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SOME NEW INEQUALITIES FOR DIFFERENTIABLE ARITHMETIC-HARMONICALLY CONVEX FUNCTIONS

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Abstract. In this study, by using an integral identity together with both the Hölder and the power-mean inequalities for integrals we establish several new inequalities for differentiable arithmetic-harmonically-convex function. Also, we give some applications for special means.

1. Preliminaries and Fundamentals

Throughout, we denote any real interval by $I \subseteq \mathbb{R}$ and any functions defined on I by $f : I \subseteq \mathbb{R} \to \mathbb{R}$. Let I° denote the interior of I. Also, we denote

$$I_f(a,b) = f(b)b - f(a)a - \int_a^b f(x)dx,$$

for brevity.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)y\right) \le tf\left(x\right) + (1-t)f\left(y\right)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

Convexity theory has appeared as a powerful technique to study a wide class of related problems in pure and applied sciences. The following double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.

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Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds.

See [2, 4], for the results of the generalization, improvement and extention of the famous integral inequality (1.1).

Definition 1.2 ([1, 5])**.** A function $f : I \subset \mathbb{R} \to (0, \infty)$ is said to be arithmetic-harmonically (AH) convex function if for all $x, y \in I$ and $t \in [0, 1]$ the equality

$$f(tx + (1-t)y) \leq \frac{f(x)f(y)}{tf(y) + (1-t)f(x)} \tag{1.2}$$

holds. If the inequality (1.2) is reversed, then the function f is said to be arithmetic-harmonically (AH) concave function.

In order to establish some inequalities of Hermite-Hadamard type integral inequalities for AH-convex functions, we will use the following lemma obtained in the special case of identity given in [3].

Lemma 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. We have the identity*

$$I_f(a, b) = \int_a^b x f'(x)dx. \tag{1.3}$$

In this study, we use Hölder integral inequality, power-mean integral inequality and the identity (1.3) in order to provide some inequalities for functions whose first derivatives in absolute value at certain power are arithmetic-harmonically convex.

Throught this paper, we will use the following notations for special means of two nonnegative numbers a, b with $b > a$:

1. the arithmetic mean

$$A := A(a, b) = \frac{a+b}{2}, \quad a, b > 0,$$

2. the geometric mean

$$G := G(a, b) = \sqrt{ab}, \quad a, b \geq 0,$$

3. the logarithmic mean

$$L := L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b, \\ a, & a = b, \end{cases} \quad b > 0,$$

4. the p-logarithmic mean

$$L_p := L_p(a, b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R} \setminus \{-1, 0\}, \\ a, & a = b, \end{cases} \quad a, b > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

2. Main Results for Lemma

Throughout this section we will denote $K_x = |f'(x)|$ for brevity.

Theorem 2.1. *Let $f : I \subset (0, +\infty) \to (0, +\infty)$ be a differentiable mapping on I°, and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is an arithmetic-harmonically convex function on the interval $[a, b]$, then the following inequality holds:*

$$|I_f(a,b)| \leq \begin{cases} \frac{(b-a)G^2(K_a,K_b)}{K_b-K_a}\left(\frac{bK_b-aK_a}{L(K_a,K_b)}-(b-a)\right), & K_a \neq K_b, \\ (b-a)K_bA(a,b), & K_a = K_b. \end{cases} \tag{2.1}$$

Proof. Since $|f'|$ is an arithmetic-harmonically convex function on the interval $[a, b]$, we have on setting $t = \frac{b-x}{b-a}$ and $1 - t = \frac{x-a}{b-a}$ in (1.2)

$$|f'(x)| \leq \frac{(b-a)K_aK_b}{(b-x)\,K_b + (x-a)\,K_a}, \tag{2.2}$$

for all $x \in [a, b]$. Substituting (2.2) in

$$|I_f(a,b)| \leq \int_a^b x\,|f'(x)|\,dx, \tag{2.3}$$

which folllows from (1.3), we have

$$|I_f(a,b)| \leq (b-a)K_aK_b \int_a^b \frac{x}{(b-x)\,K_b + (x-a)\,K_a} dx. \tag{2.4}$$

We distinguish two cases. If $K_a = K_b$, then (2.1) follows. Suppose $K_a \neq K_b$. Then, by the change of variable $u = (b-x)\,K_b + (x-a)\,K_a$, the integral in (2.4) becomes

$$\begin{aligned} &\frac{(b-a)K_aK_b}{(K_b-K_a)^2}\int_{(b-a)K_a}^{(b-a)K_b}\left(\frac{bK_b-aK_a}{u}-1\right)du \\ =&\frac{(b-a)K_aK_b}{K_b-K_a}\left(bK_b-aK_a\frac{\ln K_b-\ln K_a}{K_b-K_a}-(b-a)\right). \end{aligned}$$

