Kragujevac Journal of Mathematics
Volume 50(2) (2026), Pages 261–272.

STABILITY OF A SOLUTION FOR A HYBRID FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. This study focuses on examining the existence, uniqueness, and U-lam stability of a solution for a hybrid fractional equation by utilizing the derivative of Caputo-Hadamard (C-H). The primary tools used in our research are the Banach contraction mapping principle (BCMP) and Schaefer's fixed point theorem. Additionally, we provide an example to demonstrate our results.

1. Introduction

The definitions like Riemann-Liouville (1832), Grunwald-Letnikov (1867), Hadamard (1891, [14]) and Caputo (1997) are used to model problems in engineering and applied sciences and the formulations are used to model the physical systems and has given more accurate results. In 1891, Hadamard introduced the new derivative. For more details one can refer [6, 23, 25, 27] and the references cited therein. A new approach called Caputo-Hadamard derivative [22], obtained from the Hadamard derivative and is applied to solve for physically interpretable initial condition problems. For the recent results in Caputo-Hadamard derivative, one can cite [2, 7, 12, 16, 34–36] and the references therein.

The theory of fractional calculus is an interesting field to be explored in recent years. Also, this theory has many applications to describe many events in the real world and deal with a group of phenomena in several fields such as blood flow phenomena, mechanics, biophysics, automatic, aerodynamics, some branches of medicine, and electronics. For instance, the authors [10] discussed the applicability of fractional differential equations in electric circuits, and in 2019 M. Saqib et al. applied the

Key words and phrases. Existence, uniqueness, fractional derivative, stability.
2020 *Mathematics Subject Classification.* Primary: 26A33, 35A20. Secondary: 33B15.
DOI
Received: March 16, 2023.
Accepted: October 30, 2023.

fractional differential equation to heat transfer in hybrid nanofluid see [26]. For more details, one can refer to [24, 25, 37]. In addition to the great importance of studying the existence of solutions to fractional differential equations using the many theories of the fixed point, several studies have been conducted over the years to investigate how stability concepts such as the Mittag-Leffler function, exponential, and Lyapunov stability apply to various types of dynamic systems. Ulam and Hyers, on the other hand, identified previously unknown types of stability known as Ulam-stability [1]. This example is not exclusive, many similar works can be found in [3, 5, 11, 17, 31]. In 2008, Benchohra et al. [13], discussed the following boundary value problem

$$
\begin{gathered}
{}^{c}D^{p}\varphi(\vartheta) = f_1(\vartheta, \vartheta(t)), \quad \text{for } \vartheta \in [0,T], 0 < p \leq 1,\\
a_1\vartheta(0) + b_1\vartheta(T) = c_1,
\end{gathered}
$$

where ${}^{c}D^{p}$ denotes the Caputo fractional derivative of order p, $f_1 : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $a_1, b_1, c_1 \in \mathbb{R}$ such that $a_1 + b_1 \neq 0$.

In 2017, Arioua et al. [9] proved the existence of solution for the boundary value problem of nonlinear differential equation of fractional order

$$
{}^{c}D^{p}_{1^+}\vartheta(t) + f_1(t, \vartheta(t)) = 0, \quad \text{for } 1 < t < e, 2 < p \leq 3,
$$

with the fractional boundary conditions:

$$
\vartheta(1) = \vartheta'(1) = 0, \quad ({}^{c}D^{p-1}_{1^+}\vartheta)(e) = ({}^{c}D^{p-2}_{1^+}\vartheta)(e) = 0,
$$

where ${}^{c}D^{p}$ denotes the Caputo-Hadamard (C-H) fractional derivatives of order p, a continuous function $f_1 : [1,e] \times \mathbb{R} \to \mathbb{R}$.

