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COMPUTING THE $\mathcal{H}_2$-NORM OF A FRACTIONAL-ORDER SYSTEM USING THE STATE-SPACE LINEAR MODEL

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Abstract. The main purpose of the present paper is to establish an alternative approach to compute the $\mathcal{H}_2$-norm for a fractional-order transfer function of the first kind based on Caputo fractional derivative. The key idea behind this new approach is the use of the concept of the parahermitian transfer matrices and the state-space realization. Numerical examples are presented to illustrate the new approach.

1. Introduction

In the last few years, many researchers pointed out that fractional derivatives revealed to be a more adequate tool for the description of properties of various real materials and in different fields [3, 8, 9, 11–13, 15]. Among these fields, dynamic systems appear since they can be described and modelled using fractional derivatives [3, 8, 11, 15].

One of the most important problems in modelling and control of dynamic systems is to compute the impulse response energy, known, also, as the $\mathcal{H}_2$-norm, for a fractional-order transfer function. The $\mathcal{H}_2$-norm often arises in control theory and can be used to measure the precision of a rational approximation of a fractional transfer function and inversely [1, 10, 14, 15]. More than that, the $\mathcal{H}_2$-norm is a useful measure for assessing the system's performance.

In the literature, several methods have been proposed to compute the $\mathcal{H}_2$-norm for the fractional transfer function, most of them use an analytic or an algebraic formulation [1, 10, 15]. However, in this paper, an alternative method is provided to

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calculate the $\mathcal{H}_2$-norm for the fractional transfer function of the first kind associated with the fractional-order linear system. The main concept of this new approach is the use of the state-space realization consisting of parameters that are extracted from the fractional-order transfer function and then a transformation of the parahermitian matrix which let it invariant. Finally, the general expression of the $\mathcal{H}_2$-norm is derived thanks to some concepts and some conditions set out.

The rest of the paper is organized as follows. In Section 2, mathematical concepts and the definition of the $\mathcal{H}_2$-norm are recalled. Section 3 describes the new approach for computing the $\mathcal{H}_2$-norm for the fractional transfer function of the first kind associated with a fractional linear system. In Section 4, some examples are presented to show the performance of the proposed method. Concluding remarks are drawn in the last section.

2. Preliminaries

The $\mathcal{H}_2$-norm of a rational transfer function matrix appears among other systems norms [2,7]. It is used in several contexts and domains [8,10,14,15], some of them use it to measure the intensity of the response to standard excitations. An added benefit is that different systems can be compared using the $\mathcal{H}_2$-norm. The definition of the $\mathcal{H}_2$-norm is presented in the following.

Let $\{a,\, b,\, c\}$ be a state-space representation of a linear system and let $G(s) = c(s-a)^{-1}b$ for some $s \in \mathbb{C}$ be its transfer function. The $\mathcal{H}_2$-norm of such system is defined as [15]

$$\|G\|^2_{\mathcal{H}_2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(j\omega)G^*(j\omega)\, d\omega,$$

where $s = j\omega$, with $j = e^{j\frac{\pi}{2}}$ and $j^2 = -1$.

Note that, for $G(-j\omega) = G^*(j\omega)$, we get [15]

$$\int_{-\infty}^{0} G(j\omega)G^*(j\omega)d\omega = \int_{0}^{+\infty} G(j\omega)G^*(j\omega)d\omega,$$

then

$$\|G\|^2_{\mathcal{H}_2} = \frac{1}{\pi}\int_{0}^{+\infty} G(j\omega)G^*(j\omega)d\omega.$$

Let us recall that the main concept of this paper is to provide a numerical expression for computing the $\mathcal{H}_2$-norm of a rational transfer function matrix of the first kind using the state-space representation and a transformation of the parahermitian matrix which will be determined later under some conditions.

For this purpose, we will use the Schur complement [6], where the function G can be written through a block matrix as

$$S_G(s) = \left[\begin{array}{c|c} s-a & b \\ \hline -c & 0 \end{array}\right],$$

with respect to its right block entry.

