpha}\left(\int_0^1 (s-1)^3 f(s)ds + \int_0^\eta f(s)ds\right).$$

Then,

$$A = \int_0^1 \left(\frac{\alpha(s-1)^3}{6(\alpha-6)}\right) f(s)ds + \int_0^\eta \frac{f(s)}{\alpha - 6}ds.$$

So, (2.4) becomes

$$u(t) = \int_0^t \frac{(\alpha-6)(t-s)^3 + (\alpha t^3 - 1)(s-1)^3}{6(\alpha-6)} f(s)ds + \int_t^1 \frac{(\alpha t^3 - 1)(s-1)^3}{6(\alpha-6)} f(s)ds$$

$$+\int_0^\eta \frac{6t^3-6}{6(\alpha-6)} f(s)ds.$$

Giving us the Green's function (2.3). ☐

Now consider the space $X = C^2[0,1]$ equipped with the norm $\|\cdot\|_{C^2}$, defined by $\|w\|_{C^2} := \max\{\|w\|_\infty, \|w'\|_\infty, \|w''\|_\infty\}$ and $\|y\|_\infty := \max_{t\in[0,1]} |y(t)|$.

Lemma 2.2. $(X, \|\cdot\|_{C^2})$ *is a real Banach space.*

Proof. It should be clear, from the linearity of the derivative, that X is a vector space. So first we show that the space $(X, \|\cdot\|_{C^2})$ is normed with

$$\|w\|_{C^2} := \max\{\|w\|_\infty, \|w'\|_\infty, \|w''\|_\infty\}$$

and

$$\|y\|_\infty := \max_{t\in[0,1]} |y(t)|.$$

Given $w \in X$, then clearly, $\|w\|_{C^2} \geq 0$ from the definition. So suppose, $\|w\|_{C^2} = 0$. Then, $\|w\|_\infty = \|w'\|_\infty = \|w''\|_\infty = 0$, that is, $w(t) = 0$ for all $t \in \mathbb{R}$. Conversely, if $w(t) = 0$ for all $t \in \mathbb{R}$, then $\|w\|_\infty = \|w'\|_\infty = \|w''\|_\infty = 0$.

Take $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned}
\|\lambda w\|_{C^2} &= \max\{\max|\lambda w(t)|, \max|(\lambda w)'(t)|, \max|(\lambda w)''(t)\} \\
&= \max\{|\lambda|\max|w(t)|, |\lambda|\max|w'(t)|, |\lambda|\max|w''(t)|\} \\
&= |\lambda|\cdot\|w\|_{C^2}.
\end{aligned}$$

Let $x, y \in X$. To show the triangle inequality, note that:

$$\begin{aligned}
|x+y| &\leq |x|+|y| \leq \max|x| + \max|y|, \\
|x'+y'| &\leq |x'|+|y'| \leq \max|x'| + \max|y'|, \\
|x''+y''| &\leq |x''|+|y''| \leq \max|x''| + \max|y''|.
\end{aligned}$$

Taking the max of both sides gives

$$\begin{aligned}
\max|x+y| &\leq \max\{\max|x| + \max|y|\} \leq \max|x| + \max|y|, \\
\max|x'+y'| &\leq \max\{\max|x'| + \max|y'|\} \leq \max|x'| + \max|y'|, \\
\max|x''+y''| &\leq \max\{\max|x''| + \max|y''|\} \leq \max|x''| + \max|y''|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\max|x+y| &\leq \max|x| + \max|y| \\
&\leq \max\{\max|x|, \max|x'|, \max|x''|\} + \max\{\max|y|, \max|y'|, \max|y''|\}, \\
\max|x'+y'| &\leq \max|x'| + \max|y'| \\
&\leq \max\{\max|x|, \max|x'|, \max|x''|\} + \max\{\max|y|, \max|y'|, \max|y''|\}, \\
\max|x''+y''| &\leq \max|x''| + \max|y''| \\
&\leq \max\{\max|x|, \max|x'|, \max|x''|\} + \max\{\max|y|, \max|y'|, \max|y''|\},
\end{aligned}$$

giving us

$$\max\{\max|x+y|, \max|x'+y'|, \max|x''+y''|\} \le \max\{\max|x|, \max|x'|, \max|x''|\} + \max\{\max|y|, \max|y'|, \max|y''|\}.$$

That is, $\|x+y\|_{C^2} \le \|x\|_{C^2} + \|y\|_{C^2}$, thus showing us that the norm is well defined.

Lastly, we show that the space $(X, \|\cdot\|_{C^2})$ is complete in the norm

$$\|x-y\|_{C^2} = \max\{\max|x-y|, \max|x'-y'|, \max|x''-y''|\}.$$

Let (x_m) be an arbitrary Cauchy sequence in X. And let us denote $\|x-y\|_{C^2}$ by $d(x,y)$. Then, for any $\varepsilon > 0$, there exists an $N \in \mathbb{R}$ such that for all $m, n > N$

$$d(x_m, x_n) = \max\{\max|x_m - x_n|, \max|(x_m - x_n)'|, \max|(x_m - x_n)''|\} < \varepsilon.$$

So, for every fixed $t_0 \in [0,1]$ we have

$$\max\{|x_m(t_0) - x_n(t_0)|, |(x_m(t_0) - x_n(t_0))'|, |(x_m(t_0) - x_n(t_0))''|\} < \varepsilon,$$

that is,

$$|x_m(t_0) - x_n(t_0)| < \varepsilon, \quad |(x_m(t_0) - x_n(t_0))'| < \varepsilon, \quad |(x_m(t_0) - x_n(t_0))''| < \varepsilon.$$

So, the sequence of numbers $(x_1(t_0), x_2(t_0), x_3(t_0), \dots)$, $(x_1'(t_0), x_2'(t_0), x_3'(t_0), \dots)$ and $(x_1''(t_0), x_2''(t_0), x_3''(t_0), \dots)$ are Cauchy, and each of them converges (see [15, Theorem 1.4 – 4]), i.e.,

$$x_m(t_0) \to x(t_0) \in \mathbb{R}, \quad x_m'(t_0) \to x'(t_0) \in \mathbb{R}, \quad x_m''(t_0) \to x''(t_0) \in \mathbb{R},$$

as $m \to +\infty$. So, $x(t) \in C^2[0,1] = X$, and the space X is complete and therefore a Banach space. $\square$

Remark 2.1. The product space $E := X \times X$ with the norm

$$\|(u,v)\|_E := \max\{\|u\|_{C^2}, \|v\|_{C^2}\},$$

is also a Banach space.

The next lemmas provide some properties of the Green's function and its first and second derivatives.

