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THE RECIPROCAL COMPLEMENTARY WIENER NUMBER OF GRAPH OPERATIONS

R. NASIRI, A. NAKHAEI, AND A. R. SHOJAEIFARD

Abstract. The reciprocal complementary Wiener number of a connected graph G is defined as $\sum_{\{x,y\}\subseteq V(G)} \frac{1}{D+1-d_G(x,y)}$, where D is the diameter of G and $d_G(x,y)$ is the distance between vertices x and y. In this work, we study the reciprocal complementary Wiener number of various graph operations such as join, Cartesian product, composition, strong product, disjunction, symmetric difference, corona product, splice and link of graphs.

1. Introduction

Throughout this work, all graphs considered are simple, connected and finite. Let $G=(V(G),E(G))$ be a connected graph. For $x,y\in V(G)$, the *distance* $d_G(x,y)$ between the vertices x and y is equal to the length of a shortest path that connects x and y. For a vertex x in a connected nontrivial graph G, the *eccentricity* $\varepsilon_G(x)$ of x is the greatest geodesic distance between x and any other vertex of G. Also, the *diameter* $D=D(G)$ of the graph G is defined as the maximum eccentricity of any vertex in G. In other words,

$$\varepsilon_G(x)=\max\left\{d_G(x,y)|y\in V(G)\right\}, \quad D=D(G)=\max\left\{\varepsilon_G(x)|x\in V(G)\right\}.$$

In mathematical chemistry, a *molecular graph* (or *chemical graph*) is a labeled graph whose vertices correspond to the atoms of the compound and edges correspond to chemical bonds. It is natural to study mathematical properties of these graph models to find chemico-physical properties of the molecule under consideration.

Let G be a n-vertex graph with the vertex-set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and diameter D. The *reciprocal complementary distance matrix* $RCD=[rc_{ij}]$ of G is an $n\times n$

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matrix such that $rc_{ij} = \frac{1}{D+1-d_G(v_i,v_j)}$ if $i \neq j$, and 0 otherwise (see [7]). Ivanciuc et al. [5, 6] introduced the *reciprocal complementary Wiener number* of the graph G as:

$$RCW(G) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} rc_{ij} = \sum_{\{v_i,v_j\} \subseteq V(G)} \frac{1}{D + 1 - d_G(v_i, v_j)}. \tag{1.1}$$

This invariant has been successfully applied in the structure-property modeling of the molar hear capacity, standard Gibbs energy of formation and vaporization enthalpy of 134 alkanes C_6-C_{10} (see [5]).

Zhou et al. [14] gave various bounds for this quantity and Nordhaus-Gaddum-type result. Moreover, the trees with the smallest, the second smallest and the third smallest RCW, and the unicyclic and bicyclic graphs with the smallest and the second smallest RCW are characterized (see [2]). Zhu et al. [15] obtained the unique tree with $4 \leq D \leq n - 3$ and minimum reciprocal complementary Wiener number. They also specified the non-caterpillars with the smallest, the second smallest and the third smallest RCW-value. In [10], some bounds for the reciprocal complementary Wiener index of line graphs are presented.

Up to now, various topological indices have been introduced and used in the QSAR/QSPR studies. The *Wiener index* (or *Wiener number*) is the oldest and is one of the most studied topological quantities, both from a theoretical point of view and applications. This concept is defined as the sum of distances over all unordered vertex pairs in a graph G (see [12]). This invariant obtained wide attention and numerous results have been worked out, see the survey [13]. In special classes of graphs, such as trees, unicyclic and bicyclic graphs, this index has been studied in [3, 9, 11]. After it, a large number of other distance-based topological indices have been proposed and considered in the chemical and mathematico-chemical literature.