In 2018, Benhamida et al. [14] investigated the following Caputo-Hadamard fractional differential equations with the boundary conditions:

$$
\begin{gathered}
{}^{c}_{H}D^{p}\vartheta(t) = f_1(t, \vartheta(t)), \quad \text{for } t \in [1,T], 0 < p \leq 1,\\
a_1\vartheta(1) + b_1\vartheta(T) = c_1,
\end{gathered}
$$

where ${}^{c}_{H}D^{p}$ denotes the Caputo-Hadamard (C-H) fractional derivative of order p, a given continuous function $f_1 : [1,T] \times \mathbb{R} \to \mathbb{R}$ and the real constants a_1, b_1 and c_1 such that $a_1 + b_1 \neq 0$.

The present paper is a continuation of the work see [19], we consider the system of hybrid nonlinear Caputo-Hadamard (C-H) fractional differential equations:

$$
{}^{c}_{H}D^{\gamma_1}\left[\frac{\xi(\hat{\varkappa})}{\varpi(\hat{\varkappa}, \xi(\hat{\varkappa}))}\right] = \Lambda(\hat{\varkappa}, \xi(\hat{\varkappa})), \quad \hat{\varkappa} \in [1,T], 0 < \gamma_1 \leq 1, \tag{1.1}
$$

supplemented with the boundary condition;

$$
\lambda\frac{\xi(1)}{\varpi(1, \xi(1))} + \mu\frac{\xi(T)}{\varpi(T, \xi(T))} = \nu, \tag{1.2}
$$

where ${}^{c}_{H}D^{\gamma_1}$, denote the Caputo-Hadamard (C-H) fractional derivatives of orders γ_1. The given continuous functions $\Lambda : [1,T] \times \mathbb{R} \to \mathbb{R}$, with λ, μ and $\nu \in \mathbb{R}$, $\varpi : [1,T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$.

The paper is organized as follows. Section 2 provides an overview of preliminary concepts and an auxiliary lemma related to the problem. Section 3 presents the main proof for the existence of solutions to Problem (1.1)–(1.2). Section 4 examines the Ulam-Hyers stability of the fractional differential equations (1.1)–(1.2). In Section 5, an example is presented to further illustrate the findings of the study. Lastly, in Section 6, we conclude and discuss future work that can be done in this area.

2. Preliminaries

Definition 2.1 ([23]). Let h_1: $[1,+\infty) \to \mathbb{R}$ be an integrable function. The Hadamard fractional integral of h_1 of order q_1 is defind by

$$ {}_HI^{q_1}h_1(\hat{\varkappa}) = \frac{1}{\Gamma(q_1)}\int_1^{\hat{\varkappa}}\left(\ln\frac{\hat{\varkappa}}{s}\right)^{q_1-1}\frac{h_1(s)}{s}ds, \quad q_1>0, \hat{\varkappa}>1. $$

Definition 2.2 ([22]). The C-H fractional derivative of order q_1 where $q_1 \geq 0$, $n-1 < q_1 < n$, with $n=[q_1]+1$ and $h_1 \in AC^n_\delta[1,+\infty)$

$$ ({}^c_HD^{q_1}h_1)(\hat{\varkappa}) = \frac{1}{\Gamma(n-q_1)}\int_1^{\hat{\varkappa}}\left(\log\frac{\hat{\varkappa}}{s}\right)^{n-q_1-1}\delta^n h_1(s)\frac{ds}{s} ={}_H I^{n-q_1}(\delta^n h_1)(\hat{\varkappa}). $$

Lemma 2.1 ([22]). *Let $h_1 \in AC^n_\delta[1,+\infty)$ and $q_1>0$. Then*

$$ {}_HI^{q_1}({}^c_HD^{q_1}h_1)(\hat{\varkappa}) = h_1(\hat{\varkappa}) - \sum_{i=0}^{n-1}\frac{\delta^i x(1)}{i!}(\log t)^i. $$

