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ON THE JACOBSON SEMISIMPLE SEMIRINGS

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Abstract. Based on the minimal and simple representations, we introduce two types of Jacobson semisimplicity, m-semisimplicity and s-semisimplicity, of a semiring S. Every $m(s)$-semisimple semiring is a subdirect product of $m(s)$-primitive semirings. It is shown that a commutative s-primitive semiring is either a two element Boolean algebra or a field. Every s-primitive semiring is isomorphic to a 1-fold transitive subsemiring of the semiring of all endomorphisms of a semimodule over a division semiring.

1. Introduction

A semiring is an algebraic structure satisfying all the axioms of a ring, but one that every element has an additive inverse. The absence of additive inverses forces a semiring to deviate radically from behaving like a ring. For example, ideals are not in bijection with the congruences on a semiring. Further, the presence of additively idempotent semirings makes the class of the semirings abundant. Now semirings have become a part of mainstream mathematics for their importance in theoretical computer science [30], graph theory [16] and automata theory [9, 11, 13]; and for the surprising 'characteristic one analogy' of the usual algebra over fields [10, 12, 25, 26]; and for the role of additively idempotent semirings in tropical mathematics [1, 7, 14, 19, 20, 27].

In a recent paper [24], Katsov and Nam considered two Jacobson type semisimple semirings - J-semisimple semirings and J_s-semisimple semirings. They characterized J-semisimple semirings within the class of all additively cancellative semirings and additively idempotent J_s-semisimple semirings. Besides characterizing the semirings S such that $J(S) = J_s(S)$, a problem stated in [24], Mai and Tuyen [28] extended the

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study of the J_s-semisimple semirings to the class of all zerosumfree semirings. They proved that every J_s-semisimple zerosumfree commutative semiring is semiisomorphic to a subdirect product of its maximal entire quotients.

This article is a continuation of [6] under the project to develop a radical theory of semirings that can be used to study semiring in general. Based on the minimal and simple representations of a semiring, the present authors introduced and studied two Jacobson type radicals - Jacobson m-radical and s-radical of a semiring in [6]. Here, we define and study two types of Jacobson semisimple semirings - Jacobson m-semisimple and s-semisimple semirings.

Since ideals are not in bijection with the congruences on a semiring, it is not surprising that replacing ideals with the more general notion of congruences on semirings exhibits many excellent properties and several analogies with classical results on the rings [3, 22, 23]. In our approach, the annihilator $ann_S(M)$ of an S-semimodule M is considered as a congruence on S. Similarly to the semirings, subsemimodules are not in bijection with the congruences on a semimodule; which produces three variants of 'irreducibility' of semimodules – minimal semimodules, elementary semimodules, and simple semimodules [8, 21]. In [6], the authors of the present article introduced and characterized two Jacobson type Hoehnke radicals, namely, m-radical and s-radical of a semiring S as two congruences on S. Here we introduce m-semisimple, s-semisimple, m-primitive and s-primitive semirings in an obvious way. Considering radical as a congruence makes it easy to represent these semisimple semirings as a subdirect product of suitable class of primitive semirings. The two element Boolean algebra and the fields are the only commutative s-primitive semirings. Hence the study of commutative s-primitive semirings characterizes the subdirect products of the copies of the two element Boolean algebra and fields.

This paper is organized as follows. Section 2 briefly recaps the necessary definitions and associated facts on semirings and semimodules. Section 3 introduces m-primitive and s-primitive semirings and characterizes Jacobson semisimple semirings as subdirect products of primitive semirings. Every commutative $(s)m$-primitive semiring is a (congruence simple) semifield. Since every congruence simple semifield S with $|S| > 2$ is a field, every commutative s-semisimple semiring is a subdirect product of a family of semirings, each of which is either the 2-element Boolean algebra or a field. Finally, every s-primitive semiring is represented as a 1-fold transitive subsemiring of the semiring of all endomorphisms of a semimodule over a division semiring.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is a nonempty set S with two binary operations '+' and '$\cdot$' satisfying:

- $(S, +)$ is a commutative monoid with identity element 0;
- $(S, \cdot)$ is a semigroup;
- $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.