Brückler et al. [1] introduced a general distance-based topological index, called Q-index. The Q-index is defined as

$$Q(G) = \sum_{k \geq 0} f(k) D(G, k)\,, \tag{1.2}$$

where f is a function such that $f(0) = 0$, and $D(G, k)$ is the number of vertex pairs at distance k. Q is an additive function of increments associated with pairs of vertices of G. The Wiener, hyper-Wiener, Harary, and reciprocal complementary Wiener indices are all special cases of the Q-index. More precisely, by choosing $f(k) = k, \frac{k^2}{2} + \frac{k}{2}, \frac{1}{k}$ and $\frac{k^3}{6} + \frac{k^2}{2} + \frac{k}{3}$, the Q-index is equal to the Wiener, hyper-Wiener, Harary, and Tratch-Stankevich-Zefirov indices, respectively. In other special case, if consider $f(k) = \frac{1}{D+1-k}$, then the Q-index will be equal to the reciprocal complementary Wiener number. In other words, it holds

$$RCW(G) = \sum_{k=1}^{D} \frac{D(G, k)}{D + 1 - k}. \tag{1.3}$$

In this research, we study the reciprocal complementary Wiener number of various graph operations like join, Cartesian product, composition, strong product, disjunction, symmetric difference, corona product, splice and link of graphs.

2. Main Results

Throughout this paper, we consider graphs G_i with n_i vertices, m_i edges and the diameter D_i, $i = 1, 2$. Also, note that whenever we say $xy \notin E$, it is assumed that $x \neq y$. Moreover, we use standard notations of graph theory. The path, cycle, star, wheel and complete graphs with n vertices are denoted by P_n, C_n, S_n, W_n and K_n, respectively.

By applying relation (1.3), we compute RCW of some special graphs in the following example.

Example 2.1. Let P_n, K_n, S_n and W_n denote a path graph, complete graph, star graph and wheel graph with n vertices, respectively. Then

$$\begin{aligned} RCW(P_n) &= n-1, \\ RCW(K_n) &= \frac{1}{2}n(n-1), \\ RCW(S_n) &= \frac{1}{2}(n-1)^2, \\ RCW(W_n) &= \begin{cases} 6, & n=4, \\ \frac{1}{2}(n-1)(n-2), & n \geq 5. \end{cases} \end{aligned}$$

We begin by computing the reciprocal complementary Wiener number of join of graphs.

2.1. Join.

The *join* $G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. In the following lemma, we determine the reciprocal complementary Wiener number of join of graphs with respect to their numbers of vertices and edges.

Theorem 2.1. *Let G_1 and G_2 be two n_1- and n_2-vertex graphs, respectively.*

(i) *If G_1 and G_2 are complete graphs, then*

$$RCW(G_1 + G_2) = \frac{1}{2}(n_1 + n_2)(n_1 + n_2 - 1).$$

(ii) *If $\{G_1, G_2\} \neq \{K_{n_1}, K_{n_2}\}$, then*

$$RCW(G_1 + G_2) = \frac{1}{2}\Big(n_1(n_1 + n_2 - 1) + n_2(n_2 - 1) - m_1 - m_2\Big).$$

Proof. Suppose x and y are two vertices of G_1+G_2. By definition of the join of two graphs, one can easily see that

$$d_{G_1+G_2}(x,y)=\begin{cases}0, & x=y,\\ 1, & xy\in E_1 \text{ or } xy\in E_2 \text{ or } (x\in V_1 \text{ and } y\in V_2),\\ 2, & \text{otherwise.}\end{cases}$$

Assume that G_1 and G_2 are complete graphs, then $G_1+G_2=K_{n_1+n_2}$. Therefore, $RCW(G_1+G_2)=RCW(K_{n_1+n_2})=\binom{n_1+n_2}{2}$ (see Example 2.1). This completes the proof of part (i). To prove the second part, suppose that at least one of graphs G_1 or G_2 is not complete. So, we have $D=D(G_1+G_2)=2$, and