Lemma 2.3. *Let $G(t,s)$ be given by* (2.3) *and $0 < \eta < 1$, $1 < \alpha < 6$. Then,*

$$G(t,s) < h_0(s) := \frac{2(1-s)^3 + 1}{\eta(6-\alpha)} > 0.$$

Proof. Recall the Green's function

$$G(t,s) = \frac{1}{6(\alpha-6)} \begin{cases} (\alpha t^3 - 1)(s-1)^3, & 0 \le t \le \eta \le s \le 1, \\ (\alpha - 6)(t-s)^3 + (\alpha t^3 - 1)(s-1)^3 & \\ +6(t^3 - 1), & 0 \le s \le \eta \le t \le 1, \\ (\alpha - 6)(t-s)^3 + (\alpha t^3 - 1)(s-1)^3, & 0 \le \eta \le s \le t \le 1, \\ (\alpha t^3 - 1)(s-1)^3 + 6(t^3 - 1), & 0 \le t \le s \le \eta < 1. \end{cases}$$

For the first branch we have

$$\frac{(\alpha t^3-1)(s-1)^3}{6(\alpha-6)}=\frac{(\alpha t^3-1)(1-s)^3}{6(6-\alpha)}\leq\frac{6(1-s)^3}{6(6-\alpha)}<\frac{2(1-s)^3+1}{\eta(6-\alpha)}.$$

For the branch 2 we have

$$\begin{aligned}&\frac{(\alpha-6)(t-s)^3+(\alpha t^3-1)(s-1)^3+6(t^3-1)}{6(\alpha-6)}\\=&\frac{(6-\alpha)(t-s)^3+(\alpha t^3-1)(1-s)^3+6(1-t^3)}{6(6-\alpha)}\leq\frac{(6-\alpha)(1-s)^3+5(1-s)^3+6}{6(6-\alpha)}\\\leq&\frac{6(1-s)^3+6(1-s)^3+6}{6(6-\alpha)}=\frac{2(1-s)^3+1}{6-\alpha}<\frac{2(1-s)^3+1}{\eta(6-\alpha)}.\end{aligned}$$

For the branch 3 we have

$$\begin{aligned}\frac{(\alpha-6)(t-s)^3+(\alpha t^3-1)(s-1)^3}{6(\alpha-6)}=&\frac{(6-\alpha)(t-s)^3+(\alpha t^3-1)(1-s)^3}{6(6-\alpha)}\\\leq&\frac{(6-\alpha)(1-s)^3+5(1-s)^3}{6(6-\alpha)}\\\leq&\frac{6(1-s)^3+6(1-s)^3}{6(6-\alpha)}=\frac{2(1-s)^3}{6-\alpha}\\<&\frac{2(1-s)^3+1}{\eta(6-\alpha)}.\end{aligned}$$

For the branch 4 we have

$$\frac{(\alpha t^3-1)(1-s)^3+6(1-t^3)}{6(6-\alpha)}\leq\frac{6(1-s)^3+6}{6(6-\alpha)}<\frac{2(1-s)^3+1}{\eta(6-\alpha)}.$$

□

Lemma 2.4. *Let* $0<\eta<1$, $1<\alpha<6$ *and* $\frac{1}{\sqrt[3]{\alpha}}<t\leq 1$. *Then, for* $h_0=\frac{2(1-s)^3+1}{\eta(6-\alpha)}$ *and* $k_0:=\frac{(\alpha t^3-1)(1-s)^3\eta}{12(1-s)^3+6}\geq 0$, *we have*

$$G(t,s)\geq k_0(t,s)h_0(s).$$

Proof. Clearly, $\frac{G(t,s)}{h_0(s)}<1$ from Lemma 2.3.

Now, we want to take the smallest possible positive expression for k_0 at each branch.

Branch 1 ($0\leq t\leq\eta\leq s\leq 1$).

We aim to satisfy

$$k_0\leq\frac{G(t,s)}{h_0(s)}=\frac{(\alpha t^3-1)(s-1)^3(6-\alpha)\eta}{6(\alpha-6)(2(1-s)^3+1)}=\frac{(\alpha t^3-1)(1-s)^3\eta}{12(1-s)^3+6}.$$

Now notice that

$$\frac{(\alpha t^3-1)(1-s)^3\eta}{12(1-s)^3+6}\geq 0,$$

since $t\geq\frac{1}{\sqrt[3]{\alpha}}$.

Branch 2 $(0 \le s \le \eta \le t \le 1)$. we have

$$k_0 \le \frac{G(t,s)}{h_0(s)} = \frac{[(\alpha-6)(t-s)^3 + (\alpha t^3-1)(s-1)^3 + 6(t^3-1)]\eta(6-\alpha)}{6(\alpha-6)(2(1-s)^3+1)}$$
$$= \frac{[(6-\alpha)(t-s)^3 + (\alpha t^3-1)(1-s)^3 + 6(1-t^3)]\eta}{6(2(1-s)^3+1)}.$$

Again,

$$\frac{[(6-\alpha)(t-s)^3 + (\alpha t^3-1)(1-s)^3 + 6(1-t^3)]\eta}{6(2(1-s)^3+1)} \ge \frac{(\alpha t^3-1)(1-s)^3\eta}{12(1-s)^3+6} \ge 0.$$

Branch 3 $(0 \le \eta \le s \le t \le 1)$. Now,

$$k_0 \le \frac{G(t,s)}{h_0(s)} = \frac{[(\alpha-6)(t-s)^3 + (\alpha t^3-1)(s-1)^3]\eta(6-\alpha)}{6(\alpha-6)(2(1-s)^3+1)}$$
$$= \frac{[(6-\alpha)(t-s)^3 + (\alpha t^3-1)(1-s)^3]\eta}{6(2(1-s)^3+1)}$$

and

$$\frac{[(6-\alpha)(t-s)^3 + (\alpha t^3-1)(1-s)^3]\eta}{6(2(1-s)^3+1)} \ge \frac{(\alpha t^3-1)(1-s)^3\eta}{12(1-s)^3+6} \ge 0.$$

Branch 4 $(0 \le t \le s \le \eta < 1)$. We have

$$k_0 \le \frac{G(t,s)}{h_0(s)} = \frac{[(\alpha t^3-1)(s-1)^3 + 6(t^3-1)](6-\alpha)\eta}{6(\alpha-6)(2(1-s)^3+1)}$$
$$= \frac{[(\alpha t^3-1)(1-s)^3 + 6(1-t^3)]\eta}{6(2(1-s)^3+1)}$$

and

$$\frac{[(\alpha t^3-1)(1-s)^3 + 6(1-t^3)]\eta}{6(2(1-s)^3+1)} \ge \frac{(\alpha t^3-1)(1-s)^3\eta}{12(1-s)^3+6} \ge 0.$$

Hence,

$$\frac{(\alpha t^3-1)(1-s)^3}{6(6-\alpha)} \le G(t,s) < \frac{2(1-s)^3+1}{\eta(6-\alpha)}.$$

□

Lemma 2.5. *Let $\frac{\partial G}{\partial t}(t,s)$ be the first derivative of the Green's function given by* (2.3)*, with $0<\eta<1$ and $1<\alpha<6$. Then,*

$$\frac{\partial G}{\partial t}(t,s) < h_1(s) := \frac{3(1-s)^2+1}{\eta(6-\alpha)} > 0, \quad \textit{for all } t,s \in [0,1].$$

Proof. The first derivative of (2.3) is given by

$$\frac{\partial G}{\partial t}(t,s)=\frac{1}{6(\alpha-6)}\begin{cases}3\alpha t^2(s-1)^3, & 0\le t\le\eta\le s\le 1,\\ 3(\alpha-6)(t-s)^2+3\alpha t^2(s-1)^3+18t^2, & 0\le s\le\eta\le t\le 1,\\ 3(\alpha-6)(t-s)^2+3\alpha t^2(s-1)^3, & 0\le\eta\le s\le t\le 1,\\ 3\alpha t^2(s-1)^3+18t^2, & 0\le t\le s\le\eta<1.\end{cases}$$