The transfer function G is proper, thus, its conjugate transpose

$$G^*(s) = b(-s-a)^{-1}c,$$

is the Schur complement of the corresponding system matrix S_{G^*}

$$S_{G^*}(s) = \left[\begin{array}{c|c} -s-a & c \\ \hline -b & 0 \end{array}\right].$$

Using simple algebraic manipulation on the matrices S_G and S_{G^*} it follows

$$S_\phi(s) = \left[\begin{array}{cc|c} 0 & -s-a & c \\ s-a & -b^2 & 0 \\ \hline -c & 0 & 0 \end{array}\right].$$

Note that the matrix S_ϕ is also known as the parahermitian matrix where its corresponding parahermitian transfer function is

$$\phi(s) = c(s-a)^{-1}b^2(-s-a)^{-1}c. \tag{2.1}$$

3. The $\mathcal{H}_2$-Norm of Fractional-Order Systems

A generalized fractional-order state-space model consisting of the parameters $\{a,\, b,\, c,\, \alpha\}$ can be represented as

$$\begin{cases} \mathbf{D}^\alpha x(t) = ax(t) + bu(t), \\ \quad y(t) = cx(t), \end{cases} \tag{3.1}$$

where $x, u, y \in \mathbb{R}^*$ are respectively the state, the input and the output and $a \in \mathbb{R}^*_-$ and $b,\, c \in \mathbb{R}^*$ with a null initial condition and $\mathbf{D}^\alpha$, where $n-1 \leq \alpha < n$, for some $n \in \mathbb{N}^*$, is the α fractional-order derivation of the function x in the sense of the Caputo derivative, given by [4]

$$\mathbf{D}^\alpha x(t) = \frac{1}{\Gamma[n-\alpha]} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n}, \quad n \in \mathbb{N}^*.$$

For almost $s \in \mathbb{C}$ we assume that the pencil $(1, a)$ is regular which is equivalent to $(s^\alpha - a) \neq 0$.

From the system (3.1), the transfer function G can be extracted. Indeed, by the means of the Laplace transform [12], the direct input-output relation of the system (3.1) is written as

$$Y(s) = c(s^\alpha - a)^{-1} b\, U(s), \quad \text{for all } s \in \mathbb{C}.$$

However,

$$Y(s) = G(s)\, U(s), \quad \text{for all } s \in \mathbb{C},$$

then, in time domain, the fractional transfer function associated with the system (3.1) is given by

$$G(s) = c(s^\alpha - a)^{-1} b, \quad \text{for all } s \in \mathbb{C},$$

which has the generalized state-space realization consisting on $\{a, b, c, \alpha\}$. Its $\mathcal{H}_2$-norm, in the frequency domain, is then obtained from the following definition [15]

$$\|G\|^2_{\mathcal{H}_2} = \frac{1}{\pi}\int_0^{+\infty} G(j\omega)G^*(j\omega)d\omega,$$

with $s = j\omega$, $s^\alpha = (j\omega)^\alpha$ and $\omega^\alpha = \tilde{\omega}$.

Then, for the fractional system (3.1) and in the frequency domain, the so-called parahemitian transfer function (formula (2.1)) becomes

$$\phi(\tilde{\omega}) = c(j^\alpha\tilde{\omega} - a)^{-1}b^2(\bar{j}^\alpha\tilde{\omega} - a)^{-1}c,$$

which is also the Schur complement of the so-called system matrix S_ϕ

$$S_\phi(\tilde{\omega}) = \left[\begin{array}{cc|c} 0 & \bar{j}^\alpha\tilde{\omega} - a & c \\ j^\alpha\tilde{\omega} - a & -b^2 & 0 \\ \hline -c & 0 & 0 \end{array}\right].$$

It is well known that the parahermitian matrix can be transformed under row and column matrices transformations that leave the system state-space realization $\{a,\ b,\ c,\ \alpha\}$ invariant [6,16]. Therefore, the matrix $S_\phi(\tilde{\omega})$ can be transformed into the following matrix

$$S_{\tilde{\phi}}(\tilde{\omega}) = \left[\begin{array}{cc|c} 0 & \bar{j}^\alpha\tilde{\omega} - a & c \\ j^\alpha\tilde{\omega} - a & f & p\,c \\ \hline -c & -c\,p & 0 \end{array}\right],$$

where

$$f = (j^\alpha\tilde{\omega} - a)p + p(\bar{j}^\alpha\tilde{\omega} - a) - b^2,$$

with p satisfying a condition given thereafter.