$$\begin{aligned}RCW(G_1+G_2)=&\sum_{\{x,y\}\subseteq V(G_1+G_2)}\frac{1}{3-d_{G_1+G_2}(x,y)}\\ =&\sum_{\{x,y\}\subseteq V_1}\frac{1}{3-d_{G_1+G_2}(x,y)}+\sum_{\{x,y\}\subseteq V_2}\frac{1}{3-d_{G_1+G_2}(x,y)}\\ &+\sum_{\substack{x\in V_1\\ y\in V_2}}\frac{1}{3-d_{G_1+G_2}(x,y)}\\ =&\sum_{xy\in E_1}\frac{1}{3-d_{G_1+G_2}(x,y)}+\sum_{xy\notin E_1}\frac{1}{3-d_{G_1+G_2}(x,y)}\\ &+\sum_{xy\in E_2}\frac{1}{3-d_{G_1+G_2}(x,y)}+\sum_{xy\notin E_2}\frac{1}{3-d_{G_1+G_2}(x,y)}\\ &+\sum_{\substack{x\in V_1\\ y\in V_2}}\frac{1}{3-d_{G_1+G_2}(x,y)}\\ =&\frac{1}{2}\Big(n_1(n_1+n_2-1)+n_2(n_2-1)-m_1-m_2\Big).\end{aligned}$$

□

Example 2.2. We know that $\overline{K_r}+\overline{K_s}=K_{r,s}$ (in particular, $K_1+\overline{K_{n-1}}=K_{1,n-1}=S_n$) is the complete bipartite graph. From Theorem 2.1 we obtain explicit formulas for the reciprocal complementary Wiener number of the these graphs

$$RCW(K_{r,s})=\frac{1}{2}\Big(r(r+s-1)+s(s-1)\Big),\quad RCW(S_n)=\frac{1}{2}(n-1)^2.$$

2.2. **Cartesian product.** The *Cartesian product* $G_1\Box G_2$ of graphs G_1 and G_2 has the vertex set $V_1\times V_2$ and $(x,y)(u,v)$ is an edge of $G_1\Box G_2$ if ($x=u$ and $yv\in E_2$), or ($xu\in E_1$ and $y=v$). For example, the *ladder graph* $L_{2,n}$ can be obtained as the Cartesian product of two path graphs P_2 and P_n.

Now, we study the reciprocal complementary Wiener number of the Cartesian product of graphs. To do this, we need the following well-known relation related to distance properties of the Cartesian product of two graphs (see [4])

$$d_{G_1\Box G_2}\Big((x,y),(u,v)\Big)=d_{G_1}(x,u)+d_{G_2}(y,v). \tag{2.1}$$

Theorem 2.2. *Let G_1 and G_2 be two non-complete graphs. Then*

$$\begin{aligned} RCW(G_1\Box G_2) <& \frac{n_1m_2+n_2m_1}{D_1+D_2}+\frac{2m_1m_2}{D_1+D_2-1}+(n_1+2m_1)\left(RCW(G_2)-\frac{m_2}{D_2}\right)\\ &+(n_2+4m_2)\left(RCW(G_1)-\frac{m_1}{D_1}\right). \end{aligned}$$

Proof. Applying (2.1), we have $D = D(G_1\Box G_2) = D_1 + D_2$. Therefore,

$$\begin{aligned} RCW(G_1\Box G_2) =& \sum_{\{(x,y),(u,v)\}\subseteq V(G_1\Box G_2)} \frac{1}{D+1-d_{G_1\Box G_2}\big((x,y),(u,v)\big)}\\ =& \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\Box G_2)\\ yv\in E_2}} \frac{1}{D_1+D_2}\\ &+\sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\Box G_2)\\ yv\notin E_2}} \frac{1}{D_1+D_2+1-d_{G_2}(y,v)}\\ &+\sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\Box G_2)\\ xu\in E_1}} \frac{1}{D_1+D_2}\\ &+\sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\Box G_2)\\ xu\notin E_1}} \frac{1}{D_1+D_2+1-d_{G_1}(x,u)}\\ &+\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\Box G_2)\\ xu\in E_1,yv\in E_2}} \frac{1}{D_1+D_2-1}\\ &+\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\Box G_2)\\ xu\in E_1,yv\notin E_2}} \frac{1}{D_1+D_2-d_{G_2}(y,v)}\\ &+\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\Box G_2)\\ xu\notin E_1,yv\in E_2}} \frac{1}{D_1+D_2-d_{G_1}(x,u)}\\ &+\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\Box G_2)\\ xu\notin E_1,yv\notin E_2}} \frac{1}{D_1+D_2+1-d_{G_1}(x,u)-d_{G_2}(y,v)}\\ <& \frac{n_1m_2}{D_1+D_2}+n_1\left(RCW(G_2)-\frac{m_2}{D_2}\right)+\frac{n_2m_1}{D_1+D_2}\\ &+n_2\left(RCW(G_1)-\frac{m_1}{D_1}\right)+\frac{2m_1m_2}{D_1+D_2-1}\\ &+2m_1\left(RCW(G_2)-\frac{m_2}{D_2}\right)+2m_2\left(RCW(G_1)-\frac{m_1}{D_1}\right)\\ &+2m_2\left(RCW(G_1)-\frac{m_1}{D_1}\right) \end{aligned}$$

