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NONLOCAL NEUTRAL FUNCTIONAL SEQUENTIAL DIFFERENTIAL EQUATIONS WITH CONFORMABLE FRACTIONAL DERIVATIVE

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ABSTRACT. In this paper, we investigate the existence, uniqueness, and stability results of second-order neutral evolution differential equations within the framework of sequential conformable derivatives with nonlocal conditions. Utilizing Krasnoselskii's fixed-point theorem, we establish results concerning the existence of at least one solution, while the uniqueness of the solution is derived using Banach's fixed-point theorem. The final section is devoted to an example that illustrates the applicability of our findings.

1. INTRODUCTION

Differential equations with nonlocal conditions are essential in various scientific fields, including engineering and physics. Numerous researchers have explored the theory of these equations concerning different types of derivatives. Hernández [6] studied the second-order Cauchy problem with nonlocal conditions for the classical derivative. Recently, fractional differential equations have gained popularity in modeling various problems in biology, chemistry, and other applied areas [8, 9, 11–17]. In [18], Shur et al. treated a fractional Cauchy problem of order $\alpha \in (1, 2)$ with non-local conditions using the Caputo fractional derivative. Their study primarily focused on the results of the existence and uniqueness of mild solutions. For physical interpretations of the non-local conditions, we refer to references [2, 3, 10].

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The authors in [4] investigated the existence and regularity of solutions for some partial differential equations with nonlocal conditions in the α-norm for the following problem:

$$\begin{cases} \frac{d}{dt}\left(y(t)-\mathcal{F}\left(t,y(h_1(t))\right)\right)=-B\left(y(t)-\mathcal{F}\left(t,y(h_1(t))\right)\right)+\mathcal{G}\left(t,y(h_2(t))\right), & t\in[0,a],\\ y(0)=y_0+\varphi(y), \end{cases}$$

where $B : D(B) \subset Y \to Y$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space Y. The functions $\mathcal{F}$, $\mathcal{G}$, φ, h_1 and h_2 are continuous functions.

In [5], the authors considered the following fractional conformable problem:

$$\begin{cases} \frac{d^\beta}{dt^\beta}\left(y(t)-\mathcal{F}\left(t,y(h_1(t))\right)\right)=-B\left(y(t)-\mathcal{F}\left(t,y(h_1(t))\right)\right)+\mathcal{G}\left(t,y(h_2(t))\right), & t\in[0,a],\\ y(0)=y_0+\varphi(y), \end{cases}$$

where $\frac{d^\beta}{dt^\beta}$ is the conformable fractional derivative of $\beta \in (0,1)$. B is a sectorial operator which generates a strongly analytic semigroup $(T(t))$ on a Banach space Y. The functions $\mathcal{F}$, $\mathcal{G}$, φ, h_1 and h_2 are continuous functions.

Motivated by the previously mentioned publications, we are interested in a related problem: the second-order sequential Cauchy problem with non-local conditions, using the conformable derivative. More precisely, we are interested in second-order sequential conformable differential equations characterized by the following non-local conditions:

(1.1)

$$\begin{cases} \frac{d^\beta}{dt^\beta}\left[\frac{d^\beta}{dt^\beta}\left(y(t)-\mathcal{F}\left(t,y(h_1(t))\right)\right)\right]=B\left(y(t)-\mathcal{F}\left(t,y(h_1(t))\right)\right)+\mathcal{G}\left(t,y(h_2(t))\right), & t\in[0,a],\\ y(0)=y_0+\varphi(y),\\ \frac{d^\beta}{dt^\beta}\left(y(0)-\mathcal{F}\left(0,y(h_1(0))\right)\right)=y_1+\psi(y), \end{cases}$$

where $\frac{d^\beta}{dt^\beta}$ is conformable fractional derivative of order β. The operator B is the infinitesimal generator of a family of cosines $\{C(t),S(t)\}_{t\in\mathbb{R}}$ on a Banach space $(Y,\|\cdot\|)$. y_0 and y_1 are two elements in the Banach Y. The expression $\mathcal{C}=\mathcal{C}\left([0,a],Y\right)$ denotes Banach space of continuous functions y with the norm $|y|=\sup\{\|y(t)\|, t\in[0,a]\}$. The functions $\mathcal{F}:[0,a]\times\mathcal{C}\to Y$, $\mathcal{G}:[0,a]\times Y\to Y$, $\varphi:\mathcal{C}\to Y$, $\psi:\mathcal{C}\longrightarrow Y$, h_1 and h_2 are continuous functions.

This paper is summarized as follows. In Section 2, we review some tools related to the conformable derivative as well as some needed results. Section 3 will be devoted to the statements and the proof of the main results. In Section 4, as application, we investigate a second-order sequential conformal partial differential equation with a non-local condition.

