Kragujevac Journal of Mathematics
Volume 43(1) (2019), Pages 127–137.

REGULAR AND NORMAL OBJECTS IN THE CATEGORY OF PROXIMITY SPACES

MUAMMER KULA[1] AND SAMED ÖZKAN[2]

Abstract. In [2] and [7], Baran defined various generalization of the separation properties T_3 and T_4 for topological spaces to arbitrary topological categories. The main objective of this paper is to characterize each of the different notions of regular and normal objects in the category of proximity spaces as well as to examine how these generalizations are related.

1. Introduction

Proximity spaces were introduced by Efremovich during the first part of 1930s and axiomatized in 1951 [13]. He characterized the proximity relation "A is close to B" as a binary relation on subsets of a set X. A large part of the early work in proximity spaces was done by Smirnov [26]. He showed which topological spaces admit a proximity relation compatible with the given topology. Smirnov was also the first to discover relationship between proximities and uniformities.

Efremovich [14] defined the closure of a subset A of X to be the collection of all points of X "close" A. In this way he showed that a topology (completely regular) can be introduced in a proximity space. He also showed that every completely regular space X can be turned into a proximity space by using Urysohn's function.

The most comprehensive work on the theory of proximity spaces was done by Naimpally and Warrack [22]. All preliminary information on proximity spaces can be found in this source.

Baran [2] gave various generalizations of the usual separation properties for an arbitrary topological category over **Set**. He defined separation properties first at a point p, i.e., locally (see [4]), then they are generalized to point free definitions by

Key words and phrases. Topological category, proximity space, separation, regular, normal.
2010 *Mathematics Subject Classification.* Primary: 54B30. Secondary: 54E05, 54D10, 54D15.
Received: May 15, 2017.
Accepted: September 19, 2017.

using the generic element, [17] p. 39, method of topos theory. These generalizations are, for example, two notions of PreT_2 denoted by Pre$\overline{T}_2$ and PreT_2', each equivalent to the classical PreT_2 notion for topological spaces and four notions of T_4 denoted by $\overline{T}_4$, T_4', $S\overline{T}_4$ and ST_4', each equivalent to the classical T_4 notion for topological spaces.

Baran [2, 3] introduced the notions of "closedness" and "strong closedness" in set based topological categories and it is shown in [9] that these notions form an appropriate closure operator in the sense of Dikranjan and Giuli [12] in some well-known topological categories.

The main goal of this paper is

(1) to give an explicit characterization of the various notions of T_3 and T_4 objects in the topological category of proximity spaces;
(2) to investigate how these generalizations are related.

2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$, if $\mathcal{U}$ is concrete (i.e., faithful and amnestic), has small fibers, and for which every $\mathcal{U}$-source has an initial lift or, equivalently, for which each $\mathcal{U}$-sink has a final lift [1].

Note that a topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is said to be normalized, if constant objects, i.e., subterminals, have a unique structure [1, 4, 20, 21, 23].

Recall in [1] or [23], that an object $X \in \mathcal{E}$ (where $X \in \mathcal{E}$ stands for $X \in \mathrm{Ob}(\mathcal{E})$), a topological category, is discrete if and only if every map $\mathcal{U}(X) \to \mathcal{U}(Y)$ lifts to a map $X \to Y$ for each object $Y \in \mathcal{E}$ and an object $X \in \mathcal{E}$ is indiscrete if and only if every map $\mathcal{U}(Y) \to \mathcal{U}(X)$ lifts to a map $Y \to X$ for each object $Y \in \mathcal{E}$.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. A is called a subspace of X if the inclusion map $i : A \to X$ is an initial lift (i.e., an embedding) and we denote it by $A \subset X$.

Definition 2.1. [22] An Efremovich proximity (EF-proximity) space is a pair (X, δ), where X is a set and δ is a binary relation on the power set of X such that

(P1) $A\ \delta\ B$ if and only if $B\ \delta\ A$;
(P2) $A\ \delta\ (B \cup C)$ if and only if $A\ \delta\ B$ or $A\ \delta\ C$;
(P3) $A\ \delta\ B$ implies $A, B \neq \emptyset$;
(P4) $A \cap B \neq \emptyset$ implies $A\ \delta\ B$;
(P5) $A\ \overline{\delta}\ B$ implies there is an $E \subseteq X$ such that $A\ \overline{\delta}\ E$ and $(X - E)\ \overline{\delta}\ B$;

where $A\ \overline{\delta}\ B$ means it is not true that $A\ \delta\ B$.