Branch 1:

$$\frac{3\alpha t^2(s-1)^3}{6(\alpha-6)}\le\frac{3\alpha(1-s)^3}{6(6-\alpha)}\le\frac{\alpha(1-s)^2}{2(6-\alpha)}\le\frac{6(1-s)^2}{2(6-\alpha)}<\frac{3(1-s)^2+1}{\eta(6-\alpha)}.$$

Branch 2:

$$\begin{aligned}\frac{3(\alpha-6)(t-s)^2+3\alpha t^2(s-1)^3+18t^2}{6(\alpha-6)}&=\frac{3(6-\alpha)(t-s)^2+3\alpha t^2(1-s)^3-18t^2}{6(6-\alpha)}\\&\le\frac{3(6-\alpha)(1-s)^2+3\alpha(1-s)^3}{6(6-\alpha)}\\&=\frac{(6-\alpha)(1-s)^2+\alpha(1-s)^3}{2(6-\alpha)}\\&\le\frac{(6-\alpha)(1-s)^2+\alpha(1-s)^2}{2(6-\alpha)}\\&=\frac{(1-s)^2(6-\alpha+\alpha)}{2(6-\alpha)}\\&\le\frac{3(1-s)^2}{\eta(6-\alpha)}<\frac{3(1-s)^2+1}{\eta(6-\alpha)}.\end{aligned}$$

Branch 3:

$$\begin{aligned}\frac{3(\alpha-6)(t-s)^2+3\alpha t^2(s-1)^3}{6(\alpha-6)}&\le\frac{3(6-\alpha)(1-s)^2+3\alpha(1-s)^3}{6(6-\alpha)}\\&=\frac{(6-\alpha)(1-s)^2+\alpha(1-s)^3}{2(6-\alpha)}\\&\le\frac{(6-\alpha)(1-s)^2+\alpha(1-s)^2}{2(6-\alpha)}\\&=\frac{(1-s)^2(6-\alpha+\alpha)}{2(6-\alpha)}\\&\le\frac{3(1-s)^2}{\eta(6-\alpha)}<\frac{3(1-s)^2+1}{\eta(6-\alpha)}.\end{aligned}$$

Branch 4:

$$\frac{3\alpha t^2(s-1)^3+18t^2}{6(\alpha-6)}=\frac{3\alpha t^2(1-s)^3-18t^2}{6(6-\alpha)}\le\frac{3\alpha(1-s)^3}{6(6-\alpha)}$$

$$\leq \frac{\alpha(1-s)^2}{2(6-\alpha)} \leq \frac{3(1-s)^2}{\eta(6-\alpha)} < \frac{3(1-s)^2+1}{\eta(6-\alpha)}.$$

□

Lemma 2.6. *Under the assumptions $0 < \eta < 1$, $1 < \alpha < 6$, $\eta \leq s \leq 1$ and $\frac{1}{\sqrt[3]{\alpha}} < t \leq 1$, the first derivative of the Green's function satisfies*

$$\frac{\partial G}{\partial t}(t,s) \geq k_1(t,s)h_1(s),$$

with

$$h_1(s) = \frac{3(1-s)^2+1}{\eta(6-\alpha)} \quad \text{and} \quad k_1(t,s) = \frac{\alpha t^2(1-s)^3\eta}{6(1-s)^3+2}.$$

Both $h_1(s)$ and $k_1(t,s)$ are positive in their respective domains.

Proof. We analyze each branch of the Green's function derivative where the inequality holds.

Branch 1 ($0 \leq t \leq \eta \leq s \leq 1$). The derivative is

$$\frac{\partial G}{\partial t}(t,s) = \frac{3\alpha t^2(s-1)^3}{6(\alpha-6)} = \frac{\alpha t^2(1-s)^3}{2(6-\alpha)}.$$

Therefore,

$$\begin{aligned}\frac{\partial G}{\partial t}(t,s)\frac{1}{h_1(s)} &= \frac{\alpha t^2(1-s)^3}{2(6-\alpha)} \cdot \frac{\eta(6-\alpha)}{3(1-s)^2+1} \\ &= \frac{\alpha t^2(1-s)^3\eta}{2[3(1-s)^2+1]} \geq \frac{\alpha t^2(1-s)^3\eta}{6(1-s)^3+2} = k_1(t,s).\end{aligned}$$

Branch 3 ($0 \leq \eta \leq s \leq t \leq 1$). The derivative is

$$\frac{\partial G}{\partial t}(t,s) = \frac{3(\alpha-6)(t-s)^2+3\alpha t^2(s-1)^3}{6(\alpha-6)} = \frac{(6-\alpha)(t-s)^2+\alpha t^2(1-s)^3}{2(6-\alpha)}.$$

Then, it follows that

$$\begin{aligned}\frac{\partial G}{\partial t}(t,s)\frac{1}{h_1(s)} &= \frac{(6-\alpha)(t-s)^2+\alpha t^2(1-s)^3}{2(6-\alpha)} \cdot \frac{\eta(6-\alpha)}{3(1-s)^2+1} \\ &= \frac{[(6-\alpha)(t-s)^2+\alpha t^2(1-s)^3]\eta}{2[3(1-s)^2+1]}.\end{aligned}$$

Since $(6-\alpha)(t-s)^2 \geq 0$, we have:

$$\frac{[(6-\alpha)(t-s)^2+\alpha t^2(1-s)^3]\eta}{2[3(1-s)^2+1]} \geq \frac{\alpha t^2(1-s)^3\eta}{2[3(1-s)^2+1]} \geq k_1(t,s),$$

using the same inequality as in Branch 1.

On the other hand, the positivity of k_1 is ensured for $\frac{1}{\sqrt[3]{\alpha}} < t \leq 1$ and $\eta \leq s < 1$, since $\alpha > 0$, $t^2 > 0$, $\eta > 0$, $6(1-s)^3 + 2 > 0$ and $(1-s)^3 > 0$ for $s < 1$. Therefore, in both relevant branches we have

$$\frac{\partial G}{\partial t}(t,s) \geq k_1(t,s)h_1(s). \qquad \square$$

Lemma 2.7. *Let* $\frac{\partial^2 G}{\partial t^2}(t,s)$ *be the second derivative of* (2.3) *and* $0 < \eta < 1$, $1 < \alpha < 6$. *Then,*

$$\frac{\partial^2 G}{\partial t^2}(t,s) < h_2(s) := \frac{6(1-s)+1}{\eta(6-\alpha)} > 0.$$

Proof. The second derivative of (2.3) is given by

$$\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{1}{6(\alpha-6)}\begin{cases} 6\alpha t(s-1)^3, & 0 \leq t \leq \eta \leq s \leq 1, \\ 6(\alpha-6)(t-s) + 6\alpha t(s-1)^3 + 36t, & 0 \leq s \leq \eta \leq t \leq 1, \\ 6(\alpha-6)(t-s) + 6\alpha t(s-1)^3, & 0 \leq \eta \leq s \leq t \leq 1, \\ 6\alpha t(s-1)^3 + 36t, & 0 \leq t \leq s \leq \eta < 1. \end{cases}$$

Branch 1:

$$\frac{6\alpha t(s-1)^3}{6(\alpha-6)} \leq \frac{\alpha(1-s)^3}{(6-\alpha)} \leq \frac{\alpha(1-s)}{(6-\alpha)} \leq \frac{6(1-s)}{\eta(6-\alpha)} < \frac{6(1-s)+1}{\eta(6-\alpha)}.$$

