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EXISTENCE RESULT FOR Θ-HILFER-TYPE HYBRID FRACTIONAL DIFFERENTIAL LANGEVIN INCLUSION

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Abstract. The main objective of this work is to study the existence of solutions for a new class of Θ-Hilfer hybrid fractional differential Langevin inclusion. We present and discuss some of the characteristics of Θ-Hilfer fractional derivative, this kind of fractional derivative generalizes the well-known fractional derivatives, for different values of the function Θ and the parameter q. Existence result is obtained by making use of the fixed point theorem of Dhage. As application, we give an illustrative example to support our theoretical findings.

1. Introduction

Over the past few years, fractional differential equations have gained significant attention from researchers due to their effectiveness in modeling complex phenomena across various scientific and engineering disciplines, including physics, biology, chemistry, and control systems [1–4, 7, 9, 10, 24]. Recent advances in the theory of fractional differential equations have led to the incorporation of the Langevin equation within the framework of fractional calculus. Originally proposed by Paul Langevin in 1908, this equation is given by $m\frac{d^2x}{dt^2} = \lambda\frac{dx}{dt} + \eta(t)$ [16, 17, 21–23, 25, 28, 30].

Hybrid systems are fundamental in the design of embedded control mechanisms that interact directly with real-world environments. Consequently, significant research has been dedicated to advancing the theoretical foundation and solution techniques for hybrid fractional differential equations, particularly through various fixed point approaches [8, 14]. In this context, numerous forms of fixed point theorems are

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employed to establish the existence and uniqueness of solutions to different classes of fractional differential equations. Notably, Dhage's fixed point theorem proves to be particularly useful in the analysis of hybrid equations, offering greater adaptability by weakening standard conditions, thus enabling the derivation of existence results under broader circumstances [12, 13]. Several notions of fractional integrals and derivatives have been developed in the literature. Among the most widely recognized are the Riemann-Liouville and Caputo derivatives. In [18], Hilfer proposed a unified framework that generalizes these two classical forms, introducing what is now called the Hilfer fractional derivative, characterized by an order α and a parameter $\beta \in [0, 1]$. For further insights into both the Hilfer and the Θ-Hilfer fractional derivatives, one may consult the references [5, 6, 20, 27]. In [14], Dhage and Ntouyas investigate the existence of solutions for boundary value problems involving fractional hybrid differential inclusions. Their approach relies on a fixed point theorem established by Dhage within the framework of Banach algebras.

In [26], Nuchpong and collaborators examined a boundary value problem involving Langevin-type fractional differential equations incorporating the Θ-Hilfer fractional derivative, subject to nonlocal integral boundary conditions, formulated as follows:

$$\begin{cases} {}^{\mathcal{H}}D^{\chi_1,\beta_1;\Theta}\Big({}^{\mathcal{H}}D^{\chi_2,\beta_2;\Theta} + k\Big)x(t) = f(t, x(t)), \quad t \in [a, b], \\ x(a) = 0, \quad x(b) = \displaystyle\sum_{i=1}^{m} \lambda_i \mathfrak{I}^{\delta_i;\Theta} x(\tau_i), \end{cases}$$

where ${}^{\mathcal{H}}D^{\chi_i,\beta_i;\Theta}$ for $i = 1, 2$ denotes the Θ-Hilfer fractional derivative of order χ_i with $0 < \chi_i < 1$ and type $\beta_i \in [0, 1]$, such that $1 < \chi_1 + \chi_2 \leq 2$. The parameter k is a real constant, and $a \geq 0$. The function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous. The operator $\mathfrak{I}^{\delta_i;\Theta}$ represents the Θ-Riemann-Liouville fractional integral of order $\delta_i > 0$. Furthermore, the coefficients $\lambda_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$, and the points $\tau_1, \tau_2, \ldots, \tau_m$ satisfy $a \leq \tau_1 < \tau_2 < \cdots < \tau_m \leq b$. The authors established two existence results by employing Krasnoselskii's fixed point theorem and the Leray-Schauder nonlinear alternative, and also demonstrated an existence and uniqueness result based on the Banach contraction mapping principle.

In [15], Hilal and Kajouni explored a class of boundary value problems for hybrid differential equations of fractional order, where the dynamics involve Caputo fractional derivatives with order α satisfying $0 < \alpha < 1$.

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)}{f(t, x(t))}\right) = g(t, x(t)), \quad t \in [0, T], \\ a\dfrac{x(0)}{f(0, x(0))} + b\dfrac{x(T)}{f(T, x(T))} = c, \end{cases}$$

where $f \in C\Big([0, T] \times \mathbb{R}, \mathbb{R} \setminus \{0\}\Big)$, $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ and a, b, c are real constants with $a + b \neq 0$. By using Dhage fixed point theorem, the authors investigated the existence and uniqueness results for the problem.

Motivated by the mentioned works, in this paper, we extend their ideas by considering a new class of Θ-Hilfer fractional hybrid differential Langevin inclusion of the following form

$$
\begin{cases} {}^{\mathcal{H}}D^{p_1,q_1;\Theta}\left({}^{\mathcal{H}}D^{p_2,q_2;\Theta}\left[\dfrac{x(t)}{f(t,x(t))}\right] + \mu x(t)\right) \in G(t,x(t)), \quad t \in [a,b], \\ x(a) = 0, \quad x(b) = \displaystyle\sum_{i=1}^{n} \omega_i x(\eta_i), \end{cases} \tag{1.1}
$$

where ${}^{\mathcal{H}}D^{p_j,q_j;\Theta}$ for $j = 1,2$, denotes the Θ-Hilfer fractional derivative of order p_j, with $0 < p_j \leq 1$, and type parameter $q_j \in [0,1]$. The initial point is taken as $a \geq 0$, and μ is a nonzero real constant, i.e., $\mu \in \mathbb{R} \setminus 0$. The coefficients $\omega_i \in \mathbb{R}$ for $i = 1,\dots,n$, and the points $\eta_1,\dots,\eta_n$ satisfy $a < \eta_1 < \cdots < \eta_n < b$. The function f belongs to the class $C([a,b] \times \mathbb{R}, \mathbb{R} \setminus 0)$, and $G : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued mapping, where $\mathcal{P}(\mathbb{R})$ denotes the set of all nonempty subsets of $\mathbb{R}$.