A function $f : (X, \delta) \to (Y, \delta')$ between two proximity spaces is called a *proximity mapping* (*or a p-map*) if and only if $f(A)\ \delta'\ f(B)$ whenever $A\ \delta\ B$. It can easily be shown that f is a p-map if and only if, for subsets C and D of Y, $f^{-1}(C)\ \overline{\delta}\ f^{-1}(D)$ whenever $C\ \overline{\delta}'\ D$.

In a proximity space (X,δ), we write $A \ll B$ if and only if $A\ \overline{\delta}\ (X-B)$. The relation $\ll$ is called p-neighborhood relation or the strong inclusion. When $A \ll B$, we say that B is a p-neighborhood of A or A is strongly contained in B [15] or [22].

We denote the category of proximity spaces and proximity mappings by **Prox**. Hunsaker and Sharma [16] showed that the forgetful functor $\mathcal{U} : \mathbf{Prox} \to \mathbf{Set}$ is topological.

Definition 2.2. [24] Let X be a nonempty set and $P(X)$ be the set of all subsets of X. A proximity-base on X is a binary relation $\mathfrak{B}$ on $P(X)$ satisfying the axioms $(B1)$ through $(B5)$ given below:

(B1) $(\emptyset, X) \notin \mathfrak{B}$;
(B2) $A \cap B \neq \emptyset$ implies $(A,B) \in \mathfrak{B}$;
(B3) $(A,B) \in \mathfrak{B}$ if and only if $(B,A) \in \mathfrak{B}$;
(B4) if $(A,B) \in \mathfrak{B}$ and $A \subseteq A^*$, $B \subseteq B^*$ then $(A^*,B^*) \in \mathfrak{B}$;
(B5) if $(A,B) \notin \mathfrak{B}$ then there exists a set $E \subseteq X$ such that $(A,E) \notin \mathfrak{B}$ and $(X-E,B) \notin \mathfrak{B}$.

Definition 2.3. Let $\mathfrak{B}$ be a proximity-base on a set X and let a binary relation δ on $P(X)$ be defined as follows: $(A,B) \in \delta$ if, given any finite covers $\{A_i : 1 \leq i \leq n\}$ and $\{B_j : 1 \leq j \leq m\}$ of A and B respectively, then there exists a pair (i,j) such that $(A_i,B_j) \in \mathfrak{B}$. δ is a proximity on X finer than the relation $\mathfrak{B}$ [16] or [24].

Definition 2.4. Let X be a non-empty set, for each $i \in I$, (X_i,δ_i) be a proximity space and $f_i : X \to X_i$ be a source in **Set**. Define a binary relation $\mathfrak{B}$ on $P(X)$ as follows: for $A,B \in P(X)$, $A\ \mathfrak{B}\ B$ if and only if $f_i(A)\ \delta_i\ f_i(B)$, for all $i \in I$. $\mathfrak{B}$ is a proximity-base on X (Theorem 3.8 in [24]). The initial proximity structure δ on X generated by the proximity base $\mathfrak{B}$ is given by for $A,B \in P(X)$, $A\ \delta\ B$ if and only if for any finite covers $\{A_i : 1 \leq i \leq n\}$ and $\{B_j : 1 \leq j \leq m\}$ of A and B respectively, then there exists a pair (i,j) such that $(A_i,B_j) \in \mathfrak{B}$ [24].

Definition 2.5. Let (X,δ) be a proximity space, Y a non-empty set and f a function from a proximity space (X,δ) onto a set Y. The strong inclusion $\ll^*$ induced by the finest proximity δ^* (the quotient proximity) on Y making f proximally continuous is given by: for every $A,B \subset Y$, $A \ll^* B$ if and only if, for each binary rational s in $[0,1]$, there is some $C_s \subset Y$ such that $C_0 = A$, $C_1 = B$ and $s < t$ implies $f^{-1}(C_s) \ll_\delta f^{-1}(C_t)$ [15] or [27] p. 276, where $\ll_\delta$ represents the strong inclusion induced by the proximity δ on X. In addition, if $f : (X,\delta) \to (X,\delta^*)$ be a one-to-one p-quotient map, then $A\ \delta^*\ B$ if and only if $f^{-1}(A)\ \delta\ f^{-1}(B)$ (see [15] p. 591).