2. Preliminaries

We begin by recalling some fundamental concepts of conformable calculus [7].

Definition 2.1. The conformable derivative of order β for a function $\mathcal{H}$ is defined as follows:

$$\frac{d^{\beta}\mathcal{H}(t)}{dt^{\beta}} = \lim_{\varepsilon\to 0}\frac{\mathcal{H}(t+\varepsilon t^{1-\beta})-\mathcal{H}(t)}{\varepsilon}, \qquad t>0.$$

If this limit exists, we say that $\mathcal{H}$ is (β)-differentiable at t.

If $\mathcal{H}$ is (β)-differentiable and $\lim_{\varepsilon\to 0^{+}}\frac{d^{\beta}\mathcal{H}(t)}{dt^{\beta}}$ exists, we define

$$\frac{d^{\beta}\mathcal{H}(0)}{dt^{\beta}} = \lim_{\varepsilon\to 0^{+}}\frac{d^{\beta}\mathcal{H}(t)}{dt^{\beta}}.$$

The (β)-fractional integral of a function $\mathcal{H}$ is given by

$$I^{\beta}\mathcal{H}(t) = \int_0^t (t-\vartheta)^{\beta-1}\mathcal{H}(\vartheta)d\vartheta.$$

Theorem 2.1. *If $\mathcal{H}\in D(I^{\beta})$ is a continuous function, we obtain*

$$\frac{d^{\beta}(I^{\beta}\mathcal{H}(t))}{dt^{\beta}} = \mathcal{H}(t).$$

The Laplace transform associated with the conformable derivative is given by the following definition.

Definition 2.2 ([1])**.** The conformable fractional Laplace transform of $\mathcal{H}$ of order β is defined by

$$\mathcal{L}_{\beta}\left(\mathcal{H}\left(t\right)\right)\left(\lambda\right) = \int_0^{+\infty} t^{\beta-1}e^{-\lambda\frac{t^{\beta}}{\beta}}\mathcal{H}(t)dt.$$

The following proposition shows the effect of the fractional Laplace transform on the conformal derivative.

Proposition 2.1. *If $\mathcal{H}(t)$ is differentiable, we obtain*

$$\begin{aligned} I^{\beta}\left(\frac{d^{\beta}\mathcal{H}(t)}{dt^{\beta}}\right) =& \mathcal{H}(t)-\mathcal{H}(0),\\ \mathcal{L}_{\beta}\left(\frac{d^{\beta}\mathcal{H}(t)}{dt^{\beta}}\right)(\lambda) =& \lambda\mathcal{L}_{\beta}(\mathcal{H}(t))(\lambda)-\mathcal{H}(0). \end{aligned}$$

So let's recall certain results related to the theory of the cosine family [19].

Definition 2.3. A family $(C(\xi))_{\xi\in\mathbb{R}}$ of bounded linear operators on Y is defined as a strongly continuous family of cosines if and only if:

(a) $C(0)=I$;
(b) $C(\nu+\xi)+C(\nu-\xi)=2C(\nu)C(\xi)$, for all $\xi,\nu\in\mathbb{R}$;
(c) $\xi\to C(\xi)y$ is continuous for each fixed $y\in Y$.

We define also the sine family by

$$S(\xi)y := \int_0^{\xi} C(\vartheta)yd\vartheta.$$

Definition 2.4. B is the infinitesimal generator of a strongly continuous cosine family $((C(\xi))_{\xi\in\mathbb{R}}, (S(\xi))_{\xi\in\mathbb{R}})$ on Y defined by:

$$\begin{aligned} D(B) =& \{y \in Y, \xi \to C(\xi)y \text{ is a twice continuously differentiable function}\},\\ Ay =& \frac{d^2C(0)y}{d\xi^2}. \end{aligned}$$

We end this section with the following results.

Proposition 2.2. *The following assertions are true.*

(a) *There exist constants $\omega \geq 0$ and $M \geq 1$ where*

$$|S(\xi) - S(\nu)| \leq M \left| \int_\nu^\xi \exp(\omega|\vartheta|) d\vartheta \right|, \quad \text{for all } \nu, \xi \in \mathbb{R}.$$

(b) *If $y \in Y$ and $\xi, \nu \in \mathbb{R}$, then $\int_\nu^\xi S(\vartheta) y d\vartheta \in D(B)$ and*

$$B \int_\nu^\xi S(\vartheta) y d\vartheta = C(\xi)y - C(\nu)y.$$

(c) *If $\xi \mapsto C(\xi)y$ is differentiable, hence $S(\xi)y \in D(B)$ and $\frac{dC(\xi)}{d\xi} y = BS(\xi)y$.*

(d) *For λ such that $Re(\lambda) > \omega$, we get*
- *$\lambda^2 \in \rho(B)$, ($\rho(B)$: is the resolvent set of B),*
- *$\lambda(\lambda^2 I - B)^{-1} y = \int_0^{+\infty} e^{-\lambda\xi} C(\xi) y d\xi$, $y \in Y$,*
- *$(\lambda^2 I - B)^{-1} y = \int_0^{+\infty} e^{-\lambda\xi} S(\xi) y d\xi$, $y \in Y$.*

3. Main Results

Before presenting our main results, we introduce the following assumptions.

