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A LYAPUNOV TYPE INEQUALITY FOR A CLASS OF FRACTIONAL BOUNDARY VALUE PROBLEMS WITH RIEMANN-LIOUVILLE DERIVATIVE

ANABELA S. SILVA[1]

Abstract. In this paper, a Lyapunov-type inequality is obtained for a class of fractional boundary value problems involving Riemann-Liouville fractional derivative of orders $\alpha \in (1, 2)$ and $\beta \in (0, \alpha - 1)$. The study is based on the construction of a Green's function and the obtaining of its corresponding maximum value.

1. Introduction

The fractional calculus is a field of mathematics that deals with generalizing the concept of differentiation and integration to non-integer orders. This definition departs from the traditional concept of derivative and integral in a differential and integral calculus. The concept dates back to 1695, in a famous correspondence between L'Hopital and Leibniz. However, it is only in the last decades that this area of mathematics has gained special prominence, mainly due to its proven applications in various sciences and engineering, such as mechanics and biology. In fact, fractional order operators have the characteristics of being non-local and with memory capacity associated to its kernel, which allows to create models more practical and realistic than those using integer derivatives.

The Lyapunov's inequality [8], named after its author, gives a necessary condition for the existence of non-trivial solutions for a boundary value problem with an ordinary second order differential equation. The original result states that if there is a nontrivial

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solution of the boundary value problem

$$\begin{cases} x''(t) + q(t)x(t) = 0, & t \in [a,b], \\ x(a) = x(b) = 0, \end{cases}$$

and where $q : [a,b] \to \mathbb{R}$ is continuous, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}.$$

After that, several proofs and generalizations appeared in the literature (cf. e.g., [5, 7, 10, 11]). In recent years, some authors extended have obtained Lyapunov-type inequalities for boundary value problems involving the fractional derivatives. As a survey of results on Lyapunov-type inequalities for fractional differential equations, we recommend the work of Ntouyas et al. [9].

Inspired in the works of Dhar and Neugebauer [1] and Ferreira [2–4], in this paper we consider the fractional boundary value problem with (left) Riemann-Liouville fractional derivative

$$\begin{cases} (\mathcal{D}^{\alpha}_{a+}x)(t) + (\mathcal{D}^{\beta}_{a+}qx)(t) = 0, & t \in [a,b], \\ x(a) = x(b) = 0, \end{cases} \tag{1.1}$$

with $0 \le a < b$, $\alpha \in (1,2)$, $\beta \in (0, \alpha-1)$ and where q is a real and continuous function. To the best of author's knowledge, the theory presented in this paper has not been studied yet.

This paper is organized as follows: in the Section 2, some necessary definitions and results are presented; in Section 3 the problem is rewritten in the form of an integral equation, making use of Green's function. The maximum of this function is obtained here. It is then presented the main result of the paper, the Lyapunov's inequality for the problem under study. Finally, an example is presented applying the theory previously presented.

2. Preliminaries

In this section, we recall some important definitions and results.

Definition 2.1. Let $\alpha \in \mathbb{R}^+$. The (left) Riemann-Liouville fractional integral of order α of the function x on $[a,b]$ is defined by

$$I^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)ds,$$

where $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}\mathrm{d}t$ is the Euler Gamma function, provided the right-hand side is pointwise defined in $(a, +\infty)$.

Definition 2.2. The (left) Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function x on $[a,b]$ is defined by

$$\mathcal{D}^{\alpha}_{a+}x(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{a+}x)(t) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}x(s)ds,$$

where n is the smallest integer greater or equal than α (provided the right-hand side is pointwise defined in $(a,+\infty)$).

The following results can be found in [6] and [12] and are essential thorough this paper.

Lemma 2.1. *For $\alpha,\beta>0$, the Riemann-Liouville fractional integral satisfies the semigroup property*

$$I^{\alpha}_{a+}(I^{\beta}_{a+}x)(t)=I^{\beta}_{a+}(I^{\alpha}_{a+}x)(t)=(I^{\alpha+\beta}_{a+}x)(t), \tag{2.1}$$

at almost every point $t\in(a,b)$ for $x\in L^p(a,b)$, $1\le p\le+\infty$. If $\alpha+\beta\ge 1$, the relation holds for any point of $[a,b]$.