Definition 2.6. We write Δ for the diagonal in X^2, where $X \in \mathbf{Prox}$. For $X \in \mathbf{Prox}$ we define the wedge $X^2 \vee_\Delta X^2$, as the final structure, with respect to the map $X^2 \coprod X^2 \to X^2 \vee_\Delta X^2$, that is the identification of the two copies of X^2 along the diagonal Δ. An epi sink $\{i_1, i_2 : (X^2, \delta) \to (X^2 \vee_\Delta X^2, \delta')\}$, where i_1, i_2 are the canonical injections, in **Prox** is a final lift if and only if the following statement holds. For each pair A, B in the different component of $X^2 \vee_\Delta X^2$, $A\ \delta'\ B$ if and only if there exist sets C, D and U in X^2 such that $C\ \delta\ U$ and $U\ \delta\ D$ with $i_k^{-1}(A) = C$ and $i_j^{-1}(B) = D$ for $k, j = 1, 2$ and $k \neq j$. If A and B are in the same component of wedge, then $A\ \delta'\ B$ if and only if there exist sets C, D in X^2 such that $C\ \delta\ D$ and $i_k^{-1}(A) = C$ and $i_k^{-1}(B) = D$ for some $k = 1, 2$. Specially, if $i_k(E) = A$ and $i_k(F) = B$, then $(i_k(E), i_k(F)) \in \delta'$ if and only if $(i_k^{-1}(i_k(E)), i_k^{-1}(i_k(F))) = (E, F) \in \delta$. This is a special case of Definition 2.5.

Definition 2.7. Let X be a non-empty set. The discrete proximity structure δ on X is defined as follows for $A, B \subset X$: $A\ \delta\ B$ if and only if $A \cap B \neq \emptyset$ [22] p. 9.

Definition 2.8. Let X be a non-empty set. The indiscrete proximity structure δ on X is defined as follows for $A, B \subset X$: $A\ \delta\ B$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$ (see [22] p. 9).

3. Closedness and Hausdorff Objects

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p. A point x in $B \vee_p B$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = (p, p) = p_2$.

The principal p-axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed p-axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$.

The fold map at p, $\bigtriangledown_p : B \vee_p B \to B$ is given by $\bigtriangledown_p(x_i) = x$ for $i = 1, 2$ [2,3].

Note that the maps S_p and $\bigtriangledown_p$ are the unique maps arising from the above pushout diagram for which $S_p i_1 = (id, id) : B \to B^2$, $S_p i_2 = (p, id) : B \to B^2$, and $\bigtriangledown_p i_j = id, j = 1, 2$, respectively, where, $id : B \to B$ is the identity map and $p : B \to B$ is the constant map at p.

The infinite wedge product $\vee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times ...$ be the countable cartesian product of B. Define $A_p^\infty : \vee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, \dots, p, x, p, \dots)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, \dots, p, x, p, \dots)$ (infinite principal p-axis map), and $\bigtriangledown_p^\infty : \vee_p^\infty B \to B$ by $\bigtriangledown_p^\infty(x_i) = x$ for all $i \in I$ (infinite fold map) [2,3].

Note, also, that the map A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, \dots, p, id, p, \dots) : B \to B^\infty$, where the identity map, id is in the j-th place [9].

Definition 3.1. [2,3] Let $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi $\mathcal{U}$-sink $q : \mathcal{U}(X) = B \to B/F = (B \backslash F) \cup \{*\}$, where q is the epi map that is the identity on $B \backslash F$ and identifying F with a point $*$ [2].

Let p be a point in B.

(1) X *is* T_1 *at* p if and only if the initial lift of the $\mathcal{U}$-source $\{S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\bigtriangledown_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.
(2) p is closed if and only if the initial lift of the $\mathcal{U}$-source $\{A_p^\infty : \vee_p^\infty B \to \mathcal{U}(X^\infty) = B^\infty$ and $\nabla_p^\infty : \vee_p^\infty B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.
(3) $F \subset X$ is closed if and only if $\{*\}$, the image of F, is closed in X/F or $F = \emptyset$.
(4) $F \subset X$ is strongly closed if and only if X/F is T_1 at $\{*\}$ *or* $F = \emptyset$.
(5) *If* $B = F = \emptyset$, then we define F to be both closed and strongly closed.

Theorem 3.1 ([19], Theorem 4.5)**.** *Let* (X, δ) *be in* **Prox** *and* $p \in X$. $\{p\}$ *is closed in* X *if and only if for any* $B \subset X$*, if* $\{p\}\delta B$*, then* $p \in B$.

Theorem 3.2 ([19], Theorem 4.6)**.** *Let* (X, δ) *be in* **Prox**. *Then* $\emptyset \neq F \subset X$ *is closed if and only if* $x \in F$ *whenever* $\{x\}\delta F$ *for all* $x \in X$.

Theorem 3.3 ([19], Theorem 4.7)**.** *Let* (X, δ) *be in* **Prox**. *Then* $\emptyset \neq F \subset X$ *is strongly closed if and only if* $x \in F$ *whenever* $\{x\}\delta F$ *for all* $x \in X$.