(H_1) The function $\mathcal{G}(t, \cdot) : Y \to Y$ is continuous, and for all $r > 0$, there exists a function $\mu_r \in L^\infty([0, a], \mathbb{R}^+)$ such that $\sup_{\|y\|\leq r} \|\mathcal{G}(t, y)\| \leq \mu_r(t)$ for all $t \in [0, a]$.

(H_2) The function $\mathcal{G}(\cdot, y) : [0, a] \to Y$ is continuous for all $y \in Y$.

(H_3) There exists a constant $l_1 > 0$ such that $\|\mathcal{F}(t, x) - \mathcal{F}(t, y)\| \leq l_1|x - y|$ for all $x, y \in \mathcal{C}$.

(H_4) There exists a constant $l_2 > 0$ such that $\|\varphi(x) - \varphi(y)\| \leq l_2|x - y|$ for all $x, y \in \mathcal{C}$.

(H_5) There exists a constant $l_3 > 0$ such that $\|\psi(x) - \psi(y)\| \leq l_3|x - y|$ for all $x, y \in \mathcal{C}$.

3.1. **Existence and uniqueness of the mild solution.** Using the fractional Laplace transform in equation (1.1), we get

$$\begin{aligned}
&\mathcal{L}_\beta\left(\frac{d^\beta}{dt^\beta}\left[\frac{d^\beta}{dt^\beta}\left(y(t)-\mathcal{F}\left(t,y(h_1(t))\right)\right)\right]\right)(\lambda)\\
=&\lambda\left(\lambda^2-B\right)^{-1}\left(y\left(0\right)-\mathcal{F}\left(0,y(h_1(0))\right)\right)+\left(\lambda^2-B\right)^{-1}\frac{d^\beta}{dt^\beta}\left(y(0)-\mathcal{F}\left(0,y(h_1(0))\right)\right)\\
&+\left(\lambda^2-B\right)^{-1}\mathcal{L}_\beta(\mathcal{G}(t,y(h_2(t))))(\lambda)\\
=&\lambda\left(\lambda^2-B\right)^{-1}\left(y_0+\varphi(y)-\mathcal{F}\left(0,y(h_1(0))\right)\right)+\left(\lambda^2-B\right)^{-1}\left(y_1+\psi(y)\right)\\
&+\left(\lambda^2-B\right)^{-1}\mathcal{L}_\beta(\mathcal{G}(t,y(h_2(t))))(\lambda).
\end{aligned}$$

According to the inverse fractional Laplace transform, we find the Duhamel's formula

$$\begin{aligned}
y(t)=&\mathcal{F}\left(t,y(h_1(t))\right)+C\left(\frac{t^\beta}{\beta}\right)\left(y_0+\varphi(y)-\mathcal{F}(0,y(h_1(0)))\right)\\
&+S\left(\frac{t^\beta}{\beta}\right)\left(y_1+\psi(y)\right)+\int_0^t s^{\beta-1}S\left(\frac{t^\beta}{\beta}-\frac{s^\beta}{\beta}\right)\mathcal{G}(s,y(h_2(s)))ds.
\end{aligned}$$

Definition 3.1. $y\in\mathcal{C}$ is a mild solution of problem (1.1) if the following assertion is true:

$$\begin{aligned}
y(t)=&\mathcal{F}\left(t,y(h_1(t))\right)+C\left(\frac{t^\beta}{\beta}\right)\left(y_0+\varphi(y)-\mathcal{F}(0,y(h_1(0)))\right)\\
&+S\left(\frac{t^\beta}{\beta}\right)\left(y_1+\psi(y)\right)+\int_0^t s^{\beta-1}S\left(\frac{t^\beta}{\beta}-\frac{s^\beta}{\beta}\right)\mathcal{G}(s,y(h_2(s)))ds,\quad t\in[0,a].
\end{aligned}$$