Branch 2:

$$\begin{aligned} \frac{6(\alpha-6)(t-s) + 6\alpha t(s-1)^3 + 36t}{6(\alpha-6)} &= \frac{(6-\alpha)(t-s) + \alpha t(1-s)^3 - 6t}{(6-\alpha)} \\ &\leq \frac{(6-\alpha)(1-s) + \alpha(1-s)^3}{(6-\alpha)} \\ &= \frac{(1-s)(6-\alpha+\alpha(1-s)^2)}{(6-\alpha)} \\ &= \frac{(1-s)(6-\alpha+\alpha-2\alpha s+\alpha s^2)}{(6-\alpha)} \\ &\leq \frac{(1-s)(6-2\alpha+\alpha)}{(6-\alpha)} \\ &= \frac{(1-s)(6-\alpha)}{(6-\alpha)} \leq \frac{6(1-s)}{(6-\alpha)} < \frac{6(1-s)+1}{\eta(6-\alpha)}. \end{aligned}$$

Branch 3:

$$\begin{aligned} \frac{6(\alpha-6)(t-s) + 6\alpha t(s-1)^3}{6(\alpha-6)} &\leq \frac{(6-\alpha)(1-s) + \alpha(1-s)^3}{(6-\alpha)} \\ &= \frac{(1-s)(6-\alpha+\alpha(1-s)^2)}{(6-\alpha)} \\ &= \frac{(1-s)(6-\alpha+\alpha-2\alpha s+\alpha s^2)}{(6-\alpha)} \end{aligned}$$

$$\leq \frac{(1-s)(6-2\alpha+\alpha)}{(6-\alpha)} = \frac{(1-s)(6-\alpha)}{(6-\alpha)}$$
$$\leq \frac{6(1-s)}{(6-\alpha)} < \frac{6(1-s)+1}{\eta(6-\alpha)}.$$

Branch 4:
$$\frac{6\alpha t(s-1)^3+36t}{6(\alpha-6)} \leq \frac{\alpha(1-s)^3-6}{(6-\alpha)} \leq \frac{\alpha(1-s)}{(6-\alpha)} < \frac{6(1-s)+1}{\eta(6-\alpha)}.$$

□

Lemma 2.8. *Let* $0<\eta<1$, $1<\alpha<6$, $\eta \leq s \leq 1$ *and* $\frac{1}{\sqrt[3]{\alpha}} < t \leq 1$. *Then,*
$$\frac{\partial^2 G}{\partial t^2}(t,s) \geq k_2(t,s)h_2(s),$$
where
$$h_2(s) = \frac{6(1-s)+1}{\eta(6-\alpha)} > 0 \quad \text{and} \quad k_2(t,s) = \frac{\alpha t(1-s)^3\eta}{6(1-s)+1} \geq 0.$$
Moreover, $k_2(t,s) > 0$ *for* $t > \frac{1}{\sqrt[3]{\alpha}}$ *and* $s<1$.

Proof. We analyze the second derivative of the Green's function in the relevant branches.

Branch 1 ($0 \leq t \leq \eta \leq s \leq 1$). The second derivative is
$$\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{6\alpha t(s-1)^3}{6(\alpha-6)} = \frac{\alpha t(1-s)^3}{6-\alpha}.$$

So,
$$\frac{\partial^2 G}{\partial t^2}(t,s)\frac{1}{h_2(s)} = \frac{\alpha t(1-s)^3}{6-\alpha} \cdot \frac{\eta(6-\alpha)}{6(1-s)+1} = \frac{\alpha t(1-s)^3\eta}{6(1-s)+1} = k_2(t,s)$$
and
$$\frac{\partial^2 G}{\partial t^2}(t,s) = k_2(t,s)h_2(s).$$

Branch 3 ($0 \leq \eta \leq s \leq t \leq 1$). The second derivative is
$$\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{6(\alpha-6)(t-s)+6\alpha t(s-1)^3}{6(\alpha-6)} = \frac{(6-\alpha)(t-s)+\alpha t(1-s)^3}{6-\alpha}.$$

Furthermore, given that $(6-\alpha)(t-s) \geq 0$ for $t \geq s$ and $\alpha < 6$
$$\frac{\partial^2 G}{\partial t^2}(t,s)\frac{1}{h_2(s)} = \frac{(6-\alpha)(t-s)+\alpha t(1-s)^3}{6-\alpha} \cdot \frac{\eta(6-\alpha)}{6(1-s)+1}$$
$$= \frac{[(6-\alpha)(t-s)+\alpha t(1-s)^3]\eta}{6(1-s)+1} \geq \frac{\alpha t(1-s)^3\eta}{6(1-s)+1} = k_2(t,s).$$

Therefore,
$$\frac{\partial^2 G}{\partial t^2}(t,s) \geq k_2(t,s)h_2(s).$$

The additional inequality

$$\frac{\alpha t(1-s)^3}{6-\alpha} \leq \frac{\partial^2 G}{\partial t^2}(t,s)$$

holds because in Branch 1 we have equality and in Branch 3:

$$\frac{\partial^2 G}{\partial t^2}(t,s) = \frac{\alpha t(1-s)^3}{6-\alpha} + (t-s) \geq \frac{\alpha t(1-s)^3}{6-\alpha}.$$

□

Lemma 2.9. *The set*

$$K = \Big\{ w \in X : w(t) \geq 0, \min_{t\in I} w(t) \geq k_0 \|w\|_\infty, \min_{t\in I} w'(t) \geq k_1 \|w'\|_\infty,$$
$$\min_{t\in I} w''(t) \geq k_2 \|w''\|_\infty \Big\},$$

is a cone, where k_0, k_1, k_2 are given by Lemma 2.4, 2.6 *and* 2.8*, respectively, and* $I = \left(\frac{1}{\sqrt[3]{\alpha}}, 1\right]$.

Proof. It is trivial that K is non-empty and not identically zero.

Let $\gamma, \beta \in \mathbb{R}^+$ and for all $x, y \in K$. Then, $x \in K$ implies

$$x \in X : x(t) \geq 0, \min_{t\in I} x(t) \geq k_0 \|x\|_\infty, \min_{t\in I} x'(t) \geq k_1 \|x'\|_\infty, \wedge \min_{t\in I} x''(t) \geq k_2 \|x''\|_\infty,$$

and $y \in K$ implies

$$y \in X : y(t) \geq 0, \min_{t\in I} y(t) \geq k_0 \|y\|_\infty, \min_{t\in I} y'(t) \geq k_1 \|y'\|_\infty, \wedge \min_{t\in I} y''(t) \geq k_2 \|y''\|_\infty.$$

By Lemma 2.2, X is a Banach space, so for the linear combination $\gamma x + \beta y \in X$ we have

$$\begin{aligned}
\min_{t\in I} (\gamma x(t) + \beta y(t)) &= \gamma \min_{t\in I} x(t) + \beta \min_{t\in I} y(t) \\
&\geq \gamma k_0 \|x\|_\infty + \beta k_0 \|y\|_\infty = k_0 (\gamma \|x\|_\infty + \beta \|y\|_\infty) \\
&\geq k_0 \|\gamma x(t) + \beta y(t)\|_\infty, \\
\min_{t\in I} (\gamma x(t) + \beta y(t))' &= \gamma \min_{t\in I} (x(t))' + \beta \min_{t\in I} (y(t))' \\
&\geq \gamma k_1 \|x'\|_\infty + \beta k_1 \|y'\|_\infty = k_1 (\gamma \|x'\| + \beta \|y'\|_\infty) \\
&\geq k_1 \left\| (\gamma x(t) + \beta y(t))' \right\|_\infty
\end{aligned}$$

and

$$\begin{aligned}
\min_{t\in I} (\gamma x(t) + \beta y(t))'' &= \gamma \min_{t\in I} (x(t))'' + \beta \min_{t\in I} (y(t))'' \\
&\geq \gamma k_2 \|x''\|_\infty + \beta k_2 \|y''\|_\infty = k_2 (\gamma \|x''\|_\infty + \beta \|y''\|_\infty) \\
&\geq k_2 \left\| (\gamma x(t) + \beta y(t))'' \right\|_\infty .
\end{aligned}$$

So, $\gamma x + \beta y \in K$, and K is a cone. □

The regularity of the nonlinear functions f and h is given by the following definition.