The main contribution of this work lies in the exploration of a novel class of fractional derivatives known as the Θ-Hilfer fractional derivative [29]. This brand of fractional derivative generalizes the well-known fractional derivatives (Riemman-Liouville, Caputo, Θ-Riemman-Liouville, Hilfer-Hadamard, Katugampola derivative). In particular, the problem (1.1) encompasses several classical cases depending on the choice of the function Θ and the parameters q_j, $j = 1,2$, as detailed below:

- ⋆ When $\Theta(t) = t$ and $q_j = 1$, the formulation reduces to the Caputo fractional derivative case.
- ⋆ When $\Theta(t) = t$ and $q_j = 0$, the problem corresponds to the Riemann-Liouville-type formulation.
- ⋆ When $q_j = 0$, the problem becomes the Θ-Riemann-Liouville-type.
- ⋆ When $\Theta(t) = t$, the formulation coincides with the Hilfer-type derivative.
- ⋆ When $\Theta(t) = \log(t)$, the model simplifies to the Hilfer-Hadamard-type.
- ⋆ When $\Theta(t) = t^\rho$, the problem reduces to the Katugampola-type formulation.

The remainder of this paper is structured as follows. Section 2 presents essential notations, definitions, and lemmas from fractional calculus and multivalued analysis that are fundamental to our study. Section 3 focuses on establishing an existence result for the multivalued problem through the application of Dhage's fixed point theorem. Finally, Section 4 provides an illustrative example to demonstrate the applicability of the main findings.

2. Preliminaries

2.1. **Fractional Calculus.** Let $C([a,b],\mathbb{R})$ represent the Banach space consisting of all continuous functions mapping the interval $[a,b]$ into $\mathbb{R}$, equipped with the norm $\|f\| = \sup_{t\in[a,b]}\{|f(t)|\}$.

Definition 2.1 ([19])**.** Let (a,b), where $-\infty \leq a < b \leq +\infty$, denote a finite or infinite interval within the positive half-axis $(0,+\infty)$, and let $p > 0$. Furthermore, let $\Theta(t)$ be a positive, strictly increasing function defined on (a, b) with a continuous derivative $\Theta'(t)$ on (a, b). The Θ-Riemann-Liouville fractional integral of a function f with respect to Θ over $[a, b]$ is defined by

$$\mathfrak{I}_{a^+}^{p;\Theta} f(t) = \frac{1}{\Gamma(p)} \int_a^t \Theta'(t)(\Theta(t) - \Theta(s))^{p-1} f(s)ds, \tag{2.1}$$

where $\Gamma(\cdot)$ represents the Gamma function.

Definition 2.2 ([19])**.** Let $\Theta'(t) > 0$, with $p > 0$ and $n \in \mathbb{N}$, the Riemann-Liouville fractional derivative of order p of a function f with respect to the function Θ is defined by

$$\begin{aligned} D_{a^+}^{p;\Theta} f(t) &= \left(\frac{1}{\Theta'(t)} \cdot \frac{d}{dt}\right)^n \mathfrak{I}_{a^+}^{n-p;\Theta} f(t) \\ &= \frac{1}{\Gamma(n-p)} \left(\frac{1}{\Theta'(t)} \cdot \frac{d}{dt}\right)^n \int_a^t \Theta'(t)(\Theta(t) - \Theta(s))^{n-p-1} f(s)ds, \end{aligned} \tag{2.2}$$

where $n - 1 < p < n$, with $n = \lfloor p \rfloor + 1$, and $\lfloor p \rfloor$ denotes the integer part of the real number p.

Definition 2.3 ([19])**.** Let $n - 1 < p < n$ with $n \in \mathbb{N}$, and let $[a, b]$ be an interval such that $-\infty \leq a < b \leq +\infty$. Consider two functions $f, \Theta \in C^n([a, b], \mathbb{R})$ where Θ is strictly increasing and satisfies $\Theta'(t) > 0$ for all $t \in [a, b]$. The Θ-Hilfer fractional derivative of order p and type $0 \leq q \leq 1$ of the function f is defined by

$${}^{\mathcal{H}}D_{a^+}^{p,q;\Theta} f(t) = \mathfrak{I}_{a^+}^{q(n-p);\Theta} \left(\frac{1}{\Theta'(t)} \cdot \frac{d}{dt}\right)^n \mathfrak{I}_{a^+}^{(1-q)(n-p);\Theta} f(t) = \mathfrak{I}_{a^+}^{\gamma-p;\Theta} D_{a^+}^{\gamma;\Theta} f(t), \tag{2.3}$$

where $n - 1 < p < n$, with $n = [p] + 1$ and $[p]$ denoting the integer part of the real number p, and where $\gamma = p + q(n - p)$.

Lemma 2.1 ([19])**.** *If $f \in C^n([a, b], \mathbb{R})$, $n-1 < p < n$, $0 \leq q \leq 1$ and $\gamma = p+q(n-p)$, then*

$$\mathfrak{I}_{a^+}^{p;\Theta}({}^{H}D_{a^+}^{p,q;\Theta} f)(t) = f(t) - \sum_{k=1}^{n} \frac{(\Theta(t) - \Theta(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} f_{\Theta}^{[n-k]} \mathfrak{I}_{a^+}^{(1-q)(n-p);\Theta} f(a), \tag{2.4}$$

for all $t \in [a, b]$, where $f_{\Theta}^{[n-k]}(t) := \left(\frac{1}{\Theta'(t)} \cdot \frac{d}{dt}\right)^{n-k} f(t)$.