Theorem 3.1. *If $(S(t))_{t>0}$ is compact and (H_1)-(H_5) are satisfied, then, the Cauchy problem* (1.1) *has at least one mild solution provided that*

$$l_1+\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|l_3<1.$$

Proof. Choosing

$$r\geq\frac{\|\mathcal{F}(0,0)\|+\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\left(\|y_0\|+\|\varphi(0)\|+\|\mathcal{F}(0,0)\|\right)+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\times\varDelta}{1-l_1-\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)-\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|l_3},$$

with

$$\varDelta=\|y_1\|+\|\psi(0)\|+\frac{a^\beta}{\beta}|\mu|_{L^\infty}.$$

Let $B_r = \{y \in \mathcal{C},\ |y| \leq r\}$, for $y \in B_r$, we define the operators P_1 and P_2 as follows

$$P_1(y(t)) = \mathcal{F}\left(t, y(h_1(t))\right) + C\left(\frac{t^\beta}{\beta}\right)\Big(y_0 + \varphi(y) - \mathcal{F}(0, y(h_1(0)))\Big) + S\left(\frac{t^\beta}{\beta}\right)\Big(y_1 + \psi(y)\Big),$$

$$P_2(y(t)) = \int_0^t s^{\beta-1} S\left(\frac{t^\beta}{\beta} - \frac{s^\beta}{\beta}\right) \mathcal{G}(s, y(h_2(s)))ds, \quad t \in [0, \tau].$$

By using assumptions (H_1)-(H_5), we prove that $P_1(y) + P_2(z) \in B_r$ for all $y, z \in B_r$. Moreover, the operator P_1 is a contraction on B_r.

We are going to prove that the operator P_2 is compact and continuous.

- Firstly, we show that P_2 is continuous.

Let $y_n \in B_r$ such that $y_n \to y$ in B_r. Therefore, by using (H_1), we have

$$\|s^{\beta-1}\left(\mathcal{G}(s, y_n((h_2(s))) - \mathcal{G}(s, y((h_2(s))))\right)\| \leq 2\mu_r(s)s^{\beta-1}$$

and

$$\mathcal{G}(s, y_n((h_2(s))) \to \mathcal{G}(s, y((h_2(s))), \quad \text{as } n \to +\infty.$$

Also, we obtain

$$P_2(y_n(t)) - P_2(y(t)) = \int_0^t s^{\beta-1} S\left(\frac{t^\beta - s^\beta}{\beta}\right)\Big[\mathcal{G}(s, y_n(h_2(s))) - \mathcal{G}(s, y((h_2(s)))\Big]ds,$$

for $t \in [0, a]$. Accordingly, we obtain

$$|P_2(y_n) - P_2(y)| \leq \sup_{t \in [0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right| \int_0^a s^{\beta-1}\|\mathcal{G}(s, y_n(h_2(s))) - \mathcal{G}(s, y((h_2(s)))\|ds.$$

By using the Lebesgue dominated convergence theorem, we have

$$\lim_{n \to +\infty}|P_2(y_n) - P_2(y)| = 0.$$

- Secondly, we prove the compactness of P_2.

Claim 1. We show that $\{P_2(y(t)), y \in B_r\}$ is relatively compact in Y.

For $t \in]0, a[$, let $\varepsilon \in]0, t[$, and we define the operator P_2^ε by

$$P_2^\varepsilon(y(t)) = \int_0^{(t^\beta - \varepsilon^\beta)^{\frac{1}{\beta}}} s^{\beta-1} S\left(\frac{t^\beta - s^\beta}{\beta}\right) \mathcal{G}(s, y(h_2(s)))ds, \quad t \in [0, \tau], \text{ for all } y \in B_r.$$

The relative compactness of $\{P_2^\varepsilon(y(t)), y \in B_r\}$ in Y is guaranteed by the compactness of $(S(t))_{t>0}$. Using assumption (H_1), we have

$$\|P_2^\varepsilon(y(t)) - P_2(y(t))\| \leq |\mu_r|_{L^\infty([0,a],\mathbb{R}^+)} \sup_{t \in [0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right| \frac{\varepsilon^\beta}{\beta}.$$

Then, we conclude that $\{P_2(y(t)), y \in B_r\}$ is relatively compact in Y. It is clear that the set $\{P_2(y(0)), y \in B_r\}$ is compact. Therefore, $\{P_2(y(t)), y \in B_r\}$ is relatively compact in Y for all $t \in [0, a]$.

Claim 2. We prove that $P_2(B_r)$ is equicontinuous.