Lemma 2.2. *Let the function $h_1:[1,+\infty)\to\mathbb{R}$. The function ξ is a solution of the following equation*

$$ \xi(\hat{\varkappa}) =\varpi(\hat{\varkappa},\xi(\hat{\varkappa}))\Big(\frac{1}{\Gamma(\gamma)}\int_1^{\hat{\varkappa}}\left(\log\frac{\hat{\varkappa}}{s}\right)^{\gamma-1}h_1(s)\frac{ds}{s} \tag{2.1} $$

$$ -\frac{\mu}{\Gamma(\gamma)(\lambda+\mu)}\int_1^T\left(\log\frac{T}{s}\right)^{\gamma-1}h_1(s)\frac{ds}{s}+\frac{\nu}{\lambda+\mu}\Big) \tag{2.2} $$

if and only if

$$ {}^c_HD^{\gamma_1}\left[\frac{\xi(t)}{\varpi(\hat{\varkappa},\xi(t))}\right] =h_1(\hat{\varkappa}), \quad 0<\gamma_1<1, \tag{2.3} $$

and

$$ \lambda\frac{\xi(1)}{\varpi(1,\xi(1))}+\mu\frac{\xi(T)}{\varpi(T,\xi(T))}=\nu. \tag{2.4} $$

Proof. Suppose that ξ satisfies (2.3). Then

$$ \left[\frac{\xi(\hat{\varkappa})}{\varpi(\hat{\varkappa},\xi(\hat{\varkappa}))}\right] ={}_H I^{\gamma_1}h_1(\hat{\varkappa})+\alpha_1, \tag{2.5} $$

when we apply the boundary condition (2.4), we get

$$\begin{aligned}
\frac{\xi(1)}{\varpi(1,\xi(1))} =&\alpha_1,\\
\frac{\xi(T)}{\varpi(T,\xi(T))} =&{}_HI^{\gamma_1}h_1(T)+\alpha_1,\\
\lambda\frac{\xi(1)}{\varpi(1,\xi(1))}+\mu\frac{\xi(T)}{\varpi(T,\xi(T))} =&\nu,\\
\lambda\alpha_1+\mu\left[{}_HI^{\gamma_1}h_1(T)+\frac{\xi(1)}{\varpi(1,\xi(1))}\right] =&\nu,\\
\lambda\frac{\xi(1)}{\varpi(1,\xi(1))}+\mu_HI^{\gamma_1}h_1(T)+\mu\frac{\xi(1)}{\varpi(1,\xi(1))} =&\nu,\\
(\lambda+\mu)\frac{\xi(1)}{\varpi(1,\xi(1))}+\mu_HI^{\gamma_1}h_1(T) =&\nu,\\
\frac{\xi(1)}{\varpi(1,\xi(1))} =&\frac{\nu-\mu_HI^{\gamma_1}h(T)}{(\lambda+\mu)},
\end{aligned}$$

which leads to the solution (2.1) that

$$\xi(t)=\varpi(\hat{\varkappa},\xi(\hat{\varkappa}))\Big({}_HI^{\gamma_1}h_1(\hat{\varkappa})-\frac{\mu}{(\lambda+\mu)}{}_HI^{\gamma_1}h_1(T)+\frac{\nu}{\lambda+\mu}\Big).$$

Conversely, ξ has to satisfy equation (2.1) and then equation (2.3)–(2.4) hold. □

3. Main Results

Let us now consider the Banach space $\mathfrak{S}$ of all continuous functions $f:[1,T]\to\mathbb{R}$ endowed with the norm $\|\tilde{\xi}\|_\infty=\sup\{|\tilde{\xi}(\hat{\varkappa})|:1\leq\hat{\varkappa}\leq T\}$.