$$
\begin{aligned}
=&\frac{n_1m_2+n_2m_1}{D_1+D_2}+\frac{2m_1m_2}{D_1+D_2-1}+(n_1+2m_1)\left(RCW(G_2)-\frac{m_2}{D_2}\right)\\
&+(n_2+4m_2)\left(RCW(G_1)-\frac{m_1}{D_1}\right). \qquad \square
\end{aligned}
$$

Corollary 2.1. *Let $G_1 \cong K_{n_1}$ and $G_2 \ncong K_{n_2}$ be two graphs. Then*

$$
\begin{aligned}
RCW(G_1\Box G_2)=&\frac{n_1m_2+n_2\binom{n_1}{2}}{1+D_2}+\frac{2\binom{n_1}{2}m_2}{D_2}+2\binom{n_1}{2}\left(RCW(G_2)-\frac{m_2}{D_2}\right)\\
&+\sum_{yv\notin E_2}\frac{n_1}{D_2+2-d_{G_2}(y,v)}\\
<&\frac{n_1m_2+n_2\binom{n_1}{2}}{1+D_2}+\frac{2\binom{n_1}{2}m_2}{D_2}+\left(n_1+2\binom{n_1}{2}\right)\left(RCW(G_2)-\frac{m_2}{D_2}\right).
\end{aligned}
$$

Example 2.3. Consider the graph whose vertices are the n-tuples $b_1, b_2, \ldots, b_n$ with $b_i \in \{0,1\}$, let two vertices be adjacent if the corresponding tuples differ in precisely one place. Such a graph is called a hypercube of dimension n and denoted by Q_n. It is well-known fact that the hypercube Q_n can be written in the form $Q_n = \underbrace{K_2\Box K_2\Box\cdots\Box K_2}_{n\,\text{times}}$.

For $n=3$, by Corollary 2.1 we have

$$RCW(Q_3)=K_2\Box K_2\Box K_2=K_2\Box C_4=14.$$

Corollary 2.2. *Let $L_{2,n}$ be the ladder graph, then*

$$RCW(L_{2,n})=4n-1-2\sum_{k=1}^{n}\frac{1}{k}.$$

Proof.

$$
\begin{aligned}
RCW(L_{2,n})=&RCW(P_2\Box P_n)\\
=&\frac{2(n-1)+n}{n}+\frac{2(n-1)}{n-1}+2(n-2)+2\sum_{yv\notin E(P_n)}\frac{1}{n+1-d_{P_n}(y,v)}\\
=&-\frac{2}{n}+2n+1+2\sum_{k=2}^{n-1}\frac{D(P_n,k)}{n+1-k}.
\end{aligned}
$$

On the other hand, it is clear that $D(P_n,k)=n-k$, for $k=1,\ldots,n-1$. Therefore,

$$
\begin{aligned}
RCW(L_{2,n})=&-\frac{2}{n}+2n+1+2\sum_{k=2}^{n-1}\frac{n-k}{n+1-k}\\
=&4n-1-2\sum_{k=1}^{n}\frac{1}{k}. \qquad \square
\end{aligned}
$$

Corollary 2.3. *Let $G_1 \cong K_{n_1}$ and $G_2 \cong K_{n_2}$ be two complete graphs. Then*

$$RCW(G_1\Box G_2)=\frac{n_1n_2}{4}(2n_1n_2-n_1-n_2).$$

2.3. **Composition.** The *composition* $G_1[G_2]$ (also known as the graph *lexicographic product*) of simple undirected graphs G_1 and G_2 is the graph with the vertex set $V(G_1[G_2]) = V_1 \times V_2$ and any two vertices (x, y) and (u, v) are adjacent if and only if $xu \in E_1$ or ($x = u$ and $yv \in E_2$).