Let B be a nonempty set, $B^2 = B \times B$ be cartesian product of B with itself and $B^2 \vee_\Delta B^2$ be two distinct copies of B^2 identified along the diagonal. A point (x, y) in $B^2 \vee_\Delta B^2$ will be denoted by $(x, y)_1$(or $(x, y)_2$) if (x, y) is in the first (or second) component of $B^2 \vee_\Delta B^2$, respectively. Clearly $(x, y)_1 = (x, y)_2$ if and only if $x = y$ [2].
The principal axis map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map, $\nabla : B^2 \vee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$ [2].

Definition 3.2. [2,11] Let $\mathcal{U} : \mathcal{E} \to$ **SET** be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$.

(1) X is $\overline{T}_0$ if and only if the initial lift of the $\mathcal{U}$-source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.
(2) X is T_0' if and only if the initial lift of the $\mathcal{U}$-source $\{id : B^2 \vee_\Delta B^2 \to \mathcal{U}(B^2 \vee_\Delta B^2)' = B^2 \vee_\Delta B^2$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $(B^2 \vee_\Delta B^2)'$ is the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ and $\mathcal{D}(B^2)$ is the discrete structure on B^2. Here, i_1 and i_2 are the canonical injections.
(3) X is T_1 if and only if the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete.
(4) X is Pre$\overline{T}_2$ if and only if the initial lifts of the $\mathcal{U}$-source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ coincide.
(5) X is PreT_2' if and only if the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ coincide, where i_1 and i_2 are the canonical injections.

(6) X is $\overline{T}_2$ if and only if X is $\overline{T}_0$ and Pre$\overline{T}_2$ ([2], Definition 2.2.10).
(7) X is T_2' if and only if X is T_0' and PreT_2' ([2], Definition 2.2.10).
(8) X is KT_2 if and only if X is T_0' and Pre$\overline{T}_2$ ([6], Definition 1.2).
(9) X is LT_2 if and only if X is $\overline{T}_0$ and PreT_2' ([6], Definition 1.2).

Remark 3.1. 1. Note that for the category **Top** of topological spaces, $\overline{T}_0$, T_0', or T_1, or Pre$\overline{T}_2$, PreT_2', or all of the T_2's in Definition 3.2 reduce to the usual T_0, or T_1, or PreT_2 (where a topological space is called PreT_2 if for any two distinct points, if there is a neighbourhood of one missing the other, then the two points have disjoint neighbourhoods), or T_2 separation axioms, respectively [2].

2. For an arbitrary topological category.
(i) By Theorem 3.2 of [5] or Theorem 2.7 (1) of [6], $\overline{T}_0$ implies T_0' but the converse of implication is generally not true.
(ii) By Theorem 3.1 (1) of [8], if X is PreT_2', then X is Pre$\overline{T}_2$. But the converse of implication is generally not true.

Definition 3.3. [22, 25] An Efremovich proximity space (X, δ) is said to be a

- $\mathbf{T_0}$ space if $x \neq y$ for $x, y \in X$ implies that $x\overline{\delta}y$;
- $\mathbf{T_1}$ space if $x \neq y$ for $x, y \in X$ implies that $x\overline{\delta}y$;
- $\mathbf{T_2}$ **(Hausdorff)** space if $x \ \delta \ y$ for $x, y \in X$ implies that $x = y$.

Theorem 3.4. *Let (X, δ) be an Efremovich proximity space.*

(1) (X, δ) *in* **Prox** *is* $\overline{T}_0$ *if and only if for each distinct pair x and y in X,* $(\{x\}, \{y\}) \notin \delta$ ([18]*, Theorem 3.3*).
(2) *All objects* (X, δ) *in* **Prox** *are* T_0' ([18]*, Theorem 3.4*).
(3) (X, δ) *in* **Prox** *is* T_1 *if and only if for each distinct pair x and y in X,* $(\{x\}, \{y\}) \notin \delta$ ([18]*, Theorem 3.5*).
(4) *All objects* (X, δ) *in* **Prox** *are* Pre$\overline{T}_2$ ([18]*, Theorem 3.7*).
(5) (X, δ) *in* **Prox** *is* PreT_2' *if and only if for each distinct pair x and y in X,* $(\{x\}, \{y\}) \notin \delta$ ([18]*, Theorem 3.8*).
(6) (X, δ) *in* **Prox** *is* $\overline{T}_2$ *if and only if for each distinct pair x and y in X,* $(\{x\}, \{y\}) \notin \delta$ ([18]*, Theorem 3.10*).
(7) (X, δ) *in* **Prox** *is* T_2' *if and only if for each distinct pair x and y in X,* $(\{x\}, \{y\}) \notin \delta$ ([18]*, Theorem 3.11*).