Let consider thr following assumtions:

- (F_1) The function $\xi\mapsto\frac{\xi}{\varpi(\hat{\varkappa},\xi)}$ is increasing for every $\hat{\varkappa}\in[1,T]$.
- (F_2) There is numbers $L>0$ such that $|\varpi(\hat{\varkappa},\xi)|\leq L$ for all $(\hat{\varkappa},\xi)\in[1,T]\times\mathbb{R}$.
- (F_3) Let $\Lambda:[1,T]\times\mathbb{R}\to\mathbb{R}$ be continuous and bounded functions and there is constants π, such that, for all $\hat{\varkappa}\in[1,T]$ and $\rho,\varrho\in\mathbb{R}$
$$|\Lambda(\hat{\varkappa},\rho)-\Lambda(\hat{\varkappa},\varrho)|\leq\pi|\rho-\varrho|.$$
- (F_4) $\sup_{\hat{\varkappa}\in[1,T]}\Lambda(\hat{\varkappa},0)=\mathcal{M}<+\infty$.
- (F_5) There is $N>0$ such that $|\Lambda(\hat{\varkappa},\xi(\hat{\varkappa}))|\leq N$. For ease of computation, we set
$$\tau=\Big(1+\frac{|\mu|}{|\lambda+\mu|}\Big)\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)},\quad Q=\frac{|\nu|}{|\lambda+\mu|}<1.$$

Let consider the operator $\Theta : \mathfrak{S} \to \mathfrak{S}$ associated with Problem (1.1)–(1.2) as follows:

$$\begin{aligned}\Theta(\xi,\vartheta)(\hat{\varkappa}) =&\varpi(\hat{\varkappa},\xi)\Big(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(\log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}\Lambda(s,\xi(s),\vartheta(s))\frac{ds}{s}\\ &-\frac{\mu}{\Gamma(\gamma_1)(\lambda+\mu)}\int_1^{T}\Big(\log\frac{T}{s}\Big)^{\gamma_1-1}\Lambda(s)\frac{ds}{s}+\frac{\nu}{\lambda+\mu}\Big).\end{aligned}\tag{3.1}$$

Theorem 3.1. *Suppose that conditions* (F_1) *to* (F_5) *hold. Then* $\Theta\bar{\mathcal{B}}_r\subset\bar{\mathcal{B}}_r$, *where* $\bar{\mathcal{B}}_r=\{\xi\in\mathfrak{S}:\|(\xi)\|_\infty\le r\}$ *is a closed ball with* $r=L\tau\pi<1$. *Problem* (1.1)–(1.2) *has a unique solution on* $[1,T]$.

Proof. For $\xi\in\bar{\mathcal{B}}_r$ and $\hat{\varkappa}\in[1,T]$, it follows by (F_3) that

$$|\Lambda(\hat{\varkappa},\xi(\hat{\varkappa}))|\le|\Lambda(\hat{\varkappa},\xi(\hat{\varkappa}))-\Lambda(\hat{\varkappa},0)|\le\pi||\xi||_\infty.$$

Then we have

$$\begin{aligned}|\Theta(\xi)(\hat{\varkappa})|\le&L\max_{\hat{\varkappa}\in[1,T]}\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(\log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}\Big|\Lambda(s,\xi(s))-\Lambda(s,0))+\Lambda(s,0)\Big|\frac{ds}{s}\\ &-\frac{|\mu|}{\Gamma(\gamma_1)|\lambda+\mu|}\int_1^{T}\Big(\log\frac{T}{s}\Big)^{\gamma_1-1}|\Lambda(s,\xi(s))-\Lambda(s,0)+\Lambda(s,0)|\frac{ds}{s}\\ &+\frac{|\nu|}{|\lambda+\mu|}\Bigg]\\ \le&L\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(\log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}(\pi|\xi|+\mathcal{M})\frac{ds}{s}\\ &+\frac{|\mu|}{\Gamma(\gamma_1)|\lambda+\mu|}\int_1^{T}\Big(\log\frac{T}{s}\Big)^{\gamma_1-1}(\pi|\xi|+\mathcal{M})\frac{ds}{s}+Q\Bigg]\\ \le&L\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big(1+\frac{|\mu|}{|\lambda+\mu|}\Big)\Big(\pi|\xi|+\mathcal{M}\Big)+LQ.\end{aligned}$$