Lemma 2.2. *Let $x\in C([a,b])\cap L^1([a,b])$ and $\alpha>0$. It follows that*

$$\mathcal{D}^{\alpha}_{a+}I^{\alpha}_{a+}x(t)=x(t)$$

and

$$I^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}x(t)=x(t)+c_1(t-a)^{\alpha-1}+c_2(t-a)^{\alpha-2}+\cdots+c_n(t-a)^{\alpha-n},$$

for some $c_i\in\mathbb{R}$, $i=1,2,\dots,n$, where n is the smallest integer greater than or equal to α.

Among the several properties of the Riemann-Liouville integral, we highlight the following.

Property 1. For $\alpha,\beta>0$,

$$I^{\alpha}_{a+}[(s-a)^{\beta-1}](t)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}.$$

3. Main Results

The aim of this section is to obtain a necessary condition for the existence of a nontrivial solution of the fractional boundary value problem under study. For that purpose, we start by rewriting the considered problem in terms of an integral equation.

Lemma 3.1. *Let $1<\alpha<2$, $0<\beta<\alpha-1$ and $q\in C([a,b])$. Then the solution x of* (1.1) *can be represented in the integral form as*

$$x(t)=\int_a^b G(t,s)q(s)x(s)ds,$$

where the Green's function $G(t,s)$ is defined by

$$G(t,s)=\frac{1}{\Gamma(\alpha-\beta)}\begin{cases}\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}}, & a\le t\le s\le b,\\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}}-(t-s)^{\alpha-\beta-1}, & a\le s\le t\le b.\end{cases} \tag{3.1}$$

Proof. Let us first observe that according to Proposition 2.1, we have that

$$(I_{a+}^{\alpha}\mathcal{D}_{a+}^{\beta}x)(t) = I_{a+}^{\alpha-\beta}(I_{a+}^{\beta}\mathcal{D}_{a+}^{\beta}qx)(t).$$

Thus, following Lemma 2.2 and applying Property 1 we get

$$(I_{a+}^{\alpha}\mathcal{D}_{a+}^{\beta}qx)(t) = (I_{a+}^{\alpha-\beta}qx)(t) + c_1 I_{a+}^{\alpha-\beta}(t-a)^{\beta-1} = (I_{a+}^{\alpha-\beta}qx)(t) + c_1\frac{\Gamma(\beta)}{\Gamma(\alpha)}(t-a)^{\alpha-1},$$

$c_1 \in \mathbb{R}$. In this sense, applying I_{a+}^{α} to both members of equation

$$(\mathcal{D}_{a+}^{\alpha}x)(t) + (\mathcal{D}_{a+}^{\beta}qx)(t) = 0,$$

we get

$$x(t) + d_1(t-a)^{\alpha-1} + d_2(t-a)^{\alpha-2} + (I_{a+}^{\alpha-\beta}qx)(t) + c(t-a)^{\alpha-1} = 0,$$

$d_1, d_2, c \in \mathbb{R}$. It follows that, for some constants $k_1, k_2 \in \mathbb{R}$,

$$x(t) = k_1(t-a)^{\alpha-1} + k_2(t-a)^{\alpha-2} - (I_{a+}^{\alpha-\beta}qx)(t).$$

From $x(a) = 0$, it follow that $k_2 = 0$. From $x(b) = 0$, we get that

$$k_1 = \frac{1}{(b-a)^{\alpha-1}\Gamma(\alpha-\beta)}\int_a^b (b-s)^{\alpha-\beta-1}q(s)x(s)ds.$$

Therefore,

$$\begin{aligned} x(t) =& \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}\Gamma(\alpha-\beta)}\int_a^b (b-s)^{\alpha-\beta-1}q(s)x(s)ds \\ &- \frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-\beta-1}q(s)x(s)ds \\ =& \int_a^b G(t,s)q(s)x(s)ds, \end{aligned}$$

and the proof is complete. □

The following theorem concerns on the maximum of the Green's function (3.1) and constitutes an essential tool to obtain the main theorem of this paper.