Let $t_1, t_2 \in]0, a]$ such that $t_1 < t_2$, we have

$$P_2(y(t_2)) - P_2(y(t_1)) = \int_0^{t_1} s^{\beta-1}\left[S\left(\frac{t_2^\beta - s^\beta}{\beta}\right) - S\left(\frac{t_1^\beta - s^\beta}{\beta}\right)\right]\mathcal{G}(s, y(h_2(s)))ds + \int_{t_1}^{t_2} s^{\beta-1}S\left(\frac{t_2^\beta - s^\beta}{\beta}\right)\mathcal{G}(s, y(h_2(s)))ds.$$

Therefore, we obtain

$$\|P_2(y(t_2)) - P_2(y(t_1))\| \leq |\mu_r|_{L^\infty([0,a],\mathbb{R}^+)}\left[\frac{K}{\omega^2}\left(\exp\left(\frac{\omega t_2^\beta}{\beta}\right) - \exp\left(\frac{\omega t_1^\beta}{\beta}\right)\right)\right] + \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\frac{t_2^\beta - t_1^\beta}{\beta}\right).$$

We conclude that the functions $P_2(y)$, $y \in B_r$, are equicontinuous at $t \in [0, a]$. Applying the Arzelà-Ascoli theorem, we establish that P_2 is a compact operator. Finally, the Krasnoselskii's fixed point theorem completes the proof. □

To establish the uniqueness of the mild solution, we need the following assumption.

(H_6) There exists a constant $l_4 > 0$ such that $\|\mathcal{G}(t, z) - \mathcal{G}(t, y)\| \leq l_4\|z - y\|$ for all $z, y \in Y$ and $t \in [0, a]$.

Theorem 3.2. *Assume that* (H_2)-(H_6) *hold. Then, the Cauchy problem* (1.1) *has a unique mild solution provided that*

$$l_1 + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1 + l_2) + \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(l_3 + l_4\frac{a^\beta}{\beta}\right) < 1.$$

Proof. We define the operator $P : \mathcal{C} \to \mathcal{C}$ by

$$P(y(t)) = \mathcal{F}(t, y(h_1(t))) + C\left(\frac{t^\beta}{\beta}\right)\Big(y_0 + \varphi(y) - \mathcal{F}(0, y(h_1(0)))\Big) + S\left(\frac{t^\beta}{\beta}\right)\Big(y_1 + \psi(y)\Big) + \int_0^t s^{\beta-1}S\left(\frac{t^\beta - s^\beta}{\beta}\right)\mathcal{G}(s, y(h_2(s)))ds, \quad t \in [0, a].$$

Next, let $y, z \in \mathcal{C}$, we have

$$\begin{aligned}P(z(t)) - P(y(t)) =& \mathcal{F}(t, z(h_1(t))) - \mathcal{F}(t, y(h_1(t))) + C\left(\frac{t^\beta}{\beta}\right)(\varphi(z) - \varphi(y)) \\ &+ C\left(\frac{t^\beta}{\beta}\right)\Big(\mathcal{F}(0, y(h_1(0))) - \mathcal{F}(0, z(h_1(0)))\Big) \\ &+ S\left(\frac{t^\beta}{\beta}\right)(\psi(z) - \psi(y)) \\ &+ \int_0^t s^{\beta-1}S\left(\frac{t^\beta - s^\beta}{\beta}\right)\Big[\mathcal{G}(s, z(h_2(s))) - \mathcal{G}(s, y(h_2(s)))\Big]ds.\end{aligned}$$

Accordingly, we obtain

$$\begin{aligned}&\|P(z(t))-P(y(t))\|\\ \leq&\left[l_1+\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(l_3+l_4\frac{a^\beta}{\beta}\right)\right]|z-y|.\end{aligned}$$

Then, we get

$$|P(z)-P(y)|\leq\left[l_1+\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(l_3+l_4\frac{a^\beta}{\beta}\right)\right]|z-y|.$$

Therefore, P has a unique fixed point in $\mathcal{C}$. □

3.2. **Continuous dependence of the mild solution.** Now, we will give some results concerning the continuous dependence of the mild solution.

Theorem 3.3. *Assume that the conditions of Theorem* 3.2 *are satisfied. Let* y_0, z_0, y_1, $z_1 \in Y$ *and denote by* y, z *the solutions associated with* (y_0, y_1) *and* (z_0, z_1), *respectively. Then, we have*

$$\begin{aligned}|z-y|\leq&\frac{\beta}{\beta-\beta l_1-\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\beta l_3+l_4a^\beta\right)-\beta\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)}\\ &\times\left[\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\|z_0-y_0\|+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\|z_1-y_1\|\right].\end{aligned}$$