Moreover, we assume that the additive identity element 0 is absorbing, i.e., $0s = s0 = 0$ for all $s \in S$. There is no consensus whether every semiring contains a multiplicative unity 1. In this article, we follow the convention of Hebisch and Weinert [17], that a semiring does not contain 1, in general. Also, there are many articles on semirings where the existence of unity is not assumed [4, 5, 18, 21, 31, 32]. On the other hand, in Golan [15], it is assumed that every semiring contains 1. If a semiring S with multiplicative identity is such that every nonzero element has a multiplicative inverse, then S is called a *division semiring*. A commutative division semiring is called a *semifield*. A semiring S is said to be an *additively idempotent semiring* if $a+a=a$ for all $a \in S$. Both the two element Boolean algebra $\mathbb{B}$ and the max-plus algebra $\mathbb{R}_{max}$ are additively idempotent semifields.

Definition of the ideals, congruences and homomorphisms of semirings are as usual. Here it is assumed that every semiring homomorphism $\phi : S_1 \to S_2$ satisfies $\phi(0_1) = 0_2$. The kernel of a semiring homomorphism $\phi : S_1 \to S_2$ is defined by $\ker\phi = \{(a,b) \in S_1 \times S_2 \mid \phi(a) = \phi(b)\}$. Then $\ker\phi$ is a congruence on S_1 and $S_1/\ker\phi \simeq \phi(S_1)$.

A semiring S is said to be *congruence-simple* if it has no congruences other than the equality congruence $\Delta_S = \{(s,s) \mid s \in S\}$ and the universal congruence $\nabla_S = S \times S$.

Let I be a (left, right) ideal of a semiring S and μ be a (left, right) congruence relation on S. Then, I is said to be a *μ-saturated* (*left, right*) *ideal* of S if for every $s \in S$ and $i \in I$, $(s,i) \in \mu$ implies that $s \in I$. Thus, an ideal I is μ-saturated if and only if $I = \cup_{a\in I}[a]_\mu$.

A *right S-semimodule* is a commutative monoid $(M, +, 0_M)$ equipped with a right action $M \times S \to M$ that satisfies for all $m, m_1, m_2 \in M$ and $s, s_1, s_2 \in S$:

- $(m_1 + m_2)s = m_1 s + m_2 s$;
- $m(s_1 + s_2) = ms_1 + ms_2$;
- $m(s_1 s_2) = (ms_1)s_2$;
- $m0 = 0_M = 0_M s$.

Unless stated otherwise, by an *S-semimodule* M, we mean a right S-semimodule.

The *annihilator* of an S-semimodule M is defined by

$$ann_S(M) = \{(s_1, s_2) \in S \times S \mid ms_1 = ms_2 \text{ for all } m \in M\}.$$

Then $ann_S(M)$ is a congruence on S. If $\psi : S \to End_S(M)$ is the representation of S induced by the right action of S on the semimodule M, then $\ker\psi = ann_S(M)$.

If M is a right S-semimodule then for every congruence ρ on S with $\rho \subseteq ann_S(M)$, the scalar multiplication $m[s]_\rho = ms$ makes M an S/ρ-semimodule.

The following result can be proved easily, and so we omit the proof.

Lemma 2.1. *Let S be a semiring and ρ be a congruence on S.*

(a) *If M is an S/ρ-semimodule, then M becomes an S-semimodule under the scalar multiplication $ms = m[s]$. Moreover, $\rho \subseteq ann_S(M)$.*

(b) *Let M be an S-semimodule and $\rho \subseteq ann_S(M)$. Then, $ann_{S/\rho}(M) = ann_S(M)/\rho$.*

An S-semimodule M is said to be *faithful* if $ann_S(M) = \Delta_S$.

Following Chen et al. [8] we define the following.

Definition 2.1. Let M be an S-semimodule such that $MS \neq 0$. Then M is called

(i) minimal if M has no subsemimodules other than (0) and M;
(ii) simple if it is minimal and the only congruences on M are Δ_M and ∇_M where Δ_M is the equality relation on M and $\nabla_M = M \times M$.

In [21], simple semimodules have been termed as irreducible semimodules. We denote the class of all minimal and simple S-semimodules by $\mathcal{M}(S)$ and $\mathcal{S}(S)$, respectively. If R is a ring then $\mathcal{M}(S) = \mathcal{S}(S)$ [6].

In this paper, we will have many occasions to use the following characterization [6, 21] of the minimal semimodules.

Lemma 2.2. *A nonzero S-semimodule M is minimal if and only if $M = mS$ for all $m(\neq 0) \in M$.*

The classes $\mathcal{M}(S)$ and $\mathcal{S}(S)$ of minimal and simple representations of a semiring S induces the following two notions of Jacobson type radicals of a semiring [6].