Thus,

$$\begin{aligned}\|\Theta(\xi,\vartheta)\|_\infty\le&L\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big(1+\frac{|\mu|}{|\lambda+\mu|}\Big)\Big(\pi|\xi|+\mathcal{M}\Big)+LQ\\ \le&L\Big(\tau(\pi\|\xi\|_\infty+\mathcal{M})+Q\Big)\le L\Big(\tau\pi r+\tau\mathcal{M}+Q\Big)\le L\Big(\tau\pi r+\tau\mathcal{M}+Q\Big).\end{aligned}$$

From the foregoing estimates for Θ it follows that $||\Theta(\xi)(\hat{\varkappa})||_{\infty} \leq r$. Next, for $\xi_1, \xi_2 \in \mathfrak{S}$ and $\hat{\varkappa} \in [1, T]$, we get

$$\begin{aligned}
|\Theta(\xi_2)(\hat{\varkappa}) - \Theta(\xi_1)(\hat{\varkappa})| \leq & L_1 \Bigg(\frac{1}{\Gamma(\gamma_1)} \int_1^{\hat{\varkappa}} \left(\log \frac{\hat{\varkappa}}{s} \right)^{\gamma_1 - 1} |\Lambda(s, \xi_2(s)) - \Lambda(s, \xi_1(s))| \frac{ds}{s} \\
& + \frac{|\mu|}{\Gamma(\gamma_1)|\lambda + \mu|} \int_1^T \left(\log \frac{T}{s} \right)^{\gamma_1 - 1} |\Lambda(s, \xi_2(s)) - \Lambda(s, \xi_1(s))| \frac{ds}{s} \Bigg) \\
\leq & L \left(1 + \frac{\mu}{\lambda + \mu} \right) \frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1 + 1)} \left(\pi \|\xi_2 - \xi_1\|_{\infty} \right) \\
= & L\tau\pi \|\xi_2 - \xi_1\|_{\infty},
\end{aligned}$$

which implies that

$$\|\Theta(\xi_2)(\hat{\varkappa}) - \Theta(\xi_1)\|_{\infty} \leq L\tau\pi \|\xi_2 - \xi_1\|_{\infty}. \tag{3.2}$$

In view of condition $L\tau\pi < 1$, it follows that the operator Θ possesses a unique fixed point. This leads to the conclusion that Problem (1.1)–(1.2) has a unique solution on $[1, T]$. □

The fixed point theorem of Schaefer can now be used to demonstrate the existence of solutions to Problem (1.1)–(1.2).

Theorem 3.2. *Suppose the hypothesis* (F_1)-(F_4). *Then the boundary value Problem* (1.1)–(1.2) *has at least one solution on* $[1, T]$.

Proof. Several steps will be involved in the proof.

Step I. $\Theta : \mathfrak{S} \to \mathfrak{S}$ is continuous. Notice that continuity of Λ and ϖ show that Θ is bounded. Let ξ_n be a sequence of points in $\mathfrak{S}$ converging to a point $\xi \in \mathfrak{S}$. We get

$$\begin{aligned}
& |\Theta(\xi_n)(\hat{\varkappa}) - \Theta(\xi)(\hat{\varkappa})| \\
\leq & L \Bigg(\frac{1}{\Gamma(\gamma_1)} \int_1^{\hat{\varkappa}} \left(\log \frac{\hat{\varkappa}}{s} \right)^{\gamma_1 - 1} |\Lambda(s, \xi_n(s)) - \Lambda(s, \xi(s))| \frac{ds}{s} \\
& - \frac{|\mu_1|}{\Gamma(\gamma_1)|\lambda + \mu|} \int_1^T \left(\log \frac{T}{s} \right)^{\gamma_1 - 1} |\Lambda(s, \xi_n(s)) - \Lambda(s, \xi(s))| \frac{ds}{s} \Bigg) \\
\leq & L_1 \left(1 + \frac{|\mu|}{|\lambda + \mu|} \right) \frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1 + 1)} \|\Lambda(\cdot, \xi_n(\cdot)) - \Lambda(\cdot, \xi(\cdot))\|_{\infty},
\end{aligned}$$

since Λ is continuous, we have $\|\Theta(\xi_n) - \Theta(\xi)\|_{\infty} \to 0$ as $n \to +\infty$ for all $\hat{\varkappa} \in [1, T]$. Hence, it follows from the foregoing inequalities satisfied by above that Θ is continuous.