Lemma 2.2. *Let $a \geq 0$, $0 < p_i < 1$, $0 \leq q_i \leq 1$, $\gamma_i = p_i + q_i - p_i q_i$, $i = 1, 2$ and $w \in C([a, b], \mathbb{R})$. Then, the function x is a solution of the boundary value problem:*

$$\begin{cases} {}^{\mathcal{H}}D^{p_1,q_1;\Theta}\left({}^{\mathcal{H}}D^{p_2,q_2;\Theta}\left[\dfrac{x(t)}{f(t, x(t))}\right] + \mu x(t)\right) = w(t), \quad t \in [a, b], \\ x(a) = 0, \quad x(b) = \displaystyle\sum_{i=1}^{n} \omega_i x(\eta_i), \end{cases} \tag{2.5}$$

if and only if

$$\begin{aligned} x(t) =& f(t,x(t))\Bigg\{\mathfrak{I}^{p_1+p_2;\Theta}w(t) - \mu\mathfrak{I}^{p_2;\Theta}x(t) \\ &+ \frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Bigg[f(b,x(b))\mathfrak{I}^{p_1+p_2;\Theta}w(b) \\ &- \mu f(b,x(b))\mathfrak{I}^{p_2;\Theta}x(b) - \sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}w(\eta_i) \\ &+ \mu\sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_2;\Theta}x(\eta_i)\Bigg]\Bigg\}, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \Lambda =& \sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\frac{(\Theta(\eta_i)-\Theta(a))^{\gamma_1+p_2-1}}{\Gamma(\gamma_1+p_2)} \\ &- f(b,x(b))\frac{(\Theta(b)-\Theta(a))^{\gamma_1+p_2-1}}{\Gamma(\gamma_1+p_2)} \neq 0. \end{aligned} \tag{2.7}$$

Proof. Applying the Θ-Riemann-Liouville fractional integral of order p_1 to both sides of (2.5) we obtain, by using Lemma 2.1,

$$^{\mathcal{H}}D^{p_2,q_2;\Theta}\left[\frac{x(t)}{f(t,x(t))}\right] + \mu x(t) = \mathfrak{I}^{p_1;\Theta}w(t) + \frac{d_0}{\Gamma(\gamma_1)}((\Theta(t)-\Theta(a))^{\gamma_1-1}, \tag{2.8}$$

where d_0 constant and $\gamma_1 = p_1 + q_1 - p_1q_1$. Next, applying the Θ-Riemann-Liouville fractional integral of order p_2 to both sides of (2.8) we get, by using Lemma 2.1,

$$\begin{aligned} \frac{x(t)}{f(t,x(t))} =& \mathfrak{I}^{p_1+p_2;\Theta}w(t) - \mu\mathfrak{I}^{p_2;\Theta}x(t) + \frac{d_0}{\Gamma(\gamma_1+p_2)}((\Theta(t)-\Theta(a))^{\gamma_1+p_2-1} \\ &+ \frac{d_1}{\Gamma(\gamma_2)}((\Theta(t)-\Theta(a))^{\gamma_2-1}. \end{aligned} \tag{2.9}$$

From using the boundary conditions, we obtain that $d_1 = 0$ and

$$\begin{aligned} d_0 =& \frac{1}{\Lambda}\Bigg[f(b,x(b))\mathfrak{I}^{p_1+\alpha_2;\Theta}w(b) - \mu f(b,x(b))\mathfrak{I}^{p_2;\Theta}x(b) \\ &- \sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}w(\eta_i) + \mu\sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_2;\Theta}x(\eta_i)\Bigg]. \end{aligned}$$

Substituting the value of d_0 and d_1 in (2.9) we obtain the solution (2.6). The converse follows by direct computation. □

2.2. **Multi-valued Analysis.** For a normed space $(X, \|\cdot\|)$, we define: $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cp}(X) = \{Y \subset X : Y \text{ is compact}\}$, $\mathcal{P}_{cv}(X) = \{Y \subset X : Y \text{ is convex}\}$, $\mathcal{P}_{cp,cv}(X) = \{Y \subset X : Y \text{ is compact and convex}\}$.

Definition 2.4. A multivalued map $G : [a,b]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:

(i) $t \to G(t,x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \to G(t,x)$ is upper semicontinuous for almost all $t \in [a,b]$.

Furthermore, a Carathéodory function G is called $\mathbb{L}^1$-Carathéodory if:

(iii) there exists a function $\mathfrak{h} \in \mathbb{L}^1([a,b];\mathbb{R}^+)$ such that
$$\|G(t,x)\| = \sup\{|v| : v \in G(t,x)\} \leq \mathfrak{h}(t),$$
for all $x \in \mathbb{R}$ and for a.e. $t \in [a,b]$.

The following fixed point theorem due to Dhage [11] plays an important role in the proof of our main result.

Theorem 2.1. *Let X be a Banach algebra and let $\mathcal{A} : X \to X$ be a single valued operator and $\mathcal{B} : X \to \mathcal{P}_{cp,cv}(X)$ be a multi-valued operator satisfying:*

1. *$\mathcal{A}$ is single-valued Lipschitz with a Lipschitz constant l;*
2. *$\mathcal{B}$ is compact and upper semi-continuous;*
3. *$2Ml < 1$, where $M = \|\mathcal{B}(x)\|$ for each $x \in X$.*

Then, either

(i) *the operator inclusion $x \in \mathcal{A}x\mathcal{B}x$ has a solution, or*
(ii) *the set $\Delta = \{x \in X : \theta x \in \mathcal{A}x\mathcal{B}x, \theta > 1\}$ is unbounded.*

3. Main Result

In this section, we investigte the existence result to the problem (1.1), for this to simplify the computations, we use the following notations:

$$\begin{aligned}\Phi_1 =& \frac{(\Theta(b)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \frac{\widehat{f}(\Theta(b)-\Theta(a))^{\gamma_1+p_2-1}}{|\Lambda|\Gamma(\gamma_1+p_2)} \\ &\times \left[\frac{(\Theta(b)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\Theta(\eta_i)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}\right],\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\Phi_2 =& |\mu|\Bigg\{\frac{(\Theta(b)-\Theta(a))^{p_2}}{\Gamma(p_2+1)} + \frac{\widehat{f}(\Theta(b)-\Theta(a))^{\gamma_1+p_2-1}}{|\Lambda|\Gamma(\gamma_1+p_2)} \\ &\times \left[\frac{(\Theta(b)-\Theta(a))^{p_2}}{\Gamma(p_2+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\Theta(\eta_i)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}\right]\Bigg\},\end{aligned} \tag{3.2}$$

where $\widehat{f} = \max\Big\{|f(b,x(b))|, |f(\eta_i,x(\eta_i))|\Big\}$, $i = 1,\ldots,n$.