Definition 2.2. Let S be a semiring. We define

(a) m-radical of S by $rad_m(S) = \cap_{M \in \mathcal{M}(S)} ann_S(M)$;
(b) s-radical of S by $rad_s(S) = \cap_{M \in \mathcal{S}(S)} ann_S(M)$.

If there are no minimal semimodules over S, then we define $rad_m(S) = \nabla_S$. Similarly, we define $rad_s(S) = \nabla_S$ if there is no simple S-semimodules.

Both the assignments $S \mapsto rad_m(S)$ and $S \mapsto rad_s(S)$ are Hoehnke radicals on S [6].

A right congruence μ on S is said to be a *regular right congruence* if there exists $e \in S$ such that $(es, s) \in \mu$ for every $s \in S$. If μ is a regular right congruence on S, then $M = S/\mu$ is a right S-semimodule such that $MS \neq 0$.

The subsequent two results characterize the regular congruences μ on S such that the quotient semimodule S/μ is a minimal or a simple semimodule over S.

Lemma 2.3 ([6]). *Let M be an S-semimodule. Then, M is minimal if and only if there exists a regular right congruence μ on S such that $S/\mu \simeq M$ and $[0]_\mu$ is a maximal μ-saturated right ideal in S.*

Every simple semimodule is congruence-simple. Therefore, for every right congruence μ on S, if S/μ is simple, then μ is maximal. Hence, the following result follows.

Lemma 2.4 ([6]). *Let M be a S-semimodule. Then M is simple if and only if there exists a maximal regular right congruence μ on S such that $S/\mu \simeq M$ and $[0]_\mu$ is a maximal μ-saturated right ideal in S.*

A regular right congruence μ on S is said to be *m-regular* if $[0]_\mu$ is a maximal μ-saturated right ideal in S; and *s-regular* if it is a maximal regular right congruence such that $[0]_\mu$ is a maximal μ-saturated right ideal in S. We denote the set of all m-regular right congruences on S by $\mathcal{RC}_m(S)$ and the set of all s-regular right congruences on S by $\mathcal{RC}_s(S)$.

The following internal characterization of the m-radical and the s-radical of a semiring was proved in [6].

Lemma 2.5 ([6]). *Let S be a semiring. Then,*

(i) $rad_m(S) = \bigcap_{\mu \in \mathcal{RC}_m(S)} \mu$;
(ii) $rad_s(S) = \bigcap_{\mu \in \mathcal{RC}_s(S)} \mu$.

The following result has useful applications.

Lemma 2.6 ([6]). *Let R and S be two semirings. Then, $rad_m(R \times S) = rad_m(R) \times rad_m(S)$ and $rad_s(R \times S) = rad_s(R) \times rad_s(S)$.*

The reader is referred to [6] for more details on the m-radical and the s-radical of a semiring and to [17] for the undefined terms and notions concerning semirings and semimodules over semirings.

3. Jacobson Semisimple Semirings

In this section, we introduce and study the Jacobson m-semisimple and s-semisimple semirings.

Definition 3.1. A semiring S is said to be *m-semisimple* if $rad_m(S) = \Delta_S$; and *s-semisimple* if $rad_s(S) = \Delta_S$.

Since $\mathcal{RC}_s(S) \subseteq \mathcal{RC}_m(S)$, it follows that $rad_m(S) \subseteq rad_s(S)$. Therefore, every s-semisimple semiring is m-semisimple. The following example shows that the converse does not hold in general.

Example 3.1. Let $\mathbb{H}$ be the ring of all real quaternions. Since $\mathbb{H}$ is a division ring, $rad_m(\mathbb{H}) = rad_s(\mathbb{H}) = \Delta_{\mathbb{H}}$. Also, $rad_m(\mathbb{R}_{max}) = \Delta_{\mathbb{R}_{max}}$ and $\rho = \{(-\infty, -\infty)\} \cup \{(r, s) \mid r, s \in \mathbb{R}\}$ is the only s-regular congruence on $\mathbb{R}_{max}$ implies that $rad_s(\mathbb{R}_{max}) = \rho$ [6, Example 3.9]. Consider the semiring $S = \mathbb{H} \times \mathbb{R}_{max}$. Then, by the Lemma 2.6, we have

$$rad_m(S) = rad_m(\mathbb{H}) \times rad_m(\mathbb{R}_{max}) = \Delta_{\mathbb{H}} \times \Delta_{\mathbb{R}_{max}} = \Delta_S,$$
$$rad_s(S) = rad_s(\mathbb{H}) \times rad_s(\mathbb{R}_{max}) = \Delta_{\mathbb{H}} \times \rho.$$

Therefore, S is m-semisimple but not s-semisimple.