Proof. For $t\in[0,a]$, we have

$$\begin{aligned}z(t)-y(t)=&\mathcal{F}\left(t,z(h_1(t))\right)-\mathcal{F}\left(t,y(h_1(t))\right)+S\left(\frac{t^\beta}{\beta}\right)\left(z_1-y_1+\psi(z)-\psi(y)\right)\\ &+C\left(\frac{t^\beta}{\beta}\right)\Big(z_0-y_0+\varphi\left(z\right)-\varphi\left(y\right)+\mathcal{F}\left(0,y(h_1(0))\right)-\mathcal{F}\left(0,z(h_1(0))\right)\Big)\\ &+\int_0^t s^{\beta-1}S\left(\frac{t^\beta-s^\beta}{\beta}\right)\Big[\mathcal{G}(s,z(h_2(s)))-\mathcal{G}(s,y(h_2(s)))\Big]ds.\end{aligned}$$

Since we obtain

$$\begin{aligned}\|z(t)-y(t)\|\leq&l_1|z-y|+\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_1-y_1\|+\left(l_3+l_4\frac{a^\beta}{\beta}\right)|z-y|\right)\\ &+\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_0-y_0\|+(l_1+l_2)\,|z-y|\right).\end{aligned}$$

Accordingly, we show that

$$\begin{aligned}|z-y| \leq & l_1|z-y| + \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_1-y_1\| + \left(l_3 + l_4\frac{a^\beta}{\beta}\right)|z-y|\right)\\ &+ \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_0-y_0\| + (l_1+l_2)\,|z-y|\right).\end{aligned}$$

Finally, we get the following estimation:

$$\begin{aligned}|z-y| \leq & \frac{\beta}{\beta - \beta l_1 - \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\beta l_3 + l_4 a^\beta\right) - \beta\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)}\\ &\times\left[\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\|z_0-y_0\| + \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\|z_1-y_1\|\right]. \qquad \square\end{aligned}$$

Theorem 3.4. *Assume that the conditions of Theorem* 3.2 *are satisfied. Let* $y_0, z_0, y_1, z_1 \in Y$ *and denote by* y, z *the solutions associated with* (y_0, y_1) *and* (z_0, z_1)*, respectively. Then, we have*

$$|z-y| \leq \frac{\left(\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\|z_1-y_1\| + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\|z_0-y_0\|\right)\exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right)}{1-\left(l_1 + l_3\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right| + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)\right)\exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right)},$$

provided that

$$\left(l_1 + l_3\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right| + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)\right)\exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right) < 1.$$

Proof. For $t \in [0,a]$, we have

$$\begin{aligned}z(t) - y(t) =& F\left(t, z(h_1(t))\right) - F\left(t, y(h_1(t))\right) + S\left(\frac{t^\beta}{\beta}\right)\left(z_1 - y_1 + \psi(z) - \psi(y)\right)\\ &+ C\left(\frac{t^\beta}{\beta}\right)\Big(z_0 - y_0 + \varphi(z) - \varphi(y) + F(0, y(h_1(0))) - F(0, z(h_1(0)))\Big)\\ &+ \int_0^t s^{\beta-1}S\left(\frac{t^\beta - s^\beta}{\beta}\right)\Big[\mathcal{G}(s, z(h_2(s))) - \mathcal{G}(s, y(h_2(s)))\Big]ds.\end{aligned}$$

Then, we get

$$\begin{aligned}\|z(t) - y(t)\| \leq & l_1|z-y| + \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_1-y_1\| + l_3|z-y|\right)\\ &+ \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\left(\|z_0-y_0\| + (l_1+l_2)\,|z-y|\right)\\ &+ l_4\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\int_0^t s^{\beta-1}\|z(s)-y(s)\|ds.\end{aligned}$$

Therefore, we show that

$$\begin{aligned}|z-y| \leq &\Bigg(l_1|z-y| + \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right| (\|z_1-y_1\| + l_3|z-y|)\\ &+ \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right| (\|z_0-y_0\| + (l_1+l_2)|z-y|) \times \exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right)\Bigg).\end{aligned}$$

Finally, we conclude that

$$|z-y| \leq \frac{\left(\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\|z_1-y_1\| + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\|z_0-y_0\|\right)\exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right)}{1-\left(l_1+l_3\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right| + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)\right)\exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right)}. \qquad \square$$

Remark 3.1. If we take

$$C_1 = \frac{\exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right)}{1-\left(l_1+l_3\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right| + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)\right)\exp\left(l_4\frac{a^\beta}{\beta}\sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\right)},$$

$$C_2 = \frac{\beta}{\beta - \beta l_1 - \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\left(\beta l_3 + l_4 a^\beta\right) - \beta\sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|(l_1+l_2)},$$

we have that $C_1 < C_2$. Then, Theorem 3.4 is better than Theorem 3.3.