Definition 2.1. A function $g : [0,1] \times \mathbb{R}^m \to [0,+\infty)$, for m a positive integer, is L^∞-Carathéodory if

i) $g(\cdot, z)$ is measurable for each fixed $z \in \mathbb{R}^m$;
ii) $g(t, \cdot)$ is continuous for a.e. $t \in [0,1]$;
iii) for each $\rho > 0$, there exists a function $\varphi_\rho \in L^\infty([0,1])$ such that $g(t,z) \le \varphi_\rho(t)$ for $z \in [-\rho, \rho]$ and a.e. $t \in [0,1]$.

The Guo-Krasnoselskii fixed point existence theorem, for expansive and compressive cones theory, will be a key tool in the arguments.

Lemma 2.10 ([12])**.** *Let $(X, \|\cdot\|)$ be a Banach space, and $P \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$.*
If $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either

(i) $\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$, $u \in P \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \ge \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_2$,

then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

For the purposes of the main theorem, in order to guarantee non-negativity in the Green's function, as well as strict positivity for k_i, and bounded h_i, for $i = 0,1,2$, we impose the following restrictions.

$(A1)$ For fixed constants α and η, with $0 < \eta < 1$ and $1 < \alpha < 6$, we impose

$$\frac{1}{\sqrt[3]{\alpha}} < t \le 1 \quad \text{and} \quad \eta \le s < 1.$$

$(A2)$ All Lemmas 2.3–2.8 are satisfied.

The nonlinearities will verify either one of the following growth assumptions.

$(B1)$ For $i = 0,1,2$ and $j = 0,1,2$

$$\limsup_{\|x_i\| \to 0, \|y_j\| \to 0} \max_{t \in I} \frac{f(t, x_0, y_0, x_1, y_1, x_2, y_2)}{\max\{\|x_i\|, \|y_j\|\}} = 0, \tag{2.8}$$

$$\limsup_{\|x_i\| \to 0, \|y_j\| \to 0} \max_{t \in I} \frac{h(t, x_0, y_0, x_1, y_1, x_2, y_2)}{\max\{\|x_i\|, \|y_j\|\}} = 0$$

and

$$\liminf_{\|x_i\| \to +\infty, \|y_j\| \to +\infty} \min_{t \in I} \frac{f(t, x_0, y_0, x_1, y_1, x_2, y_2)}{\max\{\|x_i\|, \|y_j\|\}} = +\infty, \tag{2.9}$$

$$\liminf_{\|x_i\| \to +\infty, \|y_j\| \to +\infty} \min_{t \in I} \frac{h(t, x_0, y_0, x_1, y_1, x_2, y_2)}{\max\{\|x_i\|, \|y_j\|\}} = +\infty.$$

$(B2)$ For $i = 0,1,2$ and $j = 0,1,2$

$$\liminf_{\|x_i\| \to 0, \|y_j\| \to 0} \min_{t \in I} \frac{f(t, x_0, y_0, x_1, y_1, x_2, y_2)}{\max\{\|x_i\|, \|y_j\|\}} = +\infty, \tag{2.10}$$

$$\liminf_{\|x_i\|\to 0, \|y_j\|\to 0} \min_{t\in I} \frac{h(t, x_0, y_0, x_1, y_1, x_2, y_2)}{\max\{\|x_i\|, \|y_j\|\}} = +\infty$$

and

$$\limsup_{\|x_i\|\to +\infty, \|y_j\|\to +\infty} \max_{t\in I} \frac{f(t, x_0, y_0, x_1, y_1, x_2, y_2)}{\max\{\|x_i\|, \|y_j\|\}} = 0, \tag{2.11}$$

$$\limsup_{\|x_i\|\to +\infty, \|y_j\|\to +\infty} \max_{t\in I} \frac{h(t, x_0, y_0, x_1, y_1, x_2, y_2)}{\max\{\|x_i\|, \|y_j\|\}} = 0.$$

3. Main Result

The main result is given by the next theorem and provides the existence of at least one nonnegative solution of problem (1.1)–(1.6).

Theorem 3.1. *Let $f, h : [0,1] \times [0,+\infty)^2 \times \mathbb{R}^4 \to [0,+\infty)$ be L^∞-Carathéodory functions such that conditions $(A1)$ and $(A2)$ hold, and either $(B1)$ or $(B2)$ hold. Then, the problem (1.1)–(1.6) has at least one positive solution $(u(t), v(t)) \in E = K \times K$, that is, $u(t) > 0$, $v(t) > 0$, for all $t \in I = \left(\frac{1}{\sqrt[3]{\alpha}}, 1\right]$.*

Proof. Consider the operators $T_1 : E \to K$, $T_2 : E \to K$, $T : E \to E$ such that

$$T_1(u,v)(t) = \int_\eta^1 G(t,s) f(s, u(s), v(s), u'(s), v'(s), u''(s), v''(s)) ds,$$
$$T_2(u,v)(t) = \int_\eta^1 G(t,s) h(s, u(s), v(s), u'(s), v'(s), u''(s), v''(s)) ds$$

and

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)).$$

We need to prove that the solutions of the initial system are fixed points of the operator $T := (T_1, T_2)$.

Step 1. T_1 and T_2 are well defined in K.

It suffices to prove that T_1 is well defined in K since the proof is analogous for T_2.

Take $(u_1, u_2) \in E$. Then, by conditions $(A1)$ and $(A2)$,

$$\begin{aligned}\|T_1(u_1,u_2)\|_\infty &= \max_{t\in I} \int_\eta^1 G(t,s) f(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t)) ds \\ &\le \int_\eta^1 h_0(s) f(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t)) ds\end{aligned}$$

and

$$\begin{aligned}\min_{t\in I} T_1(u_1,u_2)(t) &\ge k_0 \int_\eta^1 h_0(s) f(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t)) ds \\ &\ge k_0 \|T_1(u_1,u_2)\|_\infty.\end{aligned}$$

On the other hand, for $i = 1, 2$,

$$\begin{aligned}\| \left(T_1\left(u_1,u_2\right)\right)^{(i)} \|_\infty &= \max_{t\in I} \left\| \int_\eta^1 \frac{\partial^i G}{\partial t^i}(t,s) f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \right\| \\ &\leq \int_\eta^1 h_i(s) f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds\end{aligned}$$

and

$$\begin{aligned}\min_{t\in I} \left(T_1\left(u_1,u_2\right)\right)^{(i)} &= \min_{t\in I} \int_\eta^1 \frac{\partial^i G}{\partial t^i}(t,s) f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \\ &\geq k_i \int_\eta^1 h_i(s) f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \\ &\geq k_i \| \left(T_1\left(u_1,u_2\right)\right)^{(i)} \|_\infty .\end{aligned}$$

So, for $i = 0, 1, 2$,

$$\min_{t\in I} T_1\left(u_1,u_2\right)(t) \geq c \| T_1\left(u_1,u_2\right) \|_{C^2},$$

with $0 < c \leq \max\{k_i : i = 0,1,2\} \leq 1$.