Let G_1 and G_2 be graphs on $n_1 > 1$ and n_2 vertices, respectively. It follows from the definition that the distance between two distinct vertices (x, y) and (u, v) of $G_1[G_2]$ is given by

$$d_{G_1[G_2]}\Big((x,y),(u,v)\Big) = \begin{cases} 0, & x = u \text{ and } y = v, \\ 1, & x = u \text{ and } yv \in E_2, \\ 2, & x = u \text{ and } yv \notin E_2, \\ d_{G_1}(x,u), & x \neq u. \end{cases}$$

Note that if $G_1 \cong K_1$ then $G_1[G_2] \cong G_2$. So, in the following lemma we study the reciprocal complementary Wiener number of composition $G_1[G_2]$ for case $n_1 > 1$.

Theorem 2.3. *Let G_1 and G_2 be two graphs on $n_1 > 1$ and n_2 vertices, respectively.*

(i) *If G_1 is a non-complete graph, then*

$$RCW(G_1[G_2]) = \frac{n_1 m_2}{D_1} + \frac{n_1\left(\binom{n_2}{2} - m_2\right)}{D_1 - 1} + n_2^2 RCW(G_1).$$

(ii) *If $G_1 \cong K_{n_1}$ and G_2 is a non-complete graph, then*

$$RCW(G_1[G_2]) = \frac{n_1 m_2}{2} + n_1\left(\binom{n_2}{2} - m_2\right) + \frac{n_2^2}{2}\binom{n_1}{2}.$$

(iii) *If G_1 and G_2 are complete graphs, then*

$$RCW(G_1[G_2]) = (n_1 + n_2^2)\binom{n_1}{2}.$$

Proof. By the definition of the composition of two graphs one can see that,

$$D = D(G_1[G_2]) = \begin{cases} 1, & G_1 \cong K_{n_1} \text{ and } G_2 \cong K_{n_2}, \\ 2, & G_1 \cong K_{n_1} \text{ and } G_2 \ncong K_{n_2}, \\ D_1 = D(G_1), & G_1 \ncong K_{n_1}. \end{cases}$$

Suppose G_1 and G_2 are non-complete graphs, then

$$\begin{aligned} RCW(G_1[G_2]) &= \sum_{\{(x,y),(u,v)\} \subseteq V(G_1[G_2])} \frac{1}{D + 1 - d_{G_1[G_2]}\Big((x,y),(u,v)\Big)} \\ &= \sum_{\substack{\{(x,y),(x,v)\} \subseteq V(G_1[G_2]) \\ yv \in E_2}} \frac{1}{D_1} + \sum_{\substack{\{(x,y),(x,v)\} \subseteq V(G_1[G_2]) \\ yv \notin E_2}} \frac{1}{D_1 - 1} \\ &\quad + \sum_{\substack{\{(x,y),(u,v)\} \subseteq V(G_1[G_2]) \\ x \neq u}} \frac{1}{D_1 + 1 - d_{G_1}(x,u)} \end{aligned}$$

$$=\frac{n_1m_2}{D_1}+\frac{n_1\left(\binom{n_2}{2}-m_2\right)}{D_1-1}+n_2^2RCW(G_1),$$

which completes part (i).

The proof is completed by a similar argument as proof of the first part. □

2.4. **Disjunction.** The *disjunction* $G_1 \wedge G_2$ of graphs G_1 and G_2 is the graph with vertex set $V_1 \times V_2$ and (x, y) is adjacent with (u, v) whenever $xu \in E_1$ or $yv \in E_2$.