Definition 3.1. A function $x \in C$ called a solution of problem (1.1) if $x(a) = 0$, $x(b) = \sum\limits_{i=1}^{n}\omega_i x(\eta_i)$ and there exists a function $v \in \mathbb{L}^1([a,b],\mathbb{R})$ with $v \in G(t,x(t))$ a.e. on $[a,b]$ such that

$$x(t) = f(t,x(t))\Bigg\{\mathfrak{I}^{p_1+p_2;\Theta}v(t) - \mu\mathfrak{I}^{p_2;\Theta}x(t)$$

$$
\begin{aligned}
&+ \frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Bigg[f(b,x(b))\mathfrak{I}^{p_1+p_2;\Theta}v(b) \\
&- \mu f(b,x(b))\mathfrak{I}^{p_2;\Theta}x(b) - \sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}v(\eta_i) \\
&+ \mu\sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_2;\Theta}x(\eta_i)\Bigg]\Bigg\}. \qquad (3.3)
\end{aligned}
$$

For each $x \in C$, define the set of selections of G by

$$
\mathcal{S}_{G,x} := \Big\{v \in \mathbb{L}^1([a,b],\mathbb{R}) : v(t) \in G(t,x(t)), t \in [a,b]\Big\}. \qquad (3.4)
$$

Lemma 3.1 ([19]). *Let X be a Banach space, and $G : [a,b]\times\mathbb{R} \to \mathcal{P}_{b,c,cl}(X)$ be a $\mathbb{L}^1$-Carathéodory multivalued map. And let Ψ be a linear continuous mapping from $L^1([a,b],X)$ to $C([a,b],X)$. Then, the operator:*

$$
\Psi\circ\mathcal{S}_{G} : C([a,b],X) \to \mathcal{P}_{b,c,cl}\Big(C([a,b],X)\Big), \quad x \to \Big(\Psi\circ\mathcal{S}_{G}\Big)(x) = \Psi\Big(\mathcal{S}_{G,x}\Big),
$$

is a closed graph operator in $C([a,b],X)\times C([a,b],X)$.

The following existence result is based on Dhage [14] fixed point theorem.

Theorem 3.1. *Assume that the following hold.*

(H1) *The function $f : [a,b]\times\mathbb{R} \to \mathbb{R}\setminus\{0\}$ is continuous and there exists a bounded function φ such that $\varphi(t) > 0$, for almost $t \in [a,b]$ and*

$$
|f(t,x(t)) - f(t,y(t))| \le \varphi(t)|x(t)-y(t)|,
$$

for almost $t \in [a,b]$ and for almost $x,y \in \mathbb{R}$.

(H2) *$G : [a,b]\times\mathbb{R} \to \mathcal{P}_{b,c,cl}(\mathbb{R})$ is $\mathbb{L}^1$-Carathéodory and has nonempty convex values, and for each fixed $x \in C([a,b],\mathbb{R})$ the set*

$$
\mathcal{S}_{G,x} = \{v \in \mathbb{L}^1([a,b],X) : v(t) \in G(t,x(t)), t \in [a,b]\}
$$

is nonempty and convex.

(H3) *$|G(t,x)| := \sup\{|v| : v \in G(t,x)\} \le p(t)g(\|x\|)\}$ for all $t \in [a,b]$ and all $x \in C([a,b],X)$, where $p \in \mathbb{L}^1([a,b],\mathbb{R}^+)$ and $g : \mathbb{R}^+ \to [0,+\infty)$ is continuous, bounded and nondecreasing function.*

(H4) *$\|\varphi\|\Big(\|p\|g(\|x\|)\Phi_1 + \Phi_2\Big) < \frac{1}{2}$, where Φ_1, Φ_2 are given by (3.1) and (3.2).*

Then, the problem (1.1) has at least one solution on $[a,b]$.

Proof. Consider the multivalued map $\mathcal{H} : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$. To reformulate problem (1.1) as a fixed point problem, we define the operator $\mathcal{H}$ by:

$$\mathcal{H}x := \left\{ h \in C : h(t) = \begin{cases} f(t,x(t))\Big\{\mathfrak{I}^{p_1+p_2;\Theta}v(t) - \mu\mathfrak{I}^{p_2;\Theta}x(t) \\ +\dfrac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Big[f(b,x(b))\mathfrak{I}^{p_1+p_2;\Theta}v(b) \\ -\mu f(b,x(b))\mathfrak{I}^{p_2;\Theta}x(b) \\ -\sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}v(\eta_i) \\ +\mu\sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_2;\Theta}x(\eta_i)\Big]\Big\}, \\ t\in[a,b];\, v\in \mathcal{S}_{G,x} \end{cases} \right\}.$$

In what follows we define two operators, $\mathcal{A} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by:

$$\mathcal{A}x(t) = f(t,x(t)),$$

and $\mathcal{B} : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ by:

$$\begin{aligned}\mathcal{B}(x)) =& \Big\{k \in C([a,b],\mathbb{R}) : k(t) = \mathfrak{I}^{p_1+p_2;\Theta}v(t) - \mu\mathfrak{I}^{p_2;\Theta}x(t) \\ &+ \frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Big[f(b,x(b))\mathfrak{I}^{p_1+p_2;\Theta}v(b) \\ &- \mu f(b,x(b))\mathfrak{I}^{p_2;\Theta}x(b) - \sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}v(\eta_i) \\ &+ \mu\sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_2;\Theta}x(\eta_i)\Big] : v\in\mathcal{S}_{G,x}\Big\}.\end{aligned}$$

Then, the operator $\mathcal{H}$ can be written as $\mathcal{H}x = \mathcal{A}x\mathcal{B}x$. Next, we will show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Theorem 2.1, for the poof we give it in several steps.