Now we give an example of an s-semisimple semiring.

Example 3.2. Both the ring $\mathbb{H}$ of all real quaternions and the semiring $\mathbb{N}^0$ of all nonnegative integers are s-semisimple. Hence, it follows from the Lemma 2.6 that the semiring $S = \mathbb{H} \times \mathbb{N}^0$ is both s-semisimple and m-semisimple.

If S is a Jacobson m-semisimple semiring, then $\cap_{M\in\mathcal{M}(S)} ann_S(M) = \Delta_S$ implies that every m-semisimple semiring is a subdirect product of the family of semirings $\{S/ann_S(M) \mid M \in \mathcal{M}(S)\}$. Also, for every $M \in \mathcal{M}(S)$, by Lemma 2.1, implies that M is a minimal and faithful $S/ann_S(M)$-semimodule. Similarly, every s-semisimple semiring S is a subdirect product of the family of semirings $\{S/ann_S(M) \mid M \in \mathcal{S}(S)\}$ where each quotient semiring $S/ann_S(M)$ has a faithful and simple semimodule M. Intending to characterize the structure of semisimple semirings, we introduce the following two notions.

Definition 3.2. Let S be a semiring. Then, S is called

(i) m-primitive if there is a faithful minimal S-semimodule M;
(ii) s-primitive if there is a faithful simple S-semimodule M.

If S is an m-primitive semiring, then there is a minimal S-semimodule M such that $ann_S(M) = \Delta_S$. Hence $rad_m(S) = \cap_{M\in\mathcal{M}(S)} ann_S(M) = \Delta_S$ and so S is m-semisimple. Similarly, every s-primitive semiring is s-semisimple.

A congruence σ on S is said to be an *m-primitive* (*s-primitive*) *congruence* if the quotient semiring S/σ is an m-primitive (s-primitive) semiring. Thus, σ is m-primitive(s-primitive) if and only if there exists a faithful minimal (simple) S/σ-semimodule M.

If ρ is a right congruence on S, we define

$$(\rho : \nabla_S) = \{(x, y) \in \nabla_S \mid (sx, sy) \in \rho \text{ for all } s \in S\}.$$

Then, $(\rho : \nabla_S)$ is a congruence on S.

Lemma 3.1. *Let σ be a congruence on S. Then, the following conditions are equivalent:*

(a) *σ is m-primitive (s-primitive);*
(b) *$\sigma = ann_S(M)$ for some minimal (simple) S-semimodule M;*
(c) *$\sigma = (\rho : \nabla_S)$ for some $\rho \in \mathcal{RC}_m(S)$ ($\rho \in \mathcal{RC}_s(S)$).*

Proof. We prove the result for m-primitive congruences. The other cases are similar.

(a) ⇒ (b) Let σ be an m-primitive congruence on S. Then there exists a faithful minimal S/σ-semimodule M. Hence, by Lemma 2.1, M is also a minimal S-semimodule such that $\sigma \subseteq ann_S(M)$ and $\Delta_{S/\sigma} = ann_{S/\sigma}(M) = ann_S(M)/\sigma$, i.e., $\sigma = ann_S(M)$.

(b) ⇒ (c) Let M be a minimal S-semimodule and $\sigma = ann_S(M)$. Then M is a minimal and faithful right S/σ-semimodule. Theorem 2.3 implies that there exists $\rho \in \mathcal{RC}_m(S)$ such that $M \simeq S/\rho$; and hence $ann_S(M) = (\rho : \nabla_S)$. Then $ann_{(S/\sigma)}(M) = ann_S(M)/\sigma = \Delta_{(S/\sigma)}$ implies that $ann_S(M) = \sigma$. Hence, $(\rho : \nabla_S) = ann_S(M) = \sigma$.