The existence of the value of

$$p = \frac{b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)},$$

solution of the equation $f = 0$ allows the matrix $S_{\tilde{\phi}}(\tilde{\omega})$ to become

$$S_{\tilde{\phi}}(\tilde{\omega}) = \left[\begin{array}{cc|c} 0 & \bar{j}^\alpha\tilde{\omega} - a & c \\ j^\alpha\tilde{\omega} - a & 0 & \dfrac{c\,b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)} \\ \hline -c & -\dfrac{c\,b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)} & 0 \end{array}\right].$$

In this case, the Schur complement of $S_{\tilde{\phi}}$ can be written as

$$\tilde{\phi}(\tilde{\omega}) = -\frac{c^2\,b^2}{2\,j\,a\,\sin\left(\frac{\alpha\pi}{2}\right)}\left[\frac{1}{\tilde{\omega} - a\,e^{-\frac{\alpha\pi}{2}j}} - \frac{1}{\tilde{\omega} - a\,e^{\frac{\alpha\pi}{2}j}}\right].$$

Thus, the $\mathcal{H}_2$-norm of the transfer function G is

$$\|G\|^2_{\mathcal{H}_2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(j\omega)G^*(j\omega)d\omega$$

$$
\begin{aligned}
&= \frac{1}{\alpha\pi} \int_0^{+\infty} \tilde{\omega}^{\frac{1}{\alpha}-1}\, \tilde{\phi}(\tilde{\omega})\, d\tilde{\omega} \\
&= -\frac{c^2\, b^2}{2\,\pi\, j\, a\, \alpha\, \sin\left(\frac{\alpha\pi}{2}\right)} \left[\int_0^{+\infty} \left(\frac{1}{\tilde{\omega} - a\, e^{-\frac{\alpha\pi}{2} j}} - \frac{1}{\tilde{\omega} - a\, e^{\frac{\alpha\pi}{2} j}}\right) \tilde{\omega}^{\frac{1}{\alpha}-1}\, d\tilde{\omega}\right].
\end{aligned}
\tag{3.2}
$$

The expression (3.2) can be readily computed for $\alpha = 1$. Nevertheless, for $\frac{1}{2} < \alpha < 2$ with $\alpha \neq 1$, we will use the Mellin integral transform [5] where the obtained result is presented in the following theorem.

Theorem 3.1. *Assuming that $\frac{1}{2} < \alpha < 2$ and $\alpha \neq 1$. Then the $\mathcal{H}_2$-norm of the fractional-order transfer function G, with generalized state-space realization $\{a, b, c, \alpha\}$, where $a \in \mathbb{R}^*_-$, $b, c \in \mathbb{R}^*$ and $(s^\alpha - a) \neq 0$ for almost $s \in \mathbb{C}$ is defined as*

$$
\|G\|^2_{\mathcal{H}_2} = -\frac{b^2\, c^2\, (-a)^{\frac{1}{\alpha}-2}\, \cot\left(\alpha\frac{\pi}{2}\right)}{\alpha\, \sin\left(\frac{\pi}{\alpha}\right)}.
$$

The above theorem which is the main result of this paper present numerical formula for computing the $\mathcal{H}_2$-norm for a fractional-order transfer function of the first kind represented by the generalized state-space realization.

Remark 3.1. For $\alpha = 1$ we get

$$
\|G\|^2_{\mathcal{H}_2} = -\frac{c^2 b^2}{2a}.
$$

Moreover, the same technique can be applied for any transfer function represented by a regular differential linear system written in a state-space as

$$
\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}
\tag{3.3}
$$

where $x \in \mathbb{R}^q$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector and $A \in \mathbb{R}^{q\times q}$, $B \in \mathbb{R}^{q\times m}$, and $C \in \mathbb{R}^{p\times q}$ with a null initial condition.

In this case, the $\mathcal{H}_2$-norm of the transfer function $G(s) = C\,(sI - A)^{-1}\, B$, with generalized state-space realization $\{A, B, C\}$ and $\det(sI - A) \neq 0$ for almost $s \in \mathbb{C}$ is defined as

$$
\|G\|^2_{\mathcal{H}_2} = \operatorname{tr}\left(CPC^T\right).
$$

The matrix P is a solution of the Lyapunov equation

$$
AP + PA^T + BB^T = 0,
$$

where A^T, B^T and C^T are respectively the transpose of the matrices A, B and C.

4. Numerical Examples

The algorithm has been tested for different examples, and compared to the existed methods in the state-of-art. The presented examples are taken from the references [10, 14] to validate our method. All examples have been performed using a Matlab code.