Step II. $\Theta : \mathfrak{S} \to \mathfrak{S}$ maps bounded sets into bounded sets. Let $\xi \in \mathcal{B}_{\nu_1^*} := \{\xi \in \mathfrak{S} : \|\xi\|_\infty \leq \nu_1^*\}$, certainly for any $\nu_1^* > 0$, we have

$$\begin{aligned}|\Theta(\xi)(\hat{\varkappa})| \leq & L\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\left(\log\frac{\hat{\varkappa}}{s}\right)^{\gamma_1-1}|\Lambda(s,\xi(s))|\frac{ds}{s} \\ & + \frac{|\mu|}{\Gamma(\gamma_1)|\lambda+\mu|}\int_1^T\left(\log\frac{T}{s}\right)^{\gamma_1-1}|\Lambda(s,\xi(s))|\frac{ds}{s} + \frac{\nu}{\lambda+\mu}\Bigg], \\ \|\Theta(\xi)\|_\infty \leq & L\left[\left(1+\frac{|\mu|}{|\lambda+\mu|}\right)\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)}N_1 + \frac{|\nu|}{|\lambda+\mu|}\right] := \mathcal{L}.\end{aligned}$$

Thus, we deduce that $\|\Theta(\xi)\|_\infty \leq \mathcal{L}$. Hence, it follows from the foregoing inequalities above hence the operator Θ is uniformly bounded.

Step III. Next we prove that Θ is equicontinuous sets. Let $r_1, r_2 \in [1, T]$ with $r_1 < r_2$,

$$\begin{aligned}|\Theta(\xi(r_2)) - \Theta(\xi(r_1))| \leq & L\Big[\frac{1}{\Gamma(\gamma_1)}\int_1^{r_1}\left((\log\frac{r_2}{s})^{\gamma_1-1} - (\log\frac{r_1}{s})^{\gamma_1-1}\right)|\Lambda(s,\xi(s))|\,\frac{ds}{s} \\ & + \frac{1}{\Gamma(\gamma_1)}\int_{r_1}^{r_2}(\log\frac{r_2}{s})^{\gamma_1-1}\,|\Lambda(s,\xi(s))|\,\frac{ds}{s}\Big] \\ \leq & \frac{LN}{\Gamma(\gamma_1+1)}\left((\log r_2)^{\gamma_1} - (\log r_1)^{\gamma_1}\right) \\ \to & 0 \quad \text{as} \quad r_1 \to r_2.\end{aligned}$$

Therefore, the operator Θ is equicontinuous and hence the operator $\Theta(\xi)$ is completely continuous.

Step IV . To show that the set $\mathcal{P} = \{\xi \in \mathfrak{S} : \xi = \gamma\Theta(\xi), 0 < \gamma < 1\}$ is bounded (Apriori bounds). Let $\xi \in \mathcal{P}$ and $\hat{\varkappa} \in [1, T]$. Then it follows from $\xi(\hat{\varkappa}) = \gamma\Theta(\xi)(\hat{\varkappa})$ that

$$\begin{aligned}|\xi(\hat{\varkappa})| \leq & L\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\left(\log\frac{\hat{\varkappa}}{s}\right)^{\gamma_1-1}|\Lambda(s,\xi(s))|\frac{ds}{s} \\ & - \frac{|\mu|}{\Gamma(\gamma_1)|\lambda+\mu|}\int_1^T\left(\log\frac{T}{s}\right)^{\gamma_1-1}|\Lambda(s,\xi(s))|\frac{ds}{s} + \frac{\nu}{\lambda+\mu}\Bigg] \\ \leq & L\left[\left(1+\frac{|\mu|}{|\lambda+\mu|}\right)\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)}N + \frac{|\nu|}{|\lambda+\mu|}\right] := R\end{aligned}$$