3.3. **Special case of nonlocal conditions.** Here, we study a special case of nonlocal conditions, this means that the functions ψ and φ are given by:

$$\varphi(y) = \sum_{i=1}^{n} c_i y(t_i) \quad \text{and} \quad \psi(y) = \sum_{i=1}^{n} b_i y(t_i),$$

where c_i, b_i, $i = 1, 2, \ldots, n$, are given constants and $0 < t_1 < t_2 < \cdots < t_n < a$.

Proposition 3.1. *Assume that (H_2), (H_3) and (H_6) hold. Then, the fractional problem* (1.1) *has a unique mild solution provided that there exists $\varepsilon_0 \in]0,1[$ such that*

$$l_1 + \sup_{t\in[0,a]}\left|C\left(\frac{t^\beta}{\beta}\right)\right|\left(\sum_{i=1}^{n}|c_i| + l_1\right) + \sup_{t\in[0,a]}\left|S\left(\frac{t^\beta}{\beta}\right)\right|\sum_{i=1}^{n}|d_i| \leq \varepsilon_0.$$

Proof. Define the operator $P : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}P(y(t)) =& \mathcal{F}(t, y(h_1(t))) + C\left(\frac{t^\beta}{\beta}\right)(y_0 + \varphi(y) - \mathcal{F}(0, y(h_1(0))))\\ &+ S\left(\frac{t^\beta}{\beta}\right)\big(y_1 + \psi(y)\big) + \int_0^t s^{\beta-1} S\left(\frac{t^\beta - s^\beta}{\beta}\right)\mathcal{G}(s, y(h_2(s)))ds, \quad t \in [0,a].\end{aligned}$$

Now, we define a new norm $|\cdot|_\beta$ in $\mathcal{C}$ by

$$|y|_\beta = \left|\exp\left(\frac{-\varepsilon(\cdot)^\beta}{\beta}\right) y\right|,$$

where

$$\varepsilon = \frac{l_4 \sup\limits_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right) \right|}{\varepsilon_0 - l_1 - \sup\limits_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right)\right| \left(\sum\limits_{i=1}^{n} |c_i| + l_1\right) - \sup\limits_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right)\right| \sum\limits_{i=1}^{n} |d_i|}.$$

Let $y, z \in \mathcal{C}$, and $t \in [0,a]$. Then,

$$\begin{aligned} P(z(t)) - P(y(t)) =& \mathcal{F}(t, z(h_1(t))) - \mathcal{F}(t, y(h_1(t))) + C\left(\frac{t^\beta}{\beta}\right)(\varphi(z) - \varphi(y)) \\ &+ C\left(\frac{t^\beta}{\beta}\right)\Big(\mathcal{F}(0, y(h_1(0))) - \mathcal{F}(0, z(h_1(0)))\Big) \\ &+ S\left(\frac{t^\beta}{\beta}\right)(\psi(z) - \psi(y)) \\ &+ \int_0^t s^{\beta-1} S\left(\frac{t^\beta - s^\beta}{\beta}\right)\Big[\mathcal{G}(s, z(h_2(s))) - \mathcal{G}(s, y(h_2(s)))\Big] ds. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \|P(z(t)) - P(y(t))\| \leq & \Bigg[l_1 \exp\left(\frac{\varepsilon t^\beta}{\beta}\right) + \exp\left(\frac{\varepsilon t^\beta}{\beta}\right) \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right)\right| \left(\sum_{i=1}^{n} |c_i| + l_1\right) \\ &+ \exp\left(\frac{\varepsilon t^\beta}{\beta}\right) \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right)\right| \sum_{i=1}^{n} |d_i| \\ &+ l_4 \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right)\right| \int_0^t s^{\beta-1} \exp\left(\frac{\varepsilon s^\beta}{\beta}\right) ds \Bigg] |z - y|_\beta. \end{aligned}$$

Accordingly, we show that

$$\begin{aligned} |P(z) - P(y)|_\beta \leq & \Bigg[l_1 + \sup_{t\in[0,a]} \left| C\left(\frac{t^\beta}{\beta}\right)\right| \left(\sum_{i=1}^{n} |c_i| + l_1\right) \\ &+ \sup_{t\in[0,a]} \left| S\left(\frac{t^\beta}{\beta}\right)\right| \left(\sum_{i=1}^{n} |d_i| + \frac{l_4}{\varepsilon}\right) \Bigg] |z - y|_\beta. \end{aligned}$$