Let G_1 and G_2 be graphs on $n_1 > 1$ and $n_2 > 1$ vertices, respectively. Clearly, the distance between two vertices (x, y) and (u, v) of $G_1 \wedge G_2$ is given by

$$d_{G_1\wedge G_2}\Big((x,y),(u,v)\Big)=\begin{cases}0, & x=u \text{ and } y=v,\\ 1, & xu\in E_1 \text{ or } yv\in E_2,\\ 2, & \text{otherwise.}\end{cases}$$

Note that if $n_i = 1$ for some $i \in \{1, 2\}$, then $G_1 \wedge G_2 \cong G_{n'_i}$, where $n'_i = 3 - i$. So, we determine the reciprocal complementary Wiener number of disjunction $G_1 \wedge G_2$ for cases $n_1 > 1$ and $n_2 > 1$.

Theorem 2.4. *Let G_1 and G_2 be graphs on $n_1 > 1$ and $n_2 > 1$ vertices, respectively.*

(i) *If G_1 and G_2 are complete graphs, then*

$$RCW(G_1\wedge G_2)=\frac{1}{2}\left[n_1\binom{n_2}{2}+n_2\binom{n_1}{2}+\binom{n_1}{2}\binom{n_2}{2}\right].$$

(ii) *If $\{G_1, G_2\} \neq \{K_{n_1}, K_{n_2}\}$, then*

$$RCW(G_1\wedge G_2)=\frac{1}{2}\left(n_1^2n_2^2+2m_1m_2-m_2n_1^2-m_1n_2^2-n_1n_2\right).$$

Proof. From definition of disjunction it is clear that if at least one of graphs G_1 and G_2 is not complete, then $D = D(G_1 \wedge G_2) = 2$, otherwise $D = 1$. To prove part (ii), assume that $\{G_1, G_2\} \neq \{K_{n_1}, K_{n_2}\}$. Hence, we can write

$$\begin{aligned}RCW(G_1\wedge G_2)&=\sum_{\{(x,y),(u,v)\}\subseteq V(G_1\wedge G_2)}\frac{1}{D+1-d_{G_1\wedge G_2}\Big((x,y),(u,v)\Big)}\\ &=\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\wedge G_2)\\ xu\in E_1}}\frac{1}{2}+\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\wedge G_2)\\ yv\in E_2}}\frac{1}{2}\\ &\quad-\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\wedge G_2)\\ xu\in E_1, yv\in E_2}}\frac{1}{2}+\sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\wedge G_2)\\ yv\notin E_2}}1\\ &\quad+\sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\wedge G_2)\\ xu\notin E_1}}1+\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\wedge G_2)\\ xu\notin E_1, yv\notin E_2}}1\\ &=\frac{1}{2}\left(n_1^2n_2^2+2m_1m_2-m_2n_1^2-m_1n_2^2-n_1n_2\right).\end{aligned}$$

A similar argument as part (ii), shows that

$$RCW(K_{n_1} \wedge K_{n_2}) = \frac{1}{2}\left[n_1\binom{n_2}{2} + n_2\binom{n_1}{2} + \binom{n_1}{2}\binom{n_2}{2}\right].$$

This completes the proof. □

2.5. **Strong product.** The *strong product* of graphs G_1 and G_2, denoted by $G_1 \boxtimes G_2$, is the graph with vertex set $V_1 \times V_2$ and $(x,y)(u,v)$ is an edge whenever ($x = u$ and $yv \in E_2$), or ($y = v$ and $xu \in E_1$), or ($xu \in E_1$ and $yv \in E_2$).

In the following result, we give a basic property about the strong product of graphs.

Lemma 2.1 ([4]). *Let G_1 and G_2 be two connected graphs, $x, u \in V(G_1)$ and $y, v \in V(G_2)$. Then* $d_{G_1 \boxtimes G_2}\big((x,y),(u,v)\big) = \max\{d_{G_1}(x,u), d_{G_2}(y,v)\}$.