Therefore, $T_1 E \subseteq K$. Analogously, $T_2 E \subseteq K$ and so $TE \subset E$.

Step 2. T_1 and T_2 are completely continuous in K, that is, T_1 and T_2 are uniformly bounded and equicontinuous on K.

• T_1 is uniformly bounded in K.

Let $(u_1,u_2) \in E$ such that $\|(u_1,u_2)\|_E \leq \rho$, for some $\rho > 0$. By conditions (A1), (A2) and Definition 2.1,

$$\begin{aligned}\|T_1\left(u_1,u_2\right)\|_\infty &= \max_{t\in I} \|T_1\left(u_1,u_2\right)(t)\| \\ &\leq \int_\eta^1 h_0(s) f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \\ &\leq \int_\eta^1 h_0(s)\varphi_\rho(s)ds < +\infty,\end{aligned}$$

and, for $i = 1, 2$,

$$\begin{aligned}\|\left(T_1\left(u_1,u_2\right)\right)^{(i)}\|_\infty &= \max_{t\in[0,1]} \left\| \int_\eta^1 \frac{\partial^{(i)} G}{\partial t^i}(t,s) f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \right\| \\ &\leq \int_\eta^1 h_i(s) f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \\ &\leq \int_\eta^1 h_i(s)\varphi_\rho(s)ds < +\infty.\end{aligned}$$

Therefore, $\|T_1(u_1,u_2)\|_{C^2} < +\infty$, and, so, T_1 is uniformly bounded in K. Analogously, T_2 is uniformly bounded in K, and so T is uniformly bounded in E.

• T_1 is equicontinuous in K.

This step will show that T_1 is equicontinuous in K. As for T_2 the proof is analogous and will be omitted.

Let t_1, $t_2 \in I$. Then,

$$\begin{aligned}
&\|T_1(u_1,u_2)(t_1) - T_1(u_1,u_2)(t_2)\| \\
&\le \int_\eta^1 \|G(t_1,s)-G(t_2,s)\| f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \\
&\le \int_\eta^1 \|G(t_1,s)-G(t_2,s)\| \varphi_\rho(s)ds \to 0 \quad \text{as } t_1 \to t_2,
\end{aligned}$$

for $i = 1, 2$,

$$\begin{aligned}
&|(T_1(u_1,u_2))^{(i)}(t_1) - (T_1(u_1,u_2))^{(i)}(t_2)| \\
&\le \int_\eta^1 \left\| \frac{\partial^i G}{\partial t^i}(t_1,s) - \frac{\partial^i G}{\partial t^i}(t_2,s) \right\| f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \\
&\le \int_\eta^1 \left\| \frac{\partial^i G}{\partial t^i}(t_1,s) - \frac{\partial^i G}{\partial t^i}(t_2,s) \right\| \varphi_\rho(s)ds \to 0 \quad \text{as } t_1 \to t_2.
\end{aligned}$$

Therefore, T_1 is equicontinuous in K. Consequently, T is equicontinuous in E.

By the Arzelà-Ascoli Theorem, T is completely continuous in E.

Assume that condition (B1) holds.

Step 3. $\|T(u_1,u_2)\|_E \le \|(u_1,u_2)\|_E$, for $(u_1,u_2) \in E \cap \partial\Omega_1$ with $\Omega_1 = \{(u_1,u_2) \in E : \|(u_1,u_2)\|_E < \rho_1\}$, for some $\rho_1 > 0$.

To prove that

$$\max\{\|T_1(u_1,u_2)\|_{C^2}, \|T_2(u_1,u_2)\|_{C^2}\} \le \|(u_1,u_2)\|_E,$$

it suffices to prove that

$$\|T_1(u_1,u_2)\|_{C^2} \le \|(u_1,u_2)\|_E \quad \text{and} \quad \|T_2(u_1,u_2)\|_{C^2} \le \|(u_1,u_2)\|_E.$$

Set $(u_1,u_2) \in E \cap \partial\Omega_1$. Then, $\|(u_1,u_2)\|_E = \rho_1$. For $i = 0, 1, 2$, we define

$$\varepsilon := \min\left\{ \frac{1}{\int_\eta^1 h_i(s)ds} \right\}. \tag{3.1}$$

By (2.8), there exists $0 < \rho_1 < 1$ such that

$$f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t)) \le \varepsilon\|(u_1,u_2)\|_E, \tag{3.2}$$

for $\|(u_1,u_2)\|_E \le \rho_1$. By (3.1), and (3.2),

$$\begin{aligned}
T_1(u_1,u_2)(t) &= \int_\eta^1 G(t,s) f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \\
&\le \int_\eta^1 h_0(s)\varepsilon\|(u_1,u_2)\|_E ds \\
&= \varepsilon\rho_1 \int_\eta^1 h_0(s)ds \le \rho_1 = \|(u_1,u_2)\|_E,
\end{aligned}$$

and, for $i = 1, 2$,

$$(T_1(u_1,u_2)(t))^{(i)} = \int_\eta^1 \frac{\partial^i G}{\partial t^i}(t,s) f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds$$

$$\leq \int_\eta^1 h_i \varepsilon \|(u_1, u_2)\|_E ds \leq \rho_1 = \|(u_1, u_2)\|_E.$$

Therefore, $\|T_1(u_1, u_2)\|_{C^2} \leq \|(u_1, u_2)\|_E$, for $(u_1, u_2) \in E \cap \partial\Omega_1$. Similarly, it can be proved that $\|T_2(u_1, u_2)\|_{C^2} \leq \|(u_1, u_2)\|_E$, and so, $\|T(u_1, u_2)\|_E \leq \|(u_1, u_2)\|_E$, for $(u_1, u_2) \in E \cap \partial\Omega_1$.

Step 4. $\|T\,(u_1, u_2)\,\|_E \geq \|(u_1, u_2)\|_E$, for $(u_1, u_2) \in E \cap \partial\Omega_2$ with $\Omega_2 = \{(u_1, u_2) \in E : \|(u_1, u_2)\|_E < \rho_2\}$, for some $\rho_2 > 0$. If there is $i_0 \in \{0, 1, 2\}$ or $j_0 \in \{0, 1, 2\}$, such that $u_1^{(i_0)}(t) \to +\infty$ and $u_2^{(j_0)}(t) \to +\infty$, then $\|(u_1, u_2)\|_E \to +\infty$.