Remark 3.2. If an Efremovich proximity space (X, δ) is $\overline{T}_0$ or T_1 or PreT_2' or $\overline{T}_2$ or T_2', then it is T_0' or Pre$\overline{T}_2$. However, the converse is not true generally. For example, let $X = \{1, 2\}$ and $\delta = \{(X, X), (\{1\}, \{1\}), (\{2\}, \{2\}), (X, \{1\}), (\{1\}, X), (X, \{2\}), (\{2\}, X), (\{1\}, \{2\}), (\{2\}, \{1\})\}$. Then (X, δ) is T_0' or Pre$\overline{T}_2$, but it is not $\overline{T}_0$ or T_1 or PreT_2' or $\overline{T}_2$ or T_2' since $(\{1\}, \{2\}) \in \delta$ but $1 \neq 2$.

Theorem 3.5. *All Efremovich proximity spaces (X, δ) are KT_2.*

Proof. It follows from Theorem 3.4 (2), (4). □

Theorem 3.6. *An Efremovich proximity space* (X, δ) *is* LT_2 *if and only if for each distinct pair* x *and* y *in* X, $(\{x\}, \{y\}) \notin \delta$.

Proof. It follows from Theorem 3.4 (1), (5). □

Remark 3.3. If an Efremovich proximity space (X, δ) is LT_2, then it is KT_2. However, the converse is not true generally.

4. Regular Objects

We now recall, [2, 7, 10], various generalizations of the usual T_3 separation axiom to arbitrary set based topological categories and characterize each of them for the topological category **Prox**.

Definition 4.1. [2, 7, 10] Let $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a non-empty subset of B.

(1) X is $\overline{T}_3$ if and only if X is T_1 and X/F is Pre$\overline{T}_2$ for all closed $F \neq \emptyset$ in $U(X)$.
(2) X is T_3' if and only if X is T_1 and X/F is PreT_2' for all closed $F \neq \emptyset$ in $U(X)$.
(3) X is $S\overline{T}_3$ if and only if X is T_1 and X/F is Pre$\overline{T}_2$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
(4) X is ST_3' if and only if X is T_1 and X/F is PreT_2' for all strongly closed $F \neq \emptyset$ in $U(X)$.
(5) X is KT_3 if and only if X is T_1 and X/F is Pre$\overline{T}_2$ if it is T_1, where $F \neq \emptyset$ in $U(X)$.
(6) X is LT_3 if and only if X is T_1 and X/F is PreT_2' if it is T_1, where $F \neq \emptyset$ in $U(X)$.

Remark 4.1. 1. For the category **Top** of topological spaces, all of the T_3's reduce to the usual T_3 separation axiom (cf. [2, 7, 10]).

2. If $\mathcal{U} : \mathcal{E} \to \boldsymbol{B}$, where $\boldsymbol{B}$ is a topos [17], then parts (3)-(6) of Definition 4.1 still make sense since each of these notions requires only finite products and finite colimits in their definitions. Furthermore, if $\boldsymbol{B}$ has infinite products and infinite wedge products, then Definition 4.1 (2), also, makes sense.

Let (X, δ) be in **Prox**, and F be a nonempty subset of X. Let $q \colon (X, \delta) \to (X/F, \delta^*)$ be the quotient map that identifying F to a point, $*$ [2].

Theorem 4.1. *If* (X, δ) *is* $\overline{T}_2$, *then* $(X/F, \delta^*)$ *is* $\overline{T}_2$.

Proof. Suppose (X, δ) is $\overline{T}_2$. Let a and b be any distinct pair of points in X/F. By Theorem 3.4 (6), we only need to show that $(\{a\}, \{b\}) \notin \delta^*$, where δ^* is the structure on X/F induced by q.

Suppose that $a \neq *$. By definition of q map, there exist $a \in X$ and $F \subset X$ such that $q(a) = a$ and $q(c) = *$ for any $c \in F$. Since $a \neq c$ for any $c \in F$ ($a \notin F$) and (X, δ) is $\overline{T}_2$, then $\{a\}\ \overline{\delta}\ \{c\}$. By the condition (P2) of Definition 2.1 we obtain $\{a\}\ \overline{\delta}\ F$. Then we have $\{a\}\ \overline{\delta}\ F = q^{-1}(\{a\})\ \overline{\delta}\ q^{-1}(\{*\})$. It follows that by p-neighborhood relation definition and Definition 2.5, for each binary rational s in $[0,1]$ there is some $C_s \subset X/F$ such that $C_0 = \{a\}$, $C_1 = \{*\}^c$ and $s < t$ implies $q^{-1}(C_s) \ll_\delta q^{-1}(C_t) = q^{-1}(\{a\}) \ll_\delta (q^{-1}(\{*\}))^c = q^{-1}(\{a\}) \ll_\delta q^{-1}(\{*\}^c)$ if and only if $\{a\} \ll^* \{*\}^c$. Hence $\{a\} \overline{\delta}^* \{*\}$, i.e., $(\{a\},\{*\}) \notin \delta^*$.