Step 1. $\mathcal{A}$ is a Lipschitz on $C([a,b],\mathbb{R})$.

Let $x,y\in C([a,b],\mathbb{R})$, for all $t\in[a,b]$, and by $(H1)$ we have

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \le |f(t,x(t)) - f(t,y(t))| \le |\varphi(t)|\cdot|x(t)-y(t)|.$$

Then, it follows by taking the supremum over t

$$\|\mathcal{A}x - \mathcal{A}y\| \le \|\varphi\|\cdot\|x-y\|.$$

Step 2 : $\mathcal{B}$ is convex for each $x\in C([a,b],\mathbb{R})$.

Indeed, if k_1,k_2 belong to $\mathcal{B}(x)$, then there exist $v_1,v_2\in\mathcal{S}_{G,x}$ such that for each $t\in[a,b]$ we have

$$k_j(t) = \mathfrak{I}^{p_1+p_2;\Theta}v_j(t) - \mu\mathfrak{I}^{p_2;\Theta}x(t)$$

$$
\begin{aligned}
&+ \frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Bigg[f(b,x(b))\mathfrak{I}^{p_1+p_2;\Theta}v_j(b)\\
&- \mu f(b,x(b))\mathfrak{I}^{p_2;\Theta}x(b) - \sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}v_j(\eta_i)\\
&+ \mu\sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_2;\Theta}x(\eta_i)\Bigg].
\end{aligned}
$$

For $j=1,2$, let $0\le\alpha\le 1$. Then, for each $t\in[a,b]$, we have

$$
\begin{aligned}
\lambda k_1(t)+(1-\alpha)k_2(t) =&\mathfrak{I}^{p_1+p_2;\Theta}(\lambda v_1(t)-(1-\alpha)v_2(t))\\
&- \mu\mathfrak{I}^{p_2;\Theta}x(t) + \frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Bigg[f(b,x(b))\mathfrak{I}^{p_1+p_2;\Theta}\\
&\times(\alpha v_1(b)-(1-\alpha)v_2(b)) - \mu f(b,x(b))\mathfrak{I}^{p_2;\Theta}x(b)\\
&- \sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}(\alpha v_1(\eta_i)\\
&- (1-\alpha)v_2(\eta_i)) + \mu\sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_2;\Theta}x(\eta_i)\Bigg].
\end{aligned}
$$

Thus, $\alpha k_1+(1-\alpha)k_2\in\mathcal{B}(x)$ (because $\mathcal{S}_{G,x}$ is convex), then $\mathcal{B}(x)$ is convex for each $x\in C([a,b],\mathbb{R})$.

Step 3. $\mathcal{B}$ maps bounded set into bounded set in $C([a,b],\mathbb{R})$. Indeed, for a positive number ρ, let $\mathcal{B}_\rho=\{x\in C([a,b],\mathbb{R}):\|x\|\le\rho\}$ be bounded ball in $C([a,b],\mathbb{R})$, and for every $i\in\{1,\dots,n\}$ we set $\widehat{f}=\max\{|f(b,x(b))|,|f(\eta_i,x(\eta_i))|\}$. Then, for each $k\in\mathcal{B}(x)$ and $x\in\mathcal{B}_\rho$ there exists $v\in\mathcal{S}_{G,x}$, such that

$$
\begin{aligned}
k(t) =&\mathfrak{I}^{p_1+p_2;\Theta}v(t)-\mu\mathfrak{I}^{p_2;\Theta}x(t)\\
&+ \frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Bigg[f(b,x(b))\mathfrak{I}^{p_1+p_2;\Theta}v(b)\\
&- \mu f(b,x(b))\mathfrak{I}^{p_2;\Theta}x(b) - \sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}v(\eta_i)\\
&+ \mu\sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_2;\Theta}x(\eta_i)\Bigg].
\end{aligned}
$$

Then, for every $t\in[a,b]$, we have

$$
\begin{aligned}
|k(t)|\le\sup_{t\in[a,b]}&\Bigg\{\mathfrak{I}^{p_1+p_2;\Theta}|v(t)|+|\mu|\mathfrak{I}^{p_2;\Theta}|x(t)|+\frac{\widehat{f}(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{|\Lambda|\Gamma(\gamma_1+p_2)}\\
&\times\Bigg[\mathfrak{I}^{p_1+p_2;\Theta}|v(b)|+\sum_{i=1}^{n}|\omega_i|\mathfrak{I}^{p_1+p_2;\Theta}|v(\eta_i)|\\
&+|\mu|\mathfrak{I}^{p_2;\Theta}|x(b)|+|\mu|\sum_{i=1}^{n}|\omega_i|\mathfrak{I}^{p_2;\Theta}|x(\eta_i)|\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\leq & \|p\| g(\|x\|)\Bigg\{\frac{(\Theta(b)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+\frac{\widehat{f}(\Theta(b)-(a))^{\gamma_1+p_2-1}}{|\Lambda|\Gamma(\gamma_1+p_2)} \\
& \times\left[\frac{(\Theta(b)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+\sum_{i=1}^{n}|\omega_i|\frac{(\Theta(\eta_i)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}\right]\Bigg\} \\
& +\|x\||\mu|\Bigg\{\frac{(\Theta(b)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}+\frac{\widehat{f}(\Theta(b)-\Theta(a))^{\gamma_1+p_2-1}}{|\Lambda|\Gamma(\gamma_1+p_2)} \\
& \times\left[\frac{(\Theta(b)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}+\sum_{i=1}^{n}|\omega_i|\frac{(\Theta(\eta_i)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}\right]\Bigg\} \\
\leq & \|p\| g(\|x\|)\Phi_1+\|x\|\Phi_2 \\
\leq & \|p\| g(\rho)\Phi_1+\rho\Phi_2.
\end{aligned}
$$

Then, $\|k\| \leq \|p\| g(\rho)\Phi_1+\rho\Phi_2$, where Φ_1, Φ_2 are respectively given by (3.1) and (3.2). Finally, $\mathcal{B}$ is bounded.