Taking notice of conditions (A1) and (A2), define, for $i = 0, 1, 2$,

$$\delta := \max \left\{ \frac{1}{k_i \int_\eta^1 h_i(s) ds} \right\}. \tag{3.3}$$

By (2.9), there exist $\theta > 0$ such that when $\|(u_1, u_2)\|_E \geq \theta$ we have

$$f(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t)) \geq \delta \|(u_1, u_2)\|_E. \tag{3.4}$$

Let $(u_1, u_2) \in E$ be such that $\|(u_1, u_2)\|_E = \rho_2$, with $\rho_2 < \rho_1$. Now, from (3.3) and (3.4),

$$\begin{aligned} T_1\,(u_1, u_2)\,(t) &= \int_\eta^1 G(t, s) f(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t)) ds \\ &\geq k_0 \int_\eta^1 h_0 f(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t)) ds \\ &\geq k_0 \int_\eta^1 h_0 \delta \|(u_1, u_2)\|_E ds \\ &= k_0 \delta \rho_2 \int_\eta^1 h_0(s) ds \geq \rho_2 = \|(u_1, u_2)\|_E, \end{aligned}$$

analogously, for $i = 1, 2$, we find that

$$(T_1\,(u_1, u_2)\,(t))^{(i)} \geq \|(u_1, u_2)\|_E.$$

Therefore, $\|T_1(u_1, u_2)\|_{C^2} \geq \|(u_1, u_2)\|_E$, for $(u_1, u_2) \in E \cap \partial\Omega_2$.

Similarly, $\|T_2(u_1, u_2)\|_{C^2} \geq \|(u_1, u_2)\|_E$, for $(u_1, u_2) \in E \cap \partial\Omega_2$, and, therefore, $\|T(u_1, u_2)\|_E \geq \|(u_1, u_2)\|_E$, for $(u_1, u_2) \in E \cap \partial\Omega_2$.

By Lemma 2.10, the operator T has a fixed point in $E \cap (\overline{\Omega_2} \backslash \Omega_1)$ which in turn is a solution of our problem.

Now assume that $(B2)$ holds.

Step 5. $\|T\,(u_1, u_2)\,\|_E \geq \|(u_1, u_2)\|_E$, for $u \in E \cap \partial\Omega_3$ with $\Omega_3 = \{(u_1, u_2) \in E : \|(u_1, u_2)\|_E < \rho_3\}$, for some $\rho_3 > 0$.

Taking $\delta > 0$ as in (3.3), we see that by (2.10) there exists $0 < \rho_3 < 1$ such that

$$f(t, u(t), v(t), u'(t), v'(t), u''(t), v''(t)) \geq \delta \|(u_1, u_2)\|_E.$$

Consider $(u_1, u_2) \in E$ such that $\|(u_1, u_2)\|_E = \rho_3$. Then, applying similar inequalities as in **Step 4**, we obtain that $\|T(u_1, u_2)\|_E \geq \|(u_1, u_2)\|_E$.

Step 6. $\|T(u_1,u_2)\|_E \le \|(u_1,u_2)\|_E$, for $(u_1,u_2) \in E \cap \partial\Omega_4$ with $\Omega_4 = \{(u_1,u_2) \in E : \|(u_1,u_2)\|_E < \rho_4\}$, for some $\rho_4 > 0$.

Case 6.1 Suppose that f is bounded.

Then there is an $N > 0$ such that $f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t)) \le N$ for all $(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t)) \in [0,1] \times [0,+\infty)^2 \times \mathbb{R}^4$. Let us define

$$\rho_4 := \max\left\{\rho_3 + 1, N\int_\eta^1 h_i(s)ds : i = 0,1,2\right\}$$

and take $(u_1,u_2) \in E$ with $\|(u_1,u_2)\|_E = \rho_4$. Then,

$$\begin{aligned} T_1\left(u_1,u_2\right)(t) &= \int_\eta^1 G(t,s)f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \\ &\le N\int_\eta^1 h_0(s)ds \le \rho_4, \quad \text{for } t \in I, \end{aligned}$$

and for $i = 1,2$,

$$\begin{aligned} \left(T_1\left(u_1,u_2\right)(t)\right)^{(i)} &= \int_\eta^1 \frac{\partial^i G}{\partial t^i}(t,s)f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t))ds \\ &\le N\int_\eta^1 h_i(s)ds \le \rho_4, \quad \text{for } t \in I. \end{aligned}$$

Thus, $\|T_1(u_1,u_2)\|_{C^2} \le \|(u_1,u_2)\|_E$.

The same arguments can be applied to show that $\|T_2(u_1,u_2)\|_{C^2} \le \|(u_1,u_2)\|_E$. So, $\|T(u_1,u_2)\|_E \le \|(u_1,u_2)\|_E$.

Case 6.2 Suppose that f is unbounded.

By (2.11), for every $M > 0$ such that $\|(u_1,u_2)\|_E \le M$, there exists $\mu > 0$ such that

$$\max\left\{\mu\int_\eta^1 h_i(s)ds : i = 0,1,2\right\} \le 1 \tag{3.5}$$

and

$$f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t)) \le \mu\|(u_1,u_2)\|_E. \tag{3.6}$$

Define

$$\rho_4 := \max\{M, \rho_3 + 1\}.$$

Then, for $(u_1,u_2) \in E \cap \partial\Omega_4$ we have $\|(u_1,u_2)\|_E = \rho_4$ and, by (3.6),

$$f(t,u(t),v(t),u'(t),v'(t),u''(t),v''(t)) \le \mu\|(u_1,u_2)\|_E \le \mu\rho_4. \tag{3.7}$$

By (3.5),

$$T_1\left(u_1,u_2\right)(t) \le \int_\eta^1 h_0(s)\mu\rho_4 ds \le \mu\rho_4\int_\eta^1 h_0(s)ds \le \rho_4,$$

and for $i = 1,2$,

$$\left(T_1\left(u_1,u_2\right)(t)\right)^{(i)} \le \int_\eta^1 h_i(s)\mu\rho_4 ds \le \mu\rho_4\int_\eta^1 h_i(s)ds \le \rho_4.$$

Therefore, $\|T_1(u_1,u_2)\|_{C^2} \le \|(u_1,u_2)\|_E$, for $(u_1,u_2) \in E \cap \partial\Omega_4$.

In the same way we can have $\|T_2(u_1,u_2)\|_{C^2} \leq \|(u_1,u_2)\|_E$, for $(u_1,u_2) \in E \cap \partial\Omega_4$, and therefore, $\|T(u_1,u_2)\|_E \leq \|(u_1,u_2)\|_E$, for $(u_1,u_2) \in E \cap \partial\Omega_4$.

By Lemma 2.10, the operator T has a fixed point in $E \cap (\overline{\Omega_4} \backslash \Omega_3)$ that, in turn, is a solution of the problem (1.1)–(1.6).