(c) ⇒ (a) Let $\rho \in \mathcal{RC}_m(S)$ and $\sigma = (\rho : \nabla_S)$. Then S/ρ is a minimal right S-semimodule and $ann_S(S/\rho) = (\rho : \nabla_S) = \sigma$. Since σ is a semiring congruence on S and $\sigma = (\rho : \nabla_S)$, it follows that S/ρ is a minimal right S/σ-semimodule. Also, by Lemma 2.1, we have $ann_{S/\sigma}(S/\rho) = ann_S(S/\rho)/\sigma = \Delta_{S/\sigma}$. Hence, S/ρ is a minimal and faithful right S/σ-semimodule, so σ is a m-primitive congruence on S. □

From the definition, it follows that a semiring S is an m-primitive (s-primitive) semiring if and only if Δ_S is an m-primitive (s-primitive) congruence on S. Thus, we have the following.

Corollary 3.1. *Let S be a semiring. Then, S is*

(a) *m-primitive if and only if there exists $\rho \in \mathcal{RC}_m(S)$ such that $(\rho : \nabla_S) = \Delta_S$;*
(b) *s-primitive if and only if there exists $\rho \in \mathcal{RC}_s(S)$ such that $(\rho : \nabla_S) = \Delta_S$.*

A division semiring is a noncommutative generalization of a semifield. The following result shows that primitive semirings are another class of noncommutative generalizations of semifields. The m-primitive semirings generalize the semifields, whereas the s-primitive semirings generalize the congruence-simple semifields.

Theorem 3.1. *Let S be a commutative semiring. Then, S is*

(a) *m-primitive if and only if it is a semifield;*
(b) *s-primitive if and only if it is a congruence-simple semifield.*

Proof. (a) Let S be a commutative m-primitive semiring. Then, by Corollary 3.1, there is a regular right congruence ρ in $\mathcal{RC}_m(S)$ such that $(\rho : \nabla_S) = \Delta_S$. Since S is a commutative semiring, ρ becomes a congruence on S and so $\rho = (\rho : \nabla_S) = \Delta_S$. Therefore $\Delta_S \in \mathcal{RC}_m(S)$ and there is an element $e \in S$ such that $es = s = se$ for all $s \in S$. Thus, e is a multiplicative identity in S. Also $\rho = \Delta_S \in \mathcal{RC}_m(S)$ implies that (0) is maximal Δ_S-saturated ideal in S. Since every ideal in S is Δ_S-saturated, it follows that (0) and S are the only two ideals in S. Now for each non-zero element $a \in S$, aS is a non-zero ideal in S. Hence, $aS = S$, which implies that there exists an element $b \in S$ such that $ab = e = ba$. Thus, S is a semifield.

Conversely, let S be a semifield. Then $M = S$ is a minimal S-semimodule and $ann_S(M) = \{(s_1, s_2) \in S \times S \mid ss_1 = ss_2 \text{ for all } s \in S\} = \Delta_S$. Therefore, S is m-primitive.

(b) Let S be a commutative s-primitive semiring. Then S is m-primitive, and so, by (a), it is a semifield. Also, by Corollary 3.1, there exists a right congruence $\rho \in \mathcal{RC}_s(S)$ such that $(\rho : \nabla_S) = \Delta_S$. Since S is commutative, it follows that $(\rho : \nabla_S) = \rho$. Hence $\Delta_S = \rho \in \mathcal{RC}_s(S)$ which implies that $M = S/\rho \simeq S$ is a simple S-semimodule. Hence, the semifield S is congruence-simple.

Conversely, if S is a congruence-simple semifield, then S itself is a faithful simple S-semimodule. Hence, S is s-primitive. □

Theorem 3.1 tells us that the congruence-simple semifields constitute an important subclass of the semifields. Similarly to the fields, the Krull-dimension of a congruence-simple semifield is 0, whereas there are semifields, say, for example, $\mathbb{R}_{max}$ having the Krull-dimension 1 [22]. A semiring S is called *zerosumfree* if for every $a, b \in S$, we have $a+b = 0$ implies that $a = 0$ and $b = 0$. It is well known that a semifield S is either zerosumfree or is a field [15, Proposition 4.34]. Every field is a congruence-simple semifield. If S is zerosumfree, then $\rho = \{(s,t) \in S \times S \mid s \neq 0 \neq t\} \cup \{(0,0)\}$ is a congruence on S. So for S to be congruence-simple, we must have $|S| = 2$. Then S is the 2-element Boolean algebra $\mathbb{B}$. Thus, a congruence-simple semifield is either the 2-element Boolean algebra $\mathbb{B}$ or a field.