and

$$\|\xi(t)\|_\infty \leq R. \tag{3.3}$$

Hence, $\mathcal{P}$ is bounded and therefore by Theorem 3.2, Θ has a fixed point, then Problem (1.1)–(1.2) has at least one solution on $[1, T]$. The proof is completed. □

4. Stability Results for the Problem

We analyse the Ulam-Hyers stability for Problem (1.1)–(1.2) in this section. Consider the following definitions of nonlinear operators $\mathcal{Z}_1 \in \mathcal{C}([1,T],\mathbb{R}) \to \mathcal{C}([1,T],\mathbb{R})$, where ξ is defined by (2.1)

$$ {}^{c}_{H}D^{\gamma_1}\left[\frac{\xi(\hat{\varkappa})}{\varpi(\hat{\varkappa},\xi(\hat{\varkappa}))}\right] - \Lambda(\hat{\varkappa},\xi(\hat{\varkappa})) = \mathcal{Z}_1(\xi)(\hat{\varkappa}), \quad \hat{\varkappa}\in[1,T],\ 0<\gamma_1\leq 1. $$

For some $\varsigma_1 > 0$, we consider the following inequality:

$$ ||\mathcal{Z}_1(\xi)||_\infty \leq \varsigma_1. \tag{4.1} $$

Definition 4.1. The system (1.1)–(1.2) is U-H stable if $\mathcal{M}_1 > 0$, for every solution $\xi^* \in \mathcal{C}([1,T],\mathbb{R})$ of inequality (4.1) there exists a unique solution on $\xi \in \mathcal{C}([1,T],\mathbb{R})$, of Problem (1.1)–(1.2) with $||\xi-\xi^*||_\infty \leq \mathcal{M}_1\varsigma_1$.

Theorem 4.1. *Suppose that (F_4) is satisfied, then the BVP (1.1)–(1.2) is U-H stable if $L\tau\pi > 1$.*

Proof. Let $\xi \in \mathcal{C}([1,T],\mathbb{R})$, be the solution of the BVP (1.1)–(1.2) satisfying (3.1).

Let ξ be any solution satisfying (4.1):

$$ {}^{c}_{H}D^{\gamma_1}\left[\frac{\xi(\hat{\varkappa})}{\varpi(\hat{\varkappa},\xi(\hat{\varkappa}))}\right] = \Lambda(\hat{\varkappa},\xi(\hat{\varkappa})) + \mathcal{Z}_1(\xi)(\hat{\varkappa}), \quad \hat{\varkappa}\in[1,T],\ 0<\gamma_1\leq 1. $$

Therefore,

$$ \begin{aligned} \xi^*(\hat{\varkappa}) =& \Theta(\xi^*)(\hat{\varkappa}) + \varpi(\hat{\varkappa},\xi(\hat{\varkappa}))\Bigg(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\left(\log\frac{\hat{\varkappa}}{s}\right)^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{ds}{s} \\ & - \frac{\mu}{\Gamma(\gamma_1)(\lambda+\mu)}\int_1^T\left(\log\frac{T}{s}\right)^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{ds}{s} + \frac{\nu}{\lambda+\mu}\Bigg), \end{aligned} $$

it follows that

$$ \begin{aligned} |\xi^*(\hat{\varkappa}) - \Theta(\xi^*)| \leq& \varpi(\hat{\varkappa},\xi(\hat{\varkappa}))\Bigg(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\left(\log\frac{\hat{\varkappa}}{s}\right)^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{ds}{s} \\ & - \frac{\mu}{\Gamma(\gamma_1)(\lambda+\mu)}\int_1^T\left(\log\frac{T}{s}\right)^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{ds}{s} + \frac{\nu}{\lambda+\mu}\Bigg) \\ \leq& L\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)}\left(1+\frac{|\mu|}{|\lambda+\mu|}\right)\varsigma_1. \end{aligned} $$