Example 4.1. Consider the system (3.3)

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 4 & -3 \\ 1 & -3 & -1 & -3 \\ 0 & 4 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The corresponding transfer function is

$$G(s) = \begin{bmatrix} \dfrac{-s^3 - 3s^2 - 13s + 5}{s^4 + 4s^3 + 36s^2 + 92s + 43} & \dfrac{-6s^2 - 22s - 16}{s^4 + 4s^3 + 36s^2 + 92s + 43} \\[2ex] \dfrac{-s^3 - 2s^2 - 31s - 48}{s^4 + 4s^3 + 36s^2 + 92s + 43} & \dfrac{6s + 16}{s^4 + 4s^3 + 36s^2 + 92s + 43} \end{bmatrix}.$$

Thus, using Remark 3.1, the $\mathcal{H}_2$-norm of the fractional transfer function G is

$$\|G\|_{\mathcal{H}_2} = 1.1751,$$

which is the same result when using the method presented in [14].

Example 4.2. Consider the transfer function $G(s) = \frac{k}{s^\alpha + \lambda}$ associated with the following system

$$\begin{cases} \mathbf{D}^\alpha\, x(t) = -\,\lambda\, x(t) + k\, u(t), \\ \quad\; y(t) = x(t), \end{cases}$$

with $\frac{1}{2} < \alpha < 2$, $\lambda \in \mathbb{R}_+^*$ and $k \in \mathbb{R}^*$.

The transfer function G satisfies the conditions of Theorem 3.1. Thus, the $\mathcal{H}_2$-norm of the transfer function G is given by

$$\|G\|_{\mathcal{H}_2}^2 = \begin{cases} -\frac{k^2 \lambda^{\frac{1}{\alpha}-2} \cot\left(\alpha \frac{\pi}{2}\right)}{\alpha \sin\left(\frac{\pi}{\alpha}\right)}, & \text{if } \frac{1}{2} < \alpha < 2 \text{ and } \alpha \neq 1, \\ \frac{k^2}{2\lambda}, & \text{if } \alpha = 1. \end{cases}$$

For $\alpha \in \left]\frac{1}{2}, 2\right[$, the obtained results are similar to the ones in [10]. For simplicity, if we take $k = 1$ and $\lambda = 2$, the comparison between both methods is plotted versus α in Figure 1.

5. Conclusion

In this paper, an efficient algorithm is proposed to compute the $\mathcal{H}_2$-norm for a fractional transfer function of the first kind associated with a fractional differential linear system. The approach consists of using the state-space realization and the parahermitian matrix and is based on the use of transformation matrices satisfying some conditions mentioned above. The extension of the proposed method for other types of systems, which are some of the most significant applications, will be discussed in a separate paper.

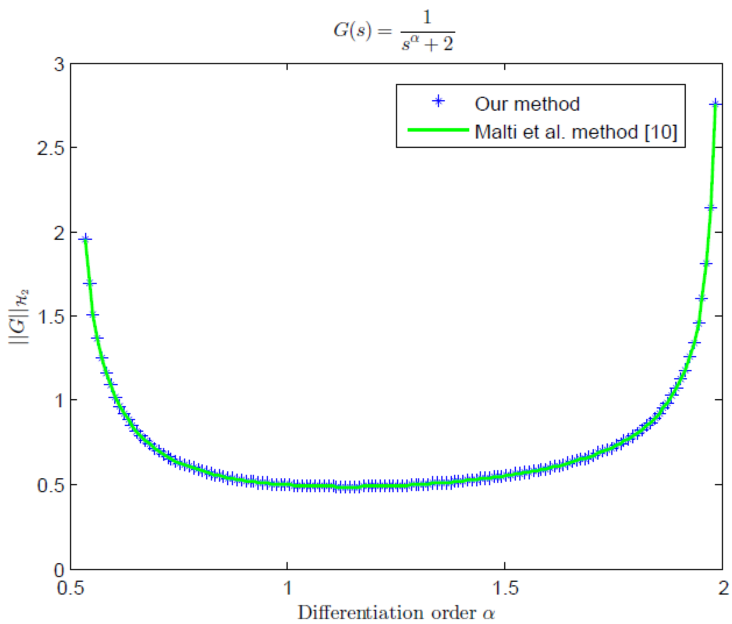

FIGURE 1. Comparison of the values of the $\mathcal{H}_2$-norm between our method and the method presented in [10] for $G(s) = \frac{1}{s^\alpha+2}$ and $\frac{1}{2} < \alpha < 2$.

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