Let $a \neq b \neq *$. By definition of q map, there exists a pair $a, b \in X$ such that $q(a) = a$ and $q(b) = b$. In this case q map can be considered as one-to-one map. Suppose that $\{a\} \ \delta^* \ \{b\}$. By definition of q map and Definition 2.5, we have $\{a\} \ \delta^* \ \{b\}$ if and only if $q^{-1}(\{a\}) \ \delta \ q^{-1}(\{b\}) = \{a\} \ \delta \ \{b\}$. But $\{a\} \ \overline{\delta} \ \{b\}$ since (X,δ) is $\overline{T}_2$. Hence $\{a\} \ \overline{\delta}^* \ \{b\}$ i.e., $(\{a\},\{b\}) \notin \delta^*$.

Consequently for each distinct points a and b in X/F, we have $(\{a\},\{b\}) \notin \delta^*$. Hence by Theorem 3.4 (6), $(X/F, \delta^*)$ is $\overline{T}_2$. □

Theorem 4.2. *If (X,δ) is T_2' (resp. $\mathrm{Pre}\overline{T}_2$ and $\mathrm{Pre}T_2'$), then $(X/F,\delta^*)$ is T_2'(resp. $\mathrm{Pre}\overline{T}_2$ and $\mathrm{Pre}T_2'$).*

Proof. It follows from Theorem 3.4 (7) (resp. (4) and (5)) and by using the same argument used in the proof of Theorem 4.1. □

Corollary 4.1. *Let (X,δ) be an Efremovich proximity space, then the following are equivalent.*

(1) *(X,δ) is $\overline{T}_3$.*
(2) *(X,δ) is T_3'.*
(3) *(X,δ) is $S\overline{T}_3$.*
(4) *(X,δ) is ST_3'.*
(5) *(X,δ) is KT_3.*
(6) *(X,δ) is LT_3.*
(7) *For each distinct pair x and y in X, $(\{x\},\{y\}) \notin \delta$.*

Proof. It follows from Theorems 3.4 (3) and 4.2. □

5. Normal Objects and Relationships

We now recall various generalizations of the usual T_4 separation axiom to arbitrary set based topological categories that are defined in [2, 7, 10], and characterize each of them for the topological category **Prox**.

Definition 5.1. [2, 7, 10] Let $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ be a topological functor and X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a non-empty subset of B.

(1) X is $\overline{T}_4$ if and only if X is T_1 and X/F is $\overline{T}_3$ for all closed $F \neq \emptyset$ in $U(X)$.
(2) X is T_4' if and only if X is T_1 and X/F is T_3' for all closed $F \neq \emptyset$ in $U(X)$.
(3) X is $S\overline{T}_4$ if and only if X is T_1 and X/F is $S\overline{T}_3$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
(4) X is ST_4' if and only if X is T_1 and X/F is ST_3' for all strongly closed $F \neq \emptyset$ in $U(X)$.

Remark 5.1.
1. For the category **Top** of topological spaces, all of the T_4's reduce to the usual T_4 separation axiom [2, 7, 10].
2. If $\mathcal{U} : \mathcal{E} \to \boldsymbol{B}$, where $\boldsymbol{B}$ is a topos [17], then Definition 5.1 still makes sense since each of these notions requires only finite products and finite colimits in their definitions.

Let (X, δ) be in **Prox**, and F be a nonempty subset of X. Let $q\colon (X, \delta) \to (X/F, \delta^*)$ be the quotient map that identifying F to a point, $*$ [2].

Theorem 5.1. *If (X, δ) is $\overline{T}_3$, then $(X/F, \delta^*)$ is $\overline{T}_3$.*

Proof. Suppose (X, δ) is $\overline{T}_3$. Let a and b be any distinct pair of points in X/F. By Corollary 4.1, we only need to show that $(\{a\}, \{b\}) \notin \delta^*$, where δ^* is the structure on X/F induced by q.