Step 4. $\mathcal{B}$ maps bounded set into equicontinuous sets of $C([a,b],\mathbb{R})$.

Let $t_1, t_2 \in [a,b]$, $t_1 < t_2$, and $x \in \mathcal{B}_\rho$, where $\mathcal{B}_\rho$ as above. Then, for each $x \in \mathcal{B}_\rho$ and $k \in \mathcal{B}(x)$, there exist $v \in \mathcal{S}_{G,x}$ and we obtain

$$
\begin{aligned}
& |k(t_2)-k(t_1)| \\
\leq & \frac{1}{\Gamma(p_1+p_2)}\Bigg|\int_a^{t_1}\Theta'(s)\Big((\Theta(t_2)-\Theta(s))^{p_1+p_2-1} \\
& -(\Theta(t_1)-\Theta(s))^{p_1+p_2-1}\Big)v(s)ds+\int_{t_1}^{t_2}\Theta'(s)(\Theta(t_2)-\Theta(s))^{p_1+p_2-1}v(s)ds\Bigg| \\
& +\frac{|\mu|}{\Gamma(p_2)}\Bigg|\int_a^{t_1}\Theta'(s)\Big((\Theta(t_2)-\Theta(s))^{p_2-1}-(\Theta(t_1)-\Theta(s))^{p_2-1}\Big)x(s)ds \\
& +\int_{t_1}^{t_2}\Theta'(s)(\Theta(t_2)-\Theta(s))^{p_2-1}x(s)ds\Bigg| \\
& +\frac{|(\Theta(t_2)-\Theta(a))^{\gamma_1+p_2-1}-(\Theta(t_1)-\Theta(a))^{\gamma_1+p_2-1}|}{|\Lambda|\Gamma(\gamma_1+p_2)} \\
& \times\Bigg[\|v(s)\|\frac{(\Theta(b)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+\|v(s)\|\sum_{i=1}^{n}|\omega_i|\frac{(\Theta(\eta_i)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} \\
& +\|x(b)\||\mu|\frac{(\Theta(b)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}+\|x(\eta_i)\||\mu|\sum_{i=1}^{n}|\omega_i|\frac{(\Theta(\eta_i)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}\Bigg] \\
\leq & \frac{\|p\|\Theta(\rho)}{\Gamma(p_1+p_2)}\Bigg|\int_a^{t_1}\Theta'(s)\Big((\Theta(t_2)-\Theta(s))^{p_1+p_2-1} \\
& -(\Theta(t_1)-\Theta(s))^{p_1+p_2-1}\Big)ds+\int_{t_1}^{t_2}\Theta'(s)(\Theta(t_2)-\Theta(s))^{p_1+p_2-1}ds\Bigg| \\
& +\frac{\rho|\mu|}{\Gamma(p_2)}\Bigg|\int_a^{t_1}\Theta'(s)\Big((\Theta(t_2)-\Theta(s))^{p_2-1}-(\Theta(t_1)-\Theta(s))^{p_2-1}\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\Theta'(s)(\Theta(t_2)-\Theta(s))^{p_2-1}ds\Big|\\
&+\frac{|(\Theta(t_2)-\Theta(a))^{\gamma_1+p_2-1}-(\Theta(t_1)-\Theta(a))^{\gamma_1+p_2-1}|}{|\Lambda|\Gamma(\gamma_1+p_2)}\\
&\times\Bigg[\|p\|\Theta(\rho)\frac{(\Theta(b)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+\|p\|\Theta(\rho)\sum_{i=1}^{n}|\omega_i|\frac{(\Theta(\eta_i)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}\\
&+\rho|\mu|\frac{(\Theta(b)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}+\rho|\mu|\sum_{i=1}^{n}|\omega_i|\frac{(\Theta(\eta_i)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}\Bigg].
\end{aligned}
$$

As $t_2\to t_1$ the right-hand side of the above inequality tends to zero, implies that $\mathcal{B}(x)$ is equicontinuous. By using Arzelá-Ascoli's theorem we deduce that $\mathcal{B}:C([a,b],\mathbb{R})\to\mathcal{P}(C([a,b],\mathbb{R}))$ is compact, therefore $\mathcal{B}$ is completely continuous.

Next, to prove that the multi-valued operator $\mathcal{B}$ is upper semicontinuous, is enough to show that $\mathcal{B}$ has a closed graph.

Step 5. $\mathcal{B}$ has a closed graph.

Let $x_n\to x_*$, $k_n\in\mathcal{B}(x_n)$ and $k_n\to k_*$, we shall prove that $k_*\in\mathcal{B}(x_*)$.

For $k_n\in\mathcal{B}(x_n)$ there exists $v_n\in\mathcal{S}_{G,x_n}$ such that for each $t\in[a,b]$,

$$
\begin{aligned}
k_n(t)=&\mathfrak{I}^{p_1+p_2;\Theta}v_n(t)-\mu\mathfrak{I}^{p_2;\Theta}x_n(t)\\
&+\frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Bigg[f(b,x_n(b))\mathfrak{I}^{p_1+p_2;\Theta}v_n(b)\\
&-\mu f(b,x_n(b))\mathfrak{I}^{p_2;\Theta}x_n(b)-\sum_{i=1}^{n}\omega_i f(\eta_i,x_n(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}v_n(\eta_i)\\
&+\mu\sum_{i=1}^{n}\omega_i f(\eta_i,x_n(\eta_i))\mathfrak{I}^{p_2;\Theta}x_n(\eta_i)\Bigg].
\end{aligned}
$$