Substituting this in (2.4) and using the definition of the logarithmic mean, we conclude (2.1) in this case. This completes the proof. □

Theorem 2.2. *Let $f : I \subset (0, +\infty) \to (0, +\infty)$ be a differentiable mapping on I°, and $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is an arithmetic-harmonically convex function on the interval $[a, b]$, then the following inequality holds:*

$$|I_f(a,b)| \leq \begin{cases} \frac{(b-a)L_p(a,b)G^2(K_a,K_b)}{\left(L(K_a,K_b)L_{q-1}^{q-1}(K_a,K_b)\right)^{\frac{1}{q}}}, & K_a \neq K_b, \\ (b-a)K_bL_p(a,b), & K_a = K_b, \end{cases} \tag{2.5}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|f'|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, we have

$$|f'(x)|^q \leq \frac{(b-a)\left(K_a K_b\right)^q}{(b-x)\,K_b^q+(x-a)\,K_a^q}, \tag{2.6}$$

for all $x\in[a,b]$. By using Hölder integral inequality in (2.3), we get

$$|I_f(a,b)| \leq \left(\int_a^b x^p dx\right)^{\frac{1}{p}}\left(\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}}. \tag{2.7}$$

By combining (2.6) and (2.7) and also using the definitions of the p-logarithmic mean and geometric mean, we obtain

$$|I_f(a,b)| \leq (b-a)G^2\left(K_a,K_b\right)L_p(a,b)\left(\int_a^b \frac{dx}{(b-x)\,K_b^q+(x-a)\,K_a^q}\right)^{\frac{1}{q}}. \tag{2.8}$$

We distinguish two cases. If $K_a=K_b$, then (2.5) follows. Suppose $K_a\neq K_b$. Then, by the change of variable $u=(b-x)\,K_b^q+(x-a)\,K_a^q$, the integral in (2.8) becomes

$$\begin{aligned}&(b-a)G^2\left(K_a,K_b\right)L_p(a,b)\left(\int_{(b-a)K_a^q}^{(b-a)K_b^q}\frac{du}{\left(K_b^q-K_a^q\right)u}\right)^{\frac{1}{q}}\\ =&(b-a)G^2\left(K_a,K_b\right)L_p(a,b)\left(\frac{\ln K_b^q-\ln K_a^q}{K_b^q-K_a^q}\right)^{\frac{1}{q}}.\end{aligned}$$

Substituting this in (2.8) and using the definitions of the logarithmic mean and the p-logarithmic mean, we conclude (2.5) in this case. This completes the proof. □

Theorem 2.3. *Let $f:I\subset(0,+\infty)\to(0,+\infty)$ be a differentiable mapping on I°, and $a,b\in I^\circ$ with $a<b$. If $|f'|^q, q\geq 1$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequality holds:*

$$|I_f(a,b)| \leq \begin{cases}\frac{(b-a)A^{1-\frac{1}{q}}(a,b)G^2(K_a,K_b)}{\left(K_b^q-K_a^q\right)^{\frac{1}{q}}}\left(\frac{bK_b^q-aK_a^q}{L(K_a,K_b)L_{q-1}^{q-1}(K_a,K_b)}-(b-a)\right)^{\frac{1}{q}}, & K_a\neq K_b,\\ (b-a)K_bA(a,b), & K_a=K_b.\end{cases} \tag{2.9}$$

Proof. Since $|f'|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, we have

$$|f'(x)|^q \leq \frac{(b-a)\left(K_a K_b\right)^q}{(b-x)\,K_b^q+(x-a)\,K_a^q}, \tag{2.10}$$

for all $x\in[a,b]$. By using well known power-mean integral inequality in (2.3), we get

$$|I_f(a,b)| \leq \left(\int_a^b x dx\right)^{1-\frac{1}{q}}\left(\int_a^b x\,|f'(x)|^q\,dx\right)^{\frac{1}{q}}. \tag{2.11}$$

By combining (2.10) and (2.11) and also using the definitions of the arithmetic mean and geometric mean, we obtain

$$|I_f(a,b)| \le (b-a)A^{1-\frac{1}{q}}(a,b)G^2\left(K_a,K_b\right)\left(\int_a^b \frac{x}{(b-x)\,K_b^q+(x-a)\,K_a^q}dx\right)^{\frac{1}{q}}. \tag{2.12}$$