However, in the following, we include an independent proof.

Theorem 3.2. *Let S be a semiring with $|S| > 2$. Then, S is a congruence-simple semifield if and only if it is a field.*

Proof. First, assume that S is a congruence-simple semifield. Denote $Z(S) = \{x \in S \mid x + y = 0 \text{ for some } y \in S\}$. Then, $Z(S)$ is an ideal of S; and so $Z(S)$ is either $\{0\}$ or S. If $Z(S) = \{0\}$, then S is zerosumfree. So, $\{(0, 0)\} \cup \{(s, t) \in S \times S \mid s \neq 0 \neq t\}$ induces a nontrivial congruence on S, which contradicts that S is congruence-simple. Hence, $Z(S) = S$ which implies that $(S, +)$ is a group. Thus, S is a field.

The converse follows trivially. □

Thus a zerosumfree semifield S with $|S| > 2$ cannot be congruence-simple. So, in particular, we have the following example.

Example 3.3. The max-plus algebra $\mathbb{R}_{max}$ is a semifield but not congruence-simple. Hence, $\mathbb{R}_{max}$ is an m-primitive semiring but not s-primitive.

The 2-element Boolean algebra $\mathbb{B}$ and the field $\mathbb{Z}_2$ of all integers modulo 2 are the only semifields of order two up to isomorphism. Hence, it turns out to be the following specific characterization of the commutative s-primitive semirings.

Corollary 3.2. *A commutative semiring S is s-primitive if and only if it is either the 2-element Boolean algebra $\mathbb{B}$ or a field.*

A semiring S is called a *subdirect product* of a family $\{S_\alpha\}_\Delta$ of semirings if there is an one-to-one semiring homomorphism $\phi : S \to \prod_\Delta S_\alpha$ such that for each $\alpha \in \Delta$, the composition $\pi_\alpha \circ \phi : S \to S_\alpha$ is onto where $\pi_\alpha : \prod_\Delta S_\alpha \to S_\alpha$ is the projection mapping.

It is well known that a semiring S is a subdirect product of a family $\{S_\alpha\}_\Delta$ of semirings if and only if there is a family $\{\rho_\alpha\}_\Delta$ of congruences on S such that $S/\rho_\alpha \simeq S_\alpha$ for every $\alpha \in \Delta$ and $\cap_\Delta \rho_\alpha = \Delta_S$.

Theorem 3.3. *A semiring S is m-semisimple (s-semisimple) if and only if it is a subdirect product of m-primitive (s-primitive) semirings.*

Proof. We prove the result for m-semisimple semirings. The proof for s-semisimple semirings is similar.

First, assume that S is a m-semisimple semiring. Then $rad_m(S) = \cap_{M \in \mathcal{M}(S)} ann_S(M) = \Delta_S$. Hence, S is a subdirect product of the family $\{S/ann_S(M) \mid M \in \mathcal{M}(S)\}$ of semirings. Lemma 3.1 implies that $ann_S(M)$ is an m-primitive congruence on S for every minimal S-semimodule M. Therefore, every semiring in the family $\{S/ann_S(M) \mid M \in \mathcal{M}(S)\}$ is an m-primitive semiring, and so S is a subdirect product of m-primitive semirings.

Conversely, let S be a subdirect product of a family of m-primitive semirings $\{S_i \mid \text{for all } i \in \Lambda\}$. Then there exists a one-to-one homomorphism $\phi : S \to \Pi_{i\in\Lambda} S_i$ such that the mapping $\pi_i \circ \phi : S \to S_i$ is onto for all $i \in \Lambda$. Thus $S/\ker(\pi_i \circ \phi) \cong S_i$ for all $i \in \Lambda$. Let M_i be a faithful minimal S_i-semimodule for each $i \in \Lambda$. Then, by the Lemma 2.1, M_i is a minimal S-semimodule where $ms = m\pi_i \circ \phi(s)$ for all $s \in S$ and $m \in M_i$. Hence, $\cap_{M\in\mathcal{M}(S)} ann_S(M) \subseteq \cap_{i\in\Lambda} ann_S(M_i)$. Now $(a, b) \in ann_S(M_i)$ implies that $m\pi_i \circ \phi(a) = m\pi_i \circ \phi(b)$ for all $m \in M_i$; and so $(\pi_i \circ \phi(a), \pi_i \circ \phi(b)) \in ann_{S_i}(M)$. Since M_i is faithful over S_i, it follows that $\pi_i \circ \phi(a) = \pi_i \circ \phi(b)$. Hence, $\cap_{i\in\Lambda} ann_S(M_i) = \Delta_S$ which implies that $rad_m(S) = \cap_{M\in\mathcal{M}(S)} ann_S(M) = \Delta_S$. Thus, S is a m-semisimple semiring. ☐