$\square$

4. Example

Consider the following fourth-order nonlinear system

$$\begin{aligned}
&u^{(4)}(t) = (t^4 + 2t^2 + 1)\left(e^{-|v'(t)u(t)|} + \sqrt{|u'(t) + v''(t)|}\right),\\
&v^{(4)}(t) = 3t^6\left[(u(t) + v(t))^4 + \cos^2(v'(t) + u''(t))\right],\\
&u(0) = u'(0) = u''(1) = 0,\\
&3u(0) = -u''\left(\frac{1}{2}\right),\\
&v(0) = v'(0) = v''(1) = 0,\\
&3v(0) = -v''\left(\frac{1}{2}\right).
\end{aligned}\tag{4.1}$$

This problem is a particular case of the system (1.1)–(1.6) with

$$\begin{aligned}
f(t,a_1,a_2,a_3,a_4,a_5,a_6) &:= (t^4 + 2t^2 + 1)\left(e^{-|a_4 a_1|} + \sqrt{|a_3 + a_6|}\right),\\
h(t,a_1,a_2,a_3,a_4,a_5,a_6) &:= 3t^6\left[(a_1 + a_2)^4 + \cos^2(a_4 + a_5)\right],\\
\eta = \frac{1}{2} \quad &\text{and} \quad \alpha = 3.
\end{aligned}$$

The functions f and h are L^∞-Carathéodory, since for $\rho > 0$, whenever

$$\max\{|a_1|,|a_2|,|a_3|,|a_4|,|a_5|,|a_6|\} < \rho,$$

there exist functions $\varphi_\rho, \psi_\rho \in L^\infty([0,1])$ such that

$$\begin{aligned}
f(t,a_1,a_2,a_3,a_4,a_5,a_6) &\leq (t^4 + 2t^2 + 1)\left(1 + \sqrt{|2\rho|}\right) := \varphi_\rho,\\
h(t,a_1,a_2,a_3,a_4,a_5,a_6) &\leq 3t^6(16\rho^4 + 1) := \psi_\rho.
\end{aligned}$$

The Green's function, its first and second derivatives are given by

$$G(t,s) = -\frac{1}{18}\begin{cases}
(3t^3-1)(s-1)^3, & 0 \leq t \leq \frac{1}{2} \leq s \leq 1,\\
-3(t-s)^3 + (3t^3-1)(s-1)^3 + 6(t^3-1), & 0 \leq s \leq \frac{1}{2} \leq t \leq 1,\\
-3(t-s)^3 + (3t^3-1)(s-1)^3, & 0 \leq \frac{1}{2} \leq s \leq t \leq 1,\\
(3t^3-1)(s-1)^3 + 6(t^3-1), & 0 \leq t \leq s \leq \frac{1}{2} < 1,
\end{cases}$$

$$\frac{\partial G}{\partial t}(t,s) = -\frac{1}{18}\begin{cases} 9t^2(s-1)^3, & 0 \le t \le \frac{1}{2} \le s \le 1, \\ -9(t-s)^2 + 9t^2(s-1)^3 + 18t^2, & 0 \le s \le \frac{1}{2} \le t \le 1, \\ -9(t-s)^2 + 9t^2(s-1)^3, & 0 \le \frac{1}{2} \le s \le t \le 1, \\ 9t^2(s-1)^3 + 18t^2, & 0 \le t \le s \le \frac{1}{2} < 1, \end{cases}$$

$$\frac{\partial^2 G}{\partial t^2}(t,s) = -\frac{1}{18}\begin{cases} 18t(s-1)^3, & 0 \le t \le \frac{1}{2} \le s \le 1, \\ -18(t-s) + 18t(s-1)^3 + 36t, & 0 \le s \le \frac{1}{2} \le t \le 1, \\ -18(t-s) + 18t(s-1)^3, & 0 \le \frac{1}{2} \le s \le t \le 1, \\ 18t(s-1)^3 + 36t, & 0 \le t \le s \le \frac{1}{2} < 1. \end{cases}$$

If we assume that condition $(A1)$ is verified, that is, for $t \in \left(\frac{1}{\sqrt[3]{3}}, 1\right]$ and $\frac{1}{2} \le s < 1$, the condition $(A2)$ is also confirmed. More precisely,

$$\begin{aligned} 0 < G(t,s) &\le \frac{4(1-s)^3 + 2}{3} := h_0(s), \\ 0 < \frac{\partial G}{\partial t}(t,s) &\le \frac{6(1-s)^2 + 2}{3} := h_1(s), \\ 0 < \frac{\partial^2 G}{\partial t^2}(t,s) &\le \frac{12(1-s) + 2}{3} := h_2(s) \end{aligned}$$

and for

$$\begin{aligned} k_0 &= \frac{(3t^3 - 1)(1-s)^3}{24(1-s)^3 + 12}, \\ k_1 &= \frac{3t^2(1-s)^3}{12(1-s)^3 + 4}, \\ k_2 &= \frac{3t(1-s)^3}{12(1-s) + 2}, \end{aligned}$$

we have

$$\begin{aligned} G(t,s) &\ge k_0 h_0(s), \\ \frac{\partial G}{\partial t}(t,s) &\ge k_1 h_1(s), \\ \frac{\partial^2 G}{\partial t^2}(t,s) &\ge k_2 h_2(s). \end{aligned}$$

Furthermore, as for $i = 0, 1, 2$ and $t \in I = \left(\frac{1}{\sqrt[3]{3}}, 1\right]$

$$\begin{aligned} &\liminf_{u^{(j)}, v^{(j)} \to 0} \ \min_{t \in I} \frac{(t^4 + 2t^2 + 1)\left(e^{-|v'(t)u(t)|} + \sqrt{|u'(t) + v''(t)|}\right)}{\max\left\{\left\|u^{(j)}\right\|, \left\|v^{(j)}\right\|\right\}} = +\infty, \\ &\liminf_{u^{(j)}, v^{(j)} \to 0} \ \min_{t \in I} \frac{3t^6\left[(u(t) + v(t))^4 + \cos^2(v'(t) + u''(t))\right]}{\max\left\{\left\|u^{(j)}\right\|, \left\|v^{(j)}\right\|\right\}} = +\infty \end{aligned}$$

and

$$\limsup_{u^{(j)},v^{(j)}\to+\infty} \max_{t\in I} \frac{(t^4+2t^2+1)\left(e^{-|v'(t)u(t)|}+\sqrt{|u'(t)+v''(t)|}\right)}{\max\left\{\|u^{(j)}\|,\|v^{(j)}\|\right\}}=0,$$

$$\limsup_{u^{(j)},v^{(j)}\to+\infty} \max_{t\in I} \frac{3t^6\left[(u(t)+v(t))^4+\cos^2(v'(t)+u''(t))\right]}{\max\left\{\|u^{(j)}\|,\|v^{(j)}\|\right\}}=0,$$

the condition $(B2)$ is confirmed.

So, by Theorem 3.1, the problem (4.1) has at least one positive solution $(u(t),v(t))\in\left(C^2[0,1]\right)^2$, that is $u(t)>0$, $v(t)>0$, for all $t\in\left(\frac{1}{\sqrt[3]{3}},1\right]$.

Acknowledgements. The authors would like to thank all those who contributed to this work, especially colleagues and reviewers for their valuable comments. The authors also acknowledge the Calouste Gulbenkian Foundation, through the Partnership for Africa Programme, for supporting this research within the 5th Edition of *Vocations for Gulbenkian Mathematics in Cape Verde*.

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[1]Faculdade de Ciências e Tecnologia, Núcleo de Matemática e Aplicações (NU-MAT), Centro de Investigação em Ciências Exatas, Universidade de Cabo Verde, Campus de Palmarejo Grande, Zona K 379 C - Praia, Cabo Verde.

[2]Centro de Investigação em Matemática e Aplicações (CIMA), Universidade de Évora, Rua Romão Ramalho, 59, 7000-671, Évora, Portugal

Email address: robert.sousa@docente.unicv.edu.cv
ORCID iD: https://orcid.org/0000-0002-0039-1561

Email address: marcofer1996@gmail.com
ORCID iD: https://orcid.org/0000-0003-4169-9789