We should prove that $v_*\in\mathcal{S}_{G,x_*}$ such that for each $t\in[a,b]$:

$$
\begin{aligned}
k_*(t)=&\mathfrak{I}^{p_1+p_2;\Theta}v_*(t)-\mu\mathfrak{I}^{p_2;\Theta}x_*(t)\\
&+\frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Bigg[f(b,x_*(b))\mathfrak{I}^{p_1+p_2;\Theta}v_*(b)\\
&-\mu f(b,x_*(b))\mathfrak{I}^{p_2;\Theta}x_*(b)-\sum_{i=1}^{n}\omega_i f(\eta_i,x_*(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}v_*(\eta_i)\\
&+\mu\sum_{i=1}^{n}\omega_i f(\eta_i,x_*(\eta_i))\mathfrak{I}^{p_2;\Theta}x_*(\eta_i)\Bigg],
\end{aligned}
$$

we have that

$$\|k_n(t)-k_*(t)\|\to 0,\quad \text{as } n\to+\infty.$$

Consider the operator defined by:

$$\Psi:\mathbb{L}^1([a,b],\mathbb{R})\to C([a,b],\mathbb{R}),\quad v\to\Psi(v)(t),$$

with

$$\Psi(v)(t) = \mathfrak{I}^{p_1+p_2;\Theta}v(t) + \frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Bigg(f(b,x(b))\mathfrak{I}^{p_1+p_2;\Theta}v(b) - \sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}v(\eta_i)\Bigg),$$

from Lemma 3.1, $\Psi \circ \mathcal{S}_G$ is a closed graph operator. Then, we have $k_n(t) \in \Psi(\mathcal{S}_{G,x_n})$, since $x_n \to x_*$, it follows that there exists $v_* \in \mathcal{S}_{G,x_*}$ such that for each $t \in [a,b]$:

$$\begin{aligned} k_*(t) =& \mathfrak{I}^{p_1+p_2;\Theta}v_*(t) - \mu\mathfrak{I}^{p_2;\Theta}x_*(t) \\ &+ \frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Bigg[f(b,x_*(b))\mathfrak{I}^{p_1+p_2;\Theta}v_*(b) \\ &- \mu f(b,x_*(b))\mathfrak{I}^{p_2;\Theta}x_*(b) - \sum_{i=1}^{n}\omega_i f(\eta_i,x_*(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}v_*(\eta_i) \\ &+ \mu\sum_{i=1}^{n}\omega_i f(\eta_i,x_*(\eta_i))\mathfrak{I}^{p_2;\Theta}x_*(\eta_i)\Bigg]. \end{aligned}$$

As a result we deduce that the multi-valued operator $\mathcal{B}$ is compact and upper semi-continuous.

Step 6. We show that $2Ml < 1$. From Step 3. we get

$$\begin{aligned} M :=& \|\mathcal{B}(C([a,b],\mathbb{R}))\| = \sup\{|\mathcal{B}(x)| : x \in C([a,b],\mathbb{R})\} \\ \le& \|p\|g(\|x\|)\Phi_1 + \|x\|\Phi_2. \end{aligned}$$

It follows, by using (H4),

$$\|\varphi\|\Big(\|p\|g(\|x\|)\Phi_1 + \|x\|\Phi_2\Big) < \frac{1}{2}.$$

This implies that $2Ml < 1$, where $l = \|\varphi\|$.

Finally, all the conditions of Theorem 2.1 are satisfied. To finish the proof it remains to prove that the set $\Delta = \{x \in C([a,b],\mathbb{R}) : \theta x \in \mathcal{A}x\mathcal{B}x, \theta > 1\}$ is bounded (i.e., (ii) of Theorem 2.1 is not possible).

Step 7. $\Delta = \{x \in X : \theta x \in \mathcal{A}x\mathcal{B}x, \theta > 1\}$ is bounded.

Let $x \in \Delta$. Then, $\theta x \in \mathcal{A}x\mathcal{B}x$ for some $\theta > 1$, thus there exists a function $v \in \mathcal{S}_{G,x}$ such that

$$\begin{aligned} x(t) =& \theta^{-1}f(t,x(t))\Bigg\{\mathfrak{I}^{p_1+p_2;\Theta}v(t) - \mu\mathfrak{I}^{p_2;\Theta}x(t) \\ &+ \frac{(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{\Lambda\Gamma(\gamma_1+p_2)}\Bigg[f(b,x(b))\mathfrak{I}^{p_1+p_2;\Theta}v(b) \\ &- \mu f(b,x(b))\mathfrak{I}^{p_2;\Theta}x(b) - \sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_1+p_2;\Theta}v(\eta_i) \end{aligned}$$

$$+\mu\sum_{i=1}^{n}\omega_i f(\eta_i,x(\eta_i))\mathfrak{I}^{p_2;\Theta}x(\eta_i)\Big]\Big]\Big\}.$$

Let us put $\overline{f}=\sup_{t\in[a,b]}|f(t,0)|>0$, and for every $t\in[a,b]$ we have

$$\begin{aligned}
\Big|x(t)\Big| \le & \Big|f(t,x(t))\Big|\Big\{\mathfrak{I}^{p_1+p_2;\Theta}|v(t)|+|\mu|\mathfrak{I}^{p_2;\Theta}|x(t)|+\frac{\widehat{f}(\Theta(t)-\Theta(a))^{\gamma_1+p_2-1}}{|\Lambda|\Gamma(\gamma_1+p_2)}\\
&\times\Big[\mathfrak{I}^{p_1+p_2;\Theta}|v(b)|+\sum_{i=1}^{n}|\omega_i|\mathfrak{I}^{p_1+p_2;\Theta}|v(\eta_i)|\\
&+|\mu|\mathfrak{I}^{p_2;\Theta}|x(b)|+|\mu|\sum_{i=1}^{n}|\omega_i|\mathfrak{I}^{p_2;\Theta}|x(\eta_i)|\Big]\Big\},\\
\le & \Big(\|\varphi\|\cdot\|x\|+\overline{f}\Big)\|p\|g(\|x\|)\\
&\times\Big\{\frac{(\Theta(b)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+\frac{\widehat{f}(\Theta(b)-(a))^{\gamma_1+p_2-1}}{|\Lambda|\Gamma(\gamma_1+p_2)}\\
&\times\Big[\frac{(\Theta(b)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}+\sum_{i=1}^{n}|\omega_i|\frac{(\Theta(\eta_i)-\Theta(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}\Big]\Big\}\\
&+\|x\|\cdot|\mu|\Big\{\frac{(\Theta(b)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}+\frac{\widehat{f}(\Theta(b)-\Theta(a))^{\gamma_1+p_2-1}}{|\Lambda|\Gamma(\gamma_1+p_2)}\\
&\times\Big[\frac{(\Theta(b)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}+\sum_{i=1}^{n}|\omega_i|\frac{(\Theta(\eta_i)-\Theta(a))^{p_2}}{\Gamma(p_2+1)}\Big]\Big\}\\
\le & \Big(\|\varphi\|\cdot\|x\|+\overline{f}\Big)\|p\|g(\|x\|)\Phi_1+\|x\|\Phi_2,
\end{aligned}$$