Theorem 3.1. *For any* $(t,s) \in [a,b]\times[a,b]$*, the Green's function* (3.1) *satisfies the following property*

$$\max_{t,s\in[a,b]}|G(t,s)| = \frac{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1},$$

with $\alpha \in (1,2)$ *and* $\beta \in (0, \alpha-1)$.

Proof. Let us consider

$$\begin{aligned} g_1(t,s) &= \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}}, \\ g_2(t,s) &= \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-\beta-1}, \end{aligned}$$

and consequently, the Green's function can be rewritten as

$$G(t,s) = \frac{1}{\Gamma(\alpha-\beta)} \begin{cases} g_1(t,s), & a \le t \le s \le b, \\ g_2(t,s), & a \le s \le t \le b. \end{cases}$$

For any $(t,s) \in [a,b] \times [a,b]$ with $t \le s$, it is clear that $G(t,s) = \frac{g_1(t,s)}{\Gamma(\alpha-\beta)} \ge 0$. Moreover, fixing $t \in [a,b]$ and taking the derivative with respect to s

$$\frac{\partial}{\partial s} g_1(t,s) = -\frac{(\alpha-\beta-1)(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-1}} \le 0.$$

Thus, for $a \le t \le s \le b$, $G(t,s)$ is decreasing (on variable s), which implies that

$$G(t,s) = \frac{g_1(t,s)}{\Gamma(\alpha-\beta)} \le \frac{g_1(t,t)}{\Gamma(\alpha-\beta)}.$$

Let

$$h(t) = \frac{g_1(t,t)}{\Gamma(\alpha-\beta)} = \frac{(t-a)^{\alpha-1}(b-t)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(b-a)^{\alpha-1}}.$$

Since $h(a) = h(b) = 0$, we can apply the Rolle's Theorem to conclude that there exists $c \in [a,b]$ such that $h(c) = \max_{t\in[a,b]} h(t)$. We have that

$$h'(t) = \frac{(t-a)^{\alpha-2}(b-t)^{\alpha-\beta-2}[(\alpha-1)(b-t) - (\alpha-\beta-1)(t-a)]}{\Gamma(\alpha-\beta)(b-a)^{\alpha-1}},$$

and hence, $h'(c) = 0$ for

$$c = \frac{(\alpha-1)b + (\alpha-\beta-1)a}{2\alpha-\beta-2}.$$

Note that

$$c > \frac{(\alpha-1)a + (\alpha-\beta-1)a}{2\alpha-\beta-2} = a$$

and

$$c < \frac{(\alpha-1)b + (\alpha-\beta-1)b}{2\alpha-\beta-2} = b,$$

which shows that c is well defined. Consequently, we obtain that

$$G(c,c) = h(c) = \frac{1}{\Gamma(\alpha-\beta)} \left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1} (b-a)^{\alpha-\beta-1}. \tag{3.2}$$

Consider now the case $a \le s \le t \le b$, $(t,s) \in [a,b] \times [a,b]$. In that case, we have that

$$G(t,s) = \frac{g_2(t,s)}{\Gamma(\alpha-\beta)}.$$

Fixing $t \in [a,b]$, and taking the derivative with respect to s, we have

$$\frac{\partial}{\partial s} g_2(t,s) = -\frac{(\alpha-\beta-1)(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-1}} + (\alpha-\beta-1)(t-s)^{\alpha-\beta-2}$$
$$= (\alpha-\beta-1)(b-s)^{\alpha-\beta-2}\left[\left(\frac{t-s}{b-s}\right)^{\alpha-\beta-2} - \left(\frac{t-a}{b-a}\right)^{\alpha-1}\right].$$

Let

$$v(s) = \left(\frac{t-s}{b-s}\right)^{\alpha-\beta-2} - \left(\frac{t-a}{b-a}\right)^{\alpha-1}.$$