Now, taken together the structure of an s-semisimple semiring characterized in Theorem 3.3 and the characterization of the commutative s-primitive semirings in Corollary 3.2 turn out to be an characterization of the commutative s-semisimple semirings.

Corollary 3.3. *Let S be a commutative semiring. Then, S is an s-semisimple semiring if and only if it is a subdirect product of a family of semirings that are either the 2-element Boolean algebra $\mathbb{B}$ or fields.*

Mischell and Fenoglio [29] and Basir et al. [2] independently proved that a commutative semiring S with $|S| \geqslant 2$ is congruence-simple if and only if it is either a field or the 2-element Boolean algebra $\mathbb{B}$. Hence, it follows that a commutative semiring is s-semisimple if and only if it is a subdirect product of congruence-simple commutative semirings. A semiring homomorphism $f : S_1 \to S_2$ is said to be *semi-isomorphism* if, for every $a \in S_1$, we have $f(a) = 0$ only for $a = 0$. Katsov and Nam [24] proved that a commutative semiring S is Brown-McCoy semisimple if and only if S is semi-isomorphic to a subdirect product of a family of semirings that are either the 2-element Boolean algebra $\mathbb{B}$ or fields. Hence, every commutative s-semisimple semiring is Brown-McCoy semisimple in the sense of Katsov and Nam.

Example 3.4. Consider the semiring $\mathbb{N}$ of all nonnegative integers. Then, for every prime p, the Bourne congruence $\sigma_{p\mathbb{N}}$ is a maximal regular congruence on $\mathbb{N}$ with $[0]_{\sigma_{p\mathbb{N}}} = p\mathbb{N}$. If J is a $\sigma_{p\mathbb{N}}$-saturated ideal in $\mathbb{N}$ with $p\mathbb{N} \subsetneq J$, then there exists $a \in J$ such that $0 < a < p$. By the Fermat's little theorem, we have $a^{p-1} \equiv 1 \pmod{p}$ which implies that $1 \in J$ and so $J = \mathbb{N}$. Thus, $p\mathbb{N} = [0]_{\sigma_{p\mathbb{N}}}$ is a maximal $\sigma_{p\mathbb{N}}$-saturated ideal in $\mathbb{N}$ and it follows that $\sigma_{p\mathbb{N}} \in \mathcal{RC}_s(\mathbb{N})$. Hence $rad_s(\mathbb{N}) \subseteq \cap\sigma_{p\mathbb{N}} = \Delta_{\mathbb{N}}$; and so $\mathbb{N}$ is an s-semisimple semiring.

Also, $\cap\sigma_{p\mathbb{N}} = \Delta_{\mathbb{N}}$ implies that $\mathbb{N}$ is a subdirect product of the family of fields $\mathbb{N}_p = \mathbb{N}/\sigma_{p\mathbb{N}}$, where p is a prime.

We conclude this section with a representation of s-primitive semirings as a semiring of endomorphisms on a semimodule over a division semiring.

The opposite semiring S^{op} of a semiring $(S, +, \cdot)$ is defined by $(S, +, \circ)$, where $a \circ b = b \cdot a$ for all $a, b \in S$. Hence, a semiring S is a division semiring if and only if the opposite semiring S^{op} is so.

Definition 3.3. Let M be a semimodule over a division semiring D. Then a subsemiring T of the endomorphism semiring $End_D(M)$ is called 1-fold transitive if for every non-zero $m \in M$ and $n \in M$ there exists $\alpha \in T$ such that $\alpha(m) = n$.

In the context of semirings, Schur's lemma [21] states that if M is a simple S-semimodule, then the endomorphism semiring $End_S(M)$ is a division semiring.

Let M be a right S-semimodule and $E = End_S(M)$. Then for the division semiring $D = E^{op}$, M is a right semimodule over D where the scalar multiplication is defined by $m \cdot \alpha = \alpha(m)$ for all $m \in M$ and $\alpha \in D$.