taking the supremum over t we get the constant $L>0$ such that

$$\|x\|\le\frac{\overline{f}\|p\|g(\|x\|)\Phi_1}{1-\Phi_2-\|\varphi\|\cdot\|p\|g(\|x\|)\Phi_1}:=L.$$

Finally, the set Δ is bounded. If it is not bounded, dividing the above inequality by $\delta:=\|x\|$ such that $\delta\to+\infty$, then we get

$$1\le\frac{\overline{f}\|p\|g(\delta)\Phi_1}{\delta\Big(1-\Phi_2-\|\varphi\|\cdot\|p\|g(\delta)\Phi_1\Big)}.$$

By $(H4)$ and using the fact that g is bounded then there exists $\mathcal{L}>0$ such that $\Theta(\delta)\le\mathcal{L}$, we get by letting $\delta\to+\infty$

$$1\le\lim_{\delta\to+\infty}\frac{\overline{f}\|p\|\mathcal{L}\Phi_1}{\delta\Big(1-\Phi_2-\|\varphi\|\|p\|\mathcal{L}\Phi_1\Big)}=0,$$

which is a contradiction, then the set Δ is bounded in $C([a,b],\mathbb{R})$, from Theorem 2.1, we deduce that the operator $\mathcal{H}$ has at least one fixed point which represent the solution of the problem (1.1) on $[a,b]$. $\square$

4. Example

Consider the following problem

$$
\begin{cases} {}^{\mathcal{H}}D^{\frac{3}{5},\frac{2}{3};\frac{e^t}{6}}\left[{}^{\mathcal{H}}D^{\frac{2}{5},\frac{1}{3};\frac{e^t}{6}}\left(\dfrac{x(t)}{\sin x(t)+2}\right)+\dfrac{1}{9}x(t)\right] \\ \in \left[\dfrac{|x(t)|^5}{5(|x(t)|^5+2)}+\dfrac{t+1}{10},\dfrac{|\sin x(t)|}{5(|\sin x(t)|+1)}+\dfrac{t}{5}\right], \quad t\in[0,1], \\ x(0)=0, \quad x(1)=\dfrac{3}{8}x\left(\dfrac{1}{3}\right)+\dfrac{5}{8}x\left(\dfrac{1}{2}\right), \end{cases} \tag{4.1}
$$

where $p_1=\frac{3}{5}$, $q_1=\frac{2}{3}$, $p_2=\frac{2}{5}$, $q_2=\frac{1}{3}$, $a=0$, $b=1$, $[a,b]=[0,1]$, $\mu=\frac{1}{9}$, $n=2$, $\omega_1=\frac{3}{8}$, $\omega_2=\frac{5}{8}$, $\eta_1=\frac{1}{3}$, $\eta_2=\frac{1}{2}$ and $\Theta(t)=\frac{e^t}{6}$. Set $G:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map defined by

$$
(t,x)\to G(t,x)=\left[\frac{|x|^5}{5(|x|^5+2)}+\frac{t+1}{10},\frac{|\sin x|}{5(|\sin x|+1)}+\frac{t}{5}\right].
$$

For $v\in G(t,x)$ we have

$$
|v|\le \max\left\{\frac{|x|^5}{5(|x|^5+2)}+\frac{t+1}{10},\frac{|\sin x|}{5(|\sin x|+1)}+\frac{t}{5}\right\}\le\frac{2}{5}.
$$

Thus,

$$
\|G(t,x)\|=\sup\{|v|:v\in G(t,x)\}\le\frac{2}{5}=p(t)g(\|x\|),\quad \text{for all } x\in\mathbb{R},
$$

where $p(t)=1$, $g(\|x\|)=\frac{2}{5}$. Here $f(t,x)=\sin x+2$. Therefore, for any $x,y\in\mathbb{R}$, we have

$$
|f(t,x)-f(t,y)|=|\sin x-\sin y|<|x-y|=\varphi(t)|x-y|,
$$

where $\varphi(t)=1$ for all $t\in[0,1]$.

With the given data, we get $\gamma_1=p_1+q_1-p_1q_1=\frac{13}{15}$, $|\Lambda|\simeq 0.46833393$, $\Phi_1=0.92632246$, $\Phi_2=0.123557112$. Then,

$$
\begin{aligned} \|\varphi\|\Big(\|p\|g(\|x\|)\Phi_1+\Phi_2\Big)&=1\cdot 1\cdot\left(\frac{2}{5}\times 0.92632246+0.123557112\right)\\ &=0.49624025<\frac{1}{2}. \end{aligned}
$$

Finally, all the conditions of Theorem 3.1 are satisfied, thus the problem (4.1) has at least one solution defined on $[0,1]$.

5. Conclusion

In this work, we have considered a new class of Θ-Hilfer fractional hybrid differential Langevin inclusion, by studying a new and challenging type of fractional derivative called Θ-Hilfer fractional derivative which generalizes the well-known fractional derivatives for different values of the function Θ and the parameter q. By making use of Dhage's fixed point theorem we prove the existence result for our problem. A numerical example is presented to clarify the obtained result. As a direction for future research, we aim to extend these results to study the fractional differential Langevin equation with a variable coefficient and the Θ-Hilfer generalized proportional fractional derivative, along with graphical and numerical examples.

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