Then, it follows that

$$v'(s) = (\alpha-\beta-2)\frac{t-b}{(b-s)^2}\left(\frac{t-s}{b-s}\right)^{\alpha-\beta-3},$$

which is positive since $\alpha-\beta-2<0$ and $\frac{t-b}{(b-s)^2} \leq 0$ (with $t \leq b$). Thus, we have that v is increasing, which means that

$$v(s) \geq v(a) = \left(\frac{t-a}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-a}{b-a}\right)^{\alpha-1} \geq 0,$$

since $0 \leq \left(\frac{t-a}{b-a}\right) \leq 1$ and $\alpha-\beta-2 < \alpha-1$. Thus, we can obtain that

$$\frac{\partial}{\partial s} g_2(t,s) \geq (\alpha-\beta-1)(b-s)^{\alpha-\beta-2}\left[\left(\frac{t-a}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-a}{b-a}\right)^{\alpha-1}\right] \geq 0,$$

and we conclude that $g_2(t,s)$ is increasing on variable s and thus

$$\frac{g_2(t,a)}{\Gamma(\alpha-\beta)} \leq \frac{g_2(t,s)}{\Gamma(\alpha-\beta)} \leq \frac{g_2(t,t)}{\Gamma(\alpha-\beta)},$$

for all $s \in [a,t]$. In this way, we obtain that, for $a \leq s \leq t \leq b$,

$$\max_{t\in[s,b]}\left|\frac{g_2(t,s)}{\Gamma(\alpha-\beta)}\right| = \max_{t\in[a,b]}\left\{\left|\frac{g_2(t,a)}{\Gamma(\alpha-\beta)}\right|, \left|\frac{g_2(t,t)}{\Gamma(\alpha-\beta)}\right|\right\}$$

Since $\frac{g_2(t,t)}{\Gamma(\alpha-\beta)} = \frac{g_1(t,t)}{\Gamma(\alpha-\beta)} \geq 0$, for any $t \in [a,b]$, we only need to consider $\frac{g_2(t,a)}{\Gamma(\alpha-\beta)}$. Observing that

$$\frac{g_2(t,a)}{\Gamma(\alpha-\beta)} = \frac{(t-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left[\left(\frac{t-a}{b-a}\right)^{\beta} - 1\right] \leq 0,$$

we can write that

$$\left|\frac{g_2(t,a)}{\Gamma(\alpha-\beta)}\right| = \frac{(t-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left[1 - \left(\frac{t-a}{b-a}\right)^{\beta}\right].$$

Let

$$w(t) = \left|\frac{g_2(t,a)}{\Gamma(\alpha-\beta)}\right| = \frac{(t-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left[1 - \left(\frac{t-a}{b-a}\right)^{\beta}\right].$$

We can observe that $w(a)=0$ and $w(b)=0$. Since $w(t)$ is continuous and nonnegative, using again Rolle's Theorem, the maximum is achieved for a $c^* \in [a,b]$ such that $w'(c^*)=0$. In this way, having in mind that $t \neq a$, we have

$$w'(t) = \frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-\beta)} \left(\frac{\alpha-\beta-1}{(t-a)^{\beta}} - \frac{\alpha-1}{(b-a)^{\beta}} \right).$$

Consequently, $w'(c^*)=0$ for

$$c^* = \sqrt[\beta]{\frac{\alpha-\beta-1}{\alpha-1}}(b-a)+a,$$

and as before, we can prove easily that $a<c^*<b$. We conclude

$$\begin{aligned} \frac{|g_2(c^*,a)|}{\Gamma(\alpha-\beta)} &= \frac{1}{\Gamma(\alpha-\beta)} \left(\left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-\beta-1}{\beta}} - \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}} \right) (b-a)^{\alpha-\beta-1} \\ &= \frac{1}{\Gamma(\alpha-\beta)} \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}} \left(\frac{\beta}{\alpha-\beta-1}\right)(b-a)^{\alpha-\beta-1}. \end{aligned} \tag{3.3}$$

Joining the results (3.2) and (3.3), we obtain that, for $\alpha \in (1,2)$, $\beta \in (0,\alpha-1)$,

$$\max_{t,s\in[a,b]} |G(t,s)| = \frac{\max\left\{ \left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}, \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}} \left(\frac{\beta}{\alpha-\beta-1}\right) \right\}}{\Gamma(\alpha-\beta)} (b-a)^{\alpha-\beta-1}. \tag{3.4}$$

To complete the proof, let us prove that

$$\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1} > \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}} \left(\frac{\beta}{\alpha-\beta-1}\right).$$