Consequently, based on the fixed point property of the operator Θ, provided in (3.1), we derive that

$$\begin{aligned} |\xi(\hat{\varkappa}) - \xi^*(\hat{\varkappa})| &= |\xi(\hat{\varkappa}) - \Theta(\xi^*)(\hat{\varkappa}) + \Theta(\xi^*)(\hat{\varkappa}) - \xi^*(\hat{\varkappa})| \\ &\leq |\Theta(\xi)(\hat{\varkappa}) - \Theta(\xi^*)(\hat{\varkappa})| + |\Theta(\xi^*)(\hat{\varkappa}) - \xi^*(\hat{\varkappa})| \\ &\leq L\tau\pi\|(\xi) - (\xi^*)\| + L\tau\varsigma_1. \end{aligned} \tag{4.2}$$

From (4.2) it follows that

$$\|\xi - \xi^*\| \leq L\pi\tau\|\xi - \xi^*\| + L\tau\varsigma_1 \leq \frac{L\tau\varsigma_1}{1 - L\tau\pi} \leq \mathcal{M}_1\varsigma_1,$$

with

$$\mathcal{M}_1 = \frac{L\tau}{1 - L\tau\pi}.$$

Hence, Problem (1.1)–(1.2) is U-H stable. □

5. Example

Example 5.1. Consider the following system of coupled fractional differential equations:

$$\begin{aligned} &{}^cD^{1/2}\Big(\frac{\xi(\hat{\varkappa})}{\varpi(\hat{\varkappa},\xi(\hat{\varkappa}))}\Big) = \frac{1}{100}\Big(\xi(\hat{\varkappa}) + \frac{1}{2}\Big) + \frac{5}{200}\cdot\frac{\xi(\hat{\varkappa})}{1+\xi(\hat{\varkappa})} + e^{-2}, \\ &\frac{\xi(1)}{\varpi(1,\xi(1))} + \frac{\xi(e)}{\varpi(e,\xi(e))} = 0. \end{aligned} \tag{5.1}$$

Here $\gamma_1 = \frac{1}{2}$, $T = e$, $\lambda = \mu = 1$, $\nu = 0$ and $\varpi(\hat{\varkappa},\xi) = \frac{\xi+1)}{100}\Big(\sin\xi + \frac{|\xi|}{1+|\xi|} + 3\Big) + e^{-1}$, $\Lambda(\hat{\varkappa},\xi(\hat{\varkappa})) = \frac{1}{100}\Big(\xi(\hat{\varkappa}) + \frac{1}{2}\Big) + \frac{5}{200}\cdot\frac{\xi(\hat{\varkappa})}{1+\xi(\hat{\varkappa})} + e^{-2}$, $\pi = \frac{2}{53}$.

From the given data, we find that $\tau = 1.6930$ and $L\tau\pi = 0.123456987 < 1$. By the Theorem (3.1), Problem (1.1)–(1.2), with the given $\Theta(\hat{\varkappa},\xi)$ has at least one solution on $[1,T]$.

Conclusion

Fractional differential equations are commonly used to model various natural phenomena, and their diverse types enable us to study the integration of many phenomena in different fields. In this study, we focused on examining the existence, uniqueness, and stability of solutions for Caputo-Hadamard hybrid fractional differential equations. Additionally, we extended our results for new classes of fractional boundary conditions by utilizing the Hadamard-Caputo sequential derivative and author fixed point theorem. For future research, we recommend exploring other types of fractional derivative operators such as the generalized Hilfer fractional derivative. Furthermore, those interested in this subject can also investigate the existence and uniqueness of solutions for coupled systems.

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