Hence, we conclude that

$$|P(z) - P(y)|_\beta \leq \varepsilon_0 |z - y|_\beta.$$

By using the contraction principle, we obtain the result. $\square$

4. Application

Consider the fractional partial differential equation of the following form

$$
\begin{aligned}
&\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\cdot\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\left(v(t,y)-\int_0^{\pi}c(t,y,\theta)v(\sin t,\theta)d\theta\right)\\
=&\frac{\partial^2(\cdot)}{\partial y^2}\left(v(t,y)-\int_0^{\pi}c(t,y,\theta)v(\sin t,\theta)d\theta\right)\\
&+\phi\left(t,\frac{\partial v(t,y)}{\partial y}\right)+\frac{|v(t,y)|}{1+|v(t,y)|}+\int_0^t\frac{|v(s,y)|}{1+|v(s,y)|}ds,
\end{aligned}
\tag{4.1}
$$

with the following nonlocal conditions:

$$
v(t,0)=v(t,\pi)=0 \quad \text{and} \quad v(0,y)=\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}v(0,y)=\sum_{i=1}^{n}c_iv(t_i,y), \quad y\in[0,\pi],
\tag{4.2}
$$

where $0<t_1<\cdots<t_n<1$ and $c_1,\ldots,c_n$ are given real constants such that

$$
\sum_{i=1}^{n}|c_i|<\frac{4}{10}.
$$

Let $Y=L^2([0,\pi])$ and define the operator $B:Y\to Y$ by

$$
B=\frac{\partial^2(\cdot)}{\partial y^2} \quad \text{and} \quad D(B)=\{v\in H^2(0,\pi), v(\pi)=v(0)=0\}.
$$

The operator B is the infinitesimal generator of a family of cosines $\{C(t),S(t)\}_{t\in\mathbb{R}}$. Furthermore, we have $|C(t)|\leq 1$ and $|S(t)|\leq 1$ for all $t\in[0,1]$.

We consider the following functions:

$$
\begin{aligned}
z(t)(y)=&v(t,y), \quad \mathcal{F}\big(t,z)(\cdot)=\int_0^{\pi}c(t,\cdot,\theta)z(\theta)d\theta,\\
\mathcal{G}(t,z(t))=&\frac{|z(t)|}{1+|z(t)|}+\int_0^t\frac{|z(s)|}{1+|z(s)|}ds, \quad h_1(t)=\sin(t), \quad h_2(t)=t,
\end{aligned}
$$

and

$$
\varphi(z)=\psi(z)=\sum_{i=1}^{n}c_iz(t_i).
$$

We assume that the following condition hold: $c:[0,1]\times[0,\pi]\times[0,\pi]\to\mathbb{R}$ is continuous with $c(t,\cdot,0)=c(t,\cdot,\pi)=0$. Then, (4.1) and (4.2) become as follows:

(4.3)

$$
\begin{cases}
\frac{d^{\beta}}{dt^{\beta}}\left[\frac{d^{\beta}}{dt^{\beta}}\Big(z(t)-\mathcal{F}(t,z(h_1(t)))\Big)\right]=B\Big(z(t)-\mathcal{F}\big(t,z(h_1(t))\big)\Big)+\mathcal{G}\big(t,z(h_2(t))\big),\\
z(0)=\varphi(z),\\
\frac{d^{\beta}}{dt^{\beta}}\Big(z(0)-\mathcal{F}(0,z(h_1(0)))\Big)=\psi(z).
\end{cases}
$$

Moreover, $\mathcal{F} : [0,1] \times \mathcal{C} \to X$, we have

$$\begin{aligned}\|\mathcal{F}(t,z)\|^2 &\leq \int_0^\pi \left(\int_0^\pi c(t,\eta,\theta)z(\theta)d\theta\right)^2 d\eta\\ &\leq \left(\int_0^\pi \int_0^\pi c(t,\eta,\theta)^2 d\theta d\eta\right)\int_0^\pi z^2(\theta)d\theta\\ &\leq \sup_{0\leq t\leq 1}\left(\int_0^\pi \int_0^\pi c(t,\eta,\theta)^2 d\theta d\eta\right)|z|^2.\end{aligned}$$

Finally, all the hypotheses of Proposition 3.1 are verified.

Therefore, the above fractional problem has a unique mild solution provided that

$$2\left(\sup_{0\leq t\leq 1}\left(\int_0^\pi \int_0^\pi c(t,\eta,\theta)^2 d\theta d\eta\right)\right)^{\frac{1}{2}} + \frac{4}{5} < 1.$$

5. Conclusion

In this manuscript, we have explored the existence of solutions for second-order conformal differential equations evolving sequentially with non-local conditions by utilizing Krasnoselskii's fixed-point theorem. Additionally, through the application of the Banach fixed-point theorem, we have established the uniqueness of the mild solution. Finally, we have presented a relevant example to demonstrate the practical application of our theoretical findings.

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