Suppose that $a \neq *$. By definition of q map, there exist $a \in X$ and $F \subset X$ such that $q(a) = a$ and $q(c) = *$, for any $c \in F$. Since $a \neq c$ for any $c \in F$ ($a \notin F$) and (X, δ) is $\overline{T}_3$, then $\{a\}\ \overline{\delta}\ \{c\}$. By the condition (P2) of Definition 2.1 we obtain $\{a\}\ \overline{\delta}\ F$. Then we have $\{a\}\ \overline{\delta}\ F = q^{-1}(\{a\})\ \overline{\delta}\ q^{-1}(\{*\})$. It follows that by p-neighborhood relation definition and Definition 2.5, for each binary rational s in $[0, 1]$ there is some $C_s \subset X/F$ such that $C_0 = \{a\}$, $C_1 = \{*\}^c$ and $s < t$ implies $q^{-1}(C_s) \ll_\delta q^{-1}(C_t) = q^{-1}(\{a\}) \ll_\delta (q^{-1}(\{*\}))^c = q^{-1}(\{a\}) \ll_\delta q^{-1}(\{*\}^c)$ if and only if $\{a\} \ll^* \{*\}^c$. Hence $\{a\}\ \overline{\delta}^*\ \{*\}$, i.e., $(\{a\}, \{*\}) \notin \delta^*$.

Let $a \neq b \neq *$. By definition of q map, there exists a pair $a, b \in X$ such that $q(a) = a$ and $q(b) = b$. In this case q map can be considered as one-to-one map. Suppose that $\{a\}\ \delta^*\ \{b\}$. By definition of q map and Definition 2.5, we have $\{a\}\ \delta^*\ \{b\}$ if and only if $q^{-1}(\{a\})\ \delta\ q^{-1}(\{b\}) = \{a\}\ \delta\ \{b\}$. But $\{a\}\ \overline{\delta}\ \{b\}$ since (X, δ) is $\overline{T}_3$. Hence $\{a\}\ \overline{\delta}^*\ \{b\}$ i.e., $(\{a\}, \{b\}) \notin \delta^*$.

Consequently for each distinct points a and b in X/F, we have $(\{a\}, \{b\}) \notin \delta^*$. Hence by Corollary 4.1, $(X/F, \delta^*)$ is $\overline{T}_3$. □

Theorem 5.2. *If (X, δ) is T'_3 (resp. $S\overline{T}_3$ and ST'_3), then $(X/F, \delta^*)$ is T'_3 (resp. $S\overline{T}_3$ and ST'_3).*

Proof. It follows from Corollary 4.1, and by using the same argument used in the proof of Theorem 5.1. □

Corollary 5.1. *Let (X, δ) be an Efremovich proximity space, then the following are equivalent.*

(1) *(X, δ) is $\overline{T}_4$.*
(2) *(X, δ) is T'_4.*
(3) *(X, δ) is $S\overline{T}_4$.*
(4) *(X, δ) is ST'_4.*
(5) *For each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.*

Proof. It follows from Theorems 3.4 (3), 5.1 and 5.2. □

We give explicit relationships among the generalized separation properties in the topological category of proximity spaces.

Remark 5.2. Let (X, δ) be in **Prox**.

(i) By Theorems 3.4 (2), 3.4 (4), 3.5, all Efremovich proximity spaces (X, δ) are T_0', Pre$\overline{T}_2$ and KT_2.

(ii) By Definition 3.3, Theorems 3.4, 3.5, 3.6 and Corollaries 4.1, 5.1, (X, δ) is T_0' or Pre$\overline{T}_2$ or KT_2 if (X, δ) is $\overline{T}_0$ or $\mathbf{T_0}$ or T_1 or $\mathbf{T_1}$ or PreT_2' or $\overline{T}_2$ or T_2' or $\mathbf{T_2}$ or LT_2 or all forms of T_3 ($\overline{T}_3$, T_3', $S\overline{T}_3$, ST_3', KT_3, LT_3) or all forms of T_4 ($\overline{T}_4$, T_4', $S\overline{T}_4$, ST_4'). But the converse implication is not true, in general. For example, let $X = \{x, y\}$ and $\delta = \{(X, X), (\{x\}, \{x\}), (\{y\}, \{y\}), (X, \{x\}), (\{x\}, X), (X, \{y\}), (\{y\}, X), (\{x\}, \{y\}), (\{y\}, \{x\})\}$. Then (X, δ) is T_0' or Pre$\overline{T}_2$ or KT_2, but it is not $\overline{T}_0$ or $\mathbf{T_0}$ or T_1 or $\mathbf{T_1}$ or PreT_2' or $\overline{T}_2$ or T_2' or $\mathbf{T_2}$ or LT_2 or all forms of T_3 and T_4, since $(\{x\}, \{y\}) \in \delta$ but $x \neq y$.

Remark 5.3. Let (X, δ) be in **Prox**. By Definition 3.3, Theorems 3.4, 3.6 and Corollaries 4.1, 5.1 then the following are equivalent.