To that purpose, we first observe that $\alpha-1>\beta$ and $2\alpha-\beta-2 = 2(\alpha-1)-\beta < \alpha-1$. Moreover, since $\frac{\alpha-\beta-1}{\alpha-1} < 1$ and $\alpha-\beta-1 < \frac{\alpha-\beta-1}{\beta}$ (having in mind that $0<\beta<1$), we conclude that

$$\left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\alpha-\beta-1} > \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-\beta-1}{\beta}}.$$

Applying those results, we therefore obtain

$$\begin{aligned} \left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1} &> \left(\frac{\alpha-1}{\alpha-1}\right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\alpha-\beta-1} \\ &= \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\alpha-\beta-1} \\ &> \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-\beta-1}{\beta}} \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}} \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{-1} \\
&= \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}} \left(\frac{\alpha-1}{\alpha-\beta-1}\right) \\
&> \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}} \left(\frac{\beta}{\alpha-\beta-1}\right).
\end{aligned}
$$

Thus, recalling (3.4), we conclude that

$$
\max_{t,s\in[a,b]} |G(t,s)| = \frac{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1},
$$

which completes the proof. □

3.1. **A Lyapunov-type inequality for the problem under study.** The following theorem states a Lyapunov-type inequality for the fractional boundary value problem (1.1).

Theorem 3.2. *Let x be a nontrivial continuous solution of the fractional boundary value problem* (1.1). *Then*

$$
\int_a^b |q(s)|ds > \frac{\Gamma(\alpha-\beta)(b-a)^{1+\beta-\alpha}}{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}.
$$

Proof. Suppose that x is a solution of the fractional boundary value problem (1.1). Then, according to Lemma 3.1,

$$
x(t) = \int_a^b G(t,s)q(s)x(s)ds,
$$

with G as defined in (3.1). Assume that x is a continuous and a nontrivial function on $[a,b]$. Thus, there exists an $m > 0$ such that $\max_{t\in[a,b]} |x(t)| = m$. We have that

$$
\begin{aligned}
m &< m \max_{t\in[a,b]} \int_a^b |G(t,s)||q(s)|ds \\
&\le m\frac{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1}\int_a^b |q(s)|ds.
\end{aligned}
$$

Thus, we can conclude that

$$
1 < \frac{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1}\int_a^b |q(s)|ds,
$$

which gives the desired result. □

The following corollary is a natural consequence of Theorem 3.2.

Corollary 3.1. *If x is a continuous nontrivial solution of the fractional boundary value problem*

$$\begin{cases} (\mathcal{D}_{a+}^{\alpha}x)(t) + \lambda(\mathcal{D}_{a+}^{\beta}x)(t) = 0, & t \in [a,b], \\ x(a) = x(b) = 0, \end{cases}$$

with $\lambda \in \mathbb{R}\backslash\{0\}$, then

$$|\lambda| > \frac{\Gamma(\alpha-\beta)(b-a)^{\beta-\alpha}}{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}.$$

4. Example

Consider the fractional boundary value problem

$$\begin{cases} (\mathcal{D}_{1+}^{1.5}x)(t) + (\mathcal{D}_{1+}^{0.3}ktx)(t) = 0, & t \in [1,2], \\ x(1) = x(2) = 0, \end{cases}$$

where k is a real number. In that case, for $\alpha = 1.5$ and $\beta = 0.3$, we have that

$$\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1} = \left(\frac{5}{7}\right)^{0.5}\left(\frac{2}{7}\right)^{0.2}.$$

Since

$$|k|\int_1^2 |t|\,dt = \frac{3}{4}|k|,$$

we have that if there is a nontrivial solution, then

$$|k| > \frac{4\Gamma(1.2)}{3\left(\frac{5}{7}\right)^{0.5}\left(\frac{2}{7}\right)^{0.2}} > 1.86.$$

5. Conclusion

In this paper, we obtain a Lyapunov type inequality for a boundary value problem with a fractional Riemann-Liouville derivatives. This result complements the existing results in the area. The novelty in this work is the fact that the problem considered uses different orders of the Riemann-Liouville derivative, which leads to different solutions and, consequently, different results. The inequality obtained is a necessary condition for the existence of a nontrivial continuous solutions of the problem under study.

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[1]CIDMA & Department of Mathematics,
University of Aveiro,
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal
Email address: anabela.silva@ua.pt