We distinguish two cases. If $K_a = K_b$, then (2.9) follows. Suppose $K_a \neq K_b$. Then, by the change of variable $u = (b-x)\,K_b^q + (x-a)\,K_a^q$, the integral in (2.12) becomes

$$\begin{aligned}
&\frac{(b-a)A^{1-\frac{1}{q}}(a,b)G^2\left(K_a,K_b\right)}{(K_b^q-K_a^q)^{\frac{2}{q}}}\left(\int_{(b-a)K_a^q}^{(b-a)K_b^q}\frac{bK_b^q-aK_a^q-u}{u}du\right)^{\frac{1}{q}}\\
=&\frac{(b-a)A^{1-\frac{1}{q}}(a,b)G^2\left(K_a,K_b\right)}{(K_b^q-K_a^q)^{\frac{1}{q}}}\left(\frac{(bK_b^q-aK_a^q)\,(\ln K_b^q-\ln K_a^q)}{K_b^q-K_a^q}-(b-a)\right)^{\frac{1}{q}}.
\end{aligned}$$

Substituting this in (2.12) and using the definitions of the logarithmic mean and the p-logarithmic mean, we conclude (2.9) in this case. This completes the proof. □

Corollary 2.1. *If we take $q = 1$ in the inequality* (2.9)*, we get the inequality* (2.1)*.*

3. Applications for Special Means

If $p \in (-1, 0)$, then the function $f(x) = x^p$, $x > 0$, is an arithmetic harmonically-convex [1]. Using this function we obtain following propositions.

Proposition 3.1. *Let $0 < a < b$ and $m \in (-1, 0)$. Then we have the following inequality:*

$$L_{m+1}^{m+1}(a,b) \le \frac{1}{m}\cdot\frac{G^{2m}\left(a,b\right)}{L_{m-1}^{m-1}(a,b)}\left((m+1)\,\frac{L_m^m(a,b)}{L\left(a^m,b^m\right)}-1\right). \tag{3.1}$$

Proof. We know that if $m \in (-1, 0)$ then the function $f(x) = \frac{x^{m+1}}{m+1}$, $x > 0$, is an arithmetic harmonically-convex function. Therefore, the assertion follows from the inequality (2.1), for $f : (0, +\infty) \to \mathbb{R}$, $f(x) = \frac{x^{m+1}}{m+1}$. □

Proposition 3.2. *Let $a, b \in (0, +\infty)$ with $a < b$, $q > 1$ and $m \in (-1, 0)$. Then we have the following inequality:*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \le \frac{L_p(a,b)G^{\frac{2m}{q}}\left(a,b\right)}{\left(L\left(a^{m/q},b^{m/q}\right)L_{q-1}^{q-1}\left(a^{m/q},b^{m/q}\right)\right)^{\frac{1}{q}}}.$$

Proof. The assertion follows from the inequality (2.5). Let $f(x) = \frac{q}{m+q}x^{\frac{m}{q}+1}$, $x \in (0, +\infty)$. Then $|f'(x)|^q = x^m$ is an arithmetic harmonically-convex on $(0, +\infty)$ and the result follows directly from Theorem 2.2. □

Proposition 3.3. *Let $a, b \in (0, +\infty)$ with $a < b$, $q > 1$ and $m \in (-1, 0)$. Then, we have the following inequality:*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq \frac{A^{1-\frac{1}{q}}(a,b)G^{\frac{2m}{q}}(a,b)}{\left(mL_{m-1}^{m-1}(a,b)\right)^{\frac{1}{q}}}\left(\frac{(m+1)L_m^m(a,b)}{L\left(a^{m/q}, b^{m/q}\right)L_{q-1}^{q-1}\left(a^{m/q}, b^{m/q}\right)} - 1\right)^{\frac{1}{q}}. \tag{3.2}$$

Proof. The assertion follows from the inequality (2.9). Let $f(x) = \frac{q}{m+q}x^{\frac{m}{q}+1}$, $x \in (0, +\infty)$. Then $|f'(x)|^q = x^m$ is an arithmetic harmonically-convex on $(0, +\infty)$ and the result follows directly from Theorem 2.3. □

Corollary 3.1. *If we take $q = 1$ in the inequality* (3.2)*, we get the following inequality*

$$L_{m+1}^{m+1}(a,b) \leq \frac{G^{2m}(a,b)}{mL_{m-1}^{m-1}(a,b)}\left(\frac{(m+1)L_m^m(a,b)}{L\left(a^m, b^m\right)} - 1\right), \tag{3.3}$$

which is the same as inequality (3.1)*.*

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