(i) (X, δ) is $\overline{T}_0$.
(ii) (X, δ) is $\mathbf{T_0}$.
(iii) (X, δ) is T_1.
(iv) (X, δ) is $\mathbf{T_1}$.
(v) (X, δ) is PreT_2'.
(vi) (X, δ) is $\overline{T}_2$.
(vii) (X, δ) is T_2'.
(viii) (X, δ) is LT_2.
(ix) (X, δ) is $\mathbf{T_2}$.
(x) (X, δ) is all forms of T_3 ($\overline{T}_3$, T_3', $S\overline{T}_3$, ST_3', KT_3, LT_3).
(xi) (X, δ) is all forms of T_4 ($\overline{T}_4$, T_4', $S\overline{T}_4$, ST_4').
(xii) For each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.

Acknowledgements. This research was supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under Grant No. 114F300.

We would like to thank the referee for his helpful comments and valuable suggestions.

References

[1] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, Wiley, New York, 1990.
[2] M. Baran, *Separation properties*, Indian J. Pure Appl. Math. **23** (1992), 333–341.
[3] M. Baran, *The notion of closedness in topological categories*, Comment. Math. Univ. Carolin. **34** (1993), 383–395.
[4] M. Baran, *Generalized local separation properties*, Indian J. Pure Appl. Math. **25** (1994), 615–620.
[5] M. Baran and H. Altındiş, *T_0-objects in topological categories*, J. Univ. Kuwait (Sci.) **22** (1995), 123–127.

[6] M. Baran and H. Altındiş, *T_2-objects in topological categories*, Acta Math. Hungar. **71** (1996), 41–48.
[7] M. Baran, *T_3 and T_4-objects in topological categories*, Indian J. Pure Appl. Math. **29** (1998), 59–69.
[8] M. Baran, *Completely regular objects and normal objects in topological categories*, Acta Math. Hungar. **80**(3) (1998), 211–224.
[9] M. Baran, *Closure operators in convergence spaces*, Acta Math. Hungar. **87** (2000), 33–45.
[10] M. Baran and M. Kula, *A note on separation and compactness in categories of convergence spaces*, Appl. Gen. Topol. **4**(1) (2003), 1–13.
[11] M. Baran, *PreT_2-objects in topological categories*, Appl. Categ. Structures **17** (2009), 591–602.
[12] D. Dikranjan and E. Giuli, *Closure operators I*, Topology Appl. **27** (1987), 129–143.
[13] V. A. Efremovich, *Infinitesimal spaces* (in Russian), Dokl. Akad. Nauk **76** (1951), 341–343.
[14] V. A. Efremovich, *The geometry of proximity I* (in Russian), Mat. Sb. **31** (1952), 189–200.
[15] L. M. Friedler, *Quotients of proximity spaces*, Proc. Amer. Math. Soc. **37**(2) (1973), 589–594.
[16] W. N. Hunsaker and P. L. Sharma, *Proximity spaces and topological functors*, Proc. Amer. Math. Soc. **45**(3) (1974), 419–425.
[17] P. T. Johnstone, *Topos Theory*, Academic Press, New York, 1977.
[18] M. Kula, S. Özkan and T. Maraşlı, *Pre-Hausdorff and Hausdorff proximity spaces*, Filomat **31**(12) (2017), 3837–3846.
[19] M. Kula, T. Maraşlı and S. Özkan, *A note on closedness and connectedness in the category of proximity spaces*, Filomat **28**(7) (2014), 1483–1492.
[20] M. V. Mielke, *Separation axioms and geometric realizations*, Indian J. Pure Appl. Math. **25** (1994), 711–722.
[21] M. V. Mielke, *Hausdorff separations and decidability*, in: *Symposium on Categorical Topology*, University of Cape Town, Rondebosch, 1999, 155–160.
[22] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge University Press, Cambridge, 1970.
[23] G. Preuss, *Theory of Topological Structures: An Approach to Topological Categories*, D. Reidel Publ. Co. Dordrecht, 1988.
[24] P. L. Sharma, *Proximity bases and subbases*, Pacific J. Math. **37** (1971), 515–526.
[25] J. L. Sieber, *Generalizations of some topological notions to syntopogenic spaces*, Ph. D. Thesis, The Pennslyvania State University, 1963.
[26] Y. M. Smirnov, *On proximity spaces*, Amer. Math. Soc. Transl. **38** (1964), 5–35.
[27] S. Willard, *General Topology*, Addison-Wesley, Reading, Massachusetts, 1970.

[1]Department of Mathematics,
Erciyes University,
Kayseri, Turkey
E-mail address: `kulam@erciyes.edu.tr`

[2]Department of Mathematics,
Nevşehir Hacı Bektaş Veli University,
Nevşehir, Turkey
E-mail address: `ozkans@nevsehir.edu.tr`