Theorem 3.4. *If S is a right s-primitive semiring, then S^{op} is isomorphic to a 1-fold transitive subsemiring of the semiring $End_D(M)$ of all endomorphisms on a semimodule M over a division semiring D.*

Proof. Let M be a faithful simple right S-semimodule. By Schur's Lemma for semimodules [21], the semiring $E = End_S(M)$ is a division semiring. Hence, $D = E^{op}$ is a division semiring, and so M as a right D-semimodule where $m \cdot \alpha \mapsto \alpha(m)$.

For every $a \in S$, define a mapping $\psi_a : M \to M$ by $\psi_a(m) = ma$. Then for every $\alpha \in D$, we have $\psi_a(m.\alpha) = \psi_a(\alpha(m)) = \alpha(m)a = \alpha(ma) = (ma) \cdot \alpha = \psi_a(m) \cdot \alpha$. In fact, ψ_a is an endomorphism on M considered a D-semimodule.

Also, the mapping $\psi : S^{op} \to End_D(M)$ defined by $\psi(a) = \psi_a$ is a semiring homomorphism. Moreover $\ker \psi = ann_S(M) = \Delta_S$ implies that ψ is an injective homomorphism; and so S^{op} is isomorphic to the subsemiring $T = \{\psi_a \mid a \in S\}$ of $End_D(M)$.

Since M is a simple right S-semimodule, by Lemma 2.2, for every $m(\neq 0) \in M$, $mS = M$. Then for every $n \in M$ there exists $a \in S$ such that $ma = n$ and so $\psi_a(m) = n$. Thus, T is a 1-fold transitive subsemiring of $End_D(M)$. ☐

It follows from Corollary 3.2 that the semifield $F = \mathbb{R}_{max}$ is not an s-primitive semiring. Since F contains 1, every F-endomorphism on F is of the form $\psi_a : F \to F$ given by $\psi_a(m) = am$. Hence, $F \simeq End_F(F)$ which implies that $End_F(F)$ is not s-primitive; whereas $End_F(F)$ is a 1-fold transitive subsemiring of itself. Thus, the converse of the Theorem 3.4 does not hold. However, the converse holds in the following weaker form.

Theorem 3.5. *Let D be a division semiring and M be a right D-semimodule. If T is a 1-fold transitive subsemiring of $End_D(M)$, then T^{op} is a right m-primitive semiring.*

Proof. Define $M \times T^{op} \to M$ by $m \cdot \alpha \mapsto \alpha(m)$. Then M is a right T^{op}-semimodule. Let m be a non-zero element in M. Then, for every $n \in M$, there exists $\alpha \in T$ such that $m \cdot \alpha = n$. Therefore, $mT^{op} = M$ which implies that M is minimal, by

Lemma 2.2. Now

$$\begin{aligned} ann_{T^{op}}(M) &= \{(\alpha, \beta) \in T \times T \mid m \cdot \alpha = m \cdot \beta \text{ for all } m \in M\} \\ &= \{(\alpha, \beta) \in T \times T \mid \alpha(m) = \beta(m) \text{ for all } m \in M\} \\ &= \{(\alpha, \beta) \in T \times T \mid \alpha = \beta\} \\ &= \Delta_S \end{aligned}$$

and so M is a faithful minimal T^{op}-semimodule. Therefore, T^{op} is a m-primitive semiring. □

4. Conclusion

In [6], based on the notions of minimal semimodule and simple semimodule, the Jacobson m-radical and s-radical of a semiring S have been considered as a congruence on S. In Section 3 of this article, we introduce the m-semisimple and s-semisimple semirings as the semiring that has the trivial Jacobson m-radical and s-radical, respectively. These two notions of semisimplicity effectively characterize the structure of semirings, including the additively idempotent semirings. The m-semisimple (s-semisimple) are isomorphic to a subdirect product of m-primitive (s-primitive) semirings. In particular, a commutative semiring is s-primitive if and only if it is a subdirect product of the fields and copies of the two element Boolean algebra. Finally, every s-primitive semiring is represented as a suitable subsemiring of the semiring $End_D(M)$ of all endomorphisms on a semimodule M over a division semiring D.

There is another notion of simplicity of semimodules, namely the congruence simple semimodules which are known as elementary semimodules [8]. An attempt may be taken to characterize the e-semisimple semirings which are defined based on the class of elementary semimodules.

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