Corollary 2.4. *Let $G_1 \boxtimes G_2$ be the strong product of connected graphs G_1 and G_2. Then $D = \max\{D_1, D_2\}$, where D, D_1 and D_2 are the diameter of $G_1 \boxtimes G_2$, G_1 and G_2, respectively.*

Theorem 2.5. *Let $G_1 \boxtimes G_2$ be the strong product of connected graphs G_1 and G_2. Then*

$$\begin{aligned} RCW(G_1 \boxtimes G_2) \leq{}& \frac{1}{D}(2m_1m_2 + n_1m_2 + n_2m_1) + (n_1 + 2m_1)\left(RCW(G_2) - \frac{m_2}{D_2}\right) \\ &+ (n_2 + 2m_2)\left(RCW(G_1) - \frac{m_1}{D_1}\right) + 2\left[\binom{n_1}{2} - m_1\right]\left[\binom{n_2}{2} - m_2\right]. \end{aligned}$$

The equality is satisfied if and only if G_1 or G_2 is a complete graph.

Proof.

$$\begin{aligned} RCW(G_1 \boxtimes G_2) ={}& \sum_{\{(x,y),(u,v)\} \subseteq V(G_1 \boxtimes G_2)} \frac{1}{D + 1 - d_{G_1 \boxtimes G_2}\big((x,y),(u,v)\big)} \\ ={}& \sum_{\substack{\{(x,y),(x,v)\} \subseteq V(G_1 \boxtimes G_2) \\ yv \in E_2}} \frac{1}{D} + \sum_{\substack{\{(x,y),(x,v)\} \subseteq V(G_1 \boxtimes G_2) \\ yv \notin E_2}} \frac{1}{D + 1 - d_{G_2}(y,v)} \\ &+ \sum_{\substack{\{(x,y),(u,y)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \in E_1}} \frac{1}{D} + \sum_{\substack{\{(x,y),(u,y)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \notin E_1}} \frac{1}{D + 1 - d_{G_1}(x,u)} \\ &+ \sum_{\substack{\{(x,y),(u,v)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \in E_1, yv \in E_2}} \frac{1}{D} + \sum_{\substack{\{(x,y),(u,v)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \in E_1, yv \notin E_2}} \frac{1}{D + 1 - d_{G_2}(y,v)} \\ &+ \sum_{\substack{\{(x,y),(u,v)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \notin E_1, yv \in E_2}} \frac{1}{D + 1 - d_{G_1}(x,u)} \\ &+ \sum_{\substack{\{(x,y),(u,v)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \notin E_1, yv \notin E_2}} \frac{1}{D + 1 - \max\{d_{G_1}(x,u), d_{G_2}(y,v)\}}. \end{aligned}$$

By (1.1), we have

$$\begin{aligned} n_1 RCW(G_2) &= \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\notin E_2}} \frac{1}{D_2+1-d_{G_2}(y,v)} \\ &\quad + \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\in E_2}} \frac{1}{D_2+1-\underbrace{d_{G_2}(y,v)}_{1}} \\ &= \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\notin E_2}} \frac{1}{D_2+1-d_{G_2}(y,v)} + \frac{n_1 m_2}{D_2}, \end{aligned}$$

hence,

$$\begin{aligned} \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\notin E_2}} \frac{1}{D+1-d_{G_2}(y,v)} &\leq \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\notin E_2}} \frac{1}{D_2+1-d_{G_2}(y,v)} \\ &= n_1\left(RCW(G_2)-\frac{m_2}{D_2}\right). \end{aligned}$$

Similarly, we can check that

$$\begin{aligned} \sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\boxtimes G_2)\\ xu\notin E_1}} \frac{1}{D+1-d_{G_1}(x,u)} &\leq n_2\left(RCW(G_1)-\frac{m_1}{D_1}\right), \\ \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\boxtimes G_2)\\ xu\in E_1, yv\notin E_2}} \frac{1}{D+1-d_{G_2}(y,v)} &\leq 2m_1\left(RCW(G_2)-\frac{m_2}{D_2}\right), \\ \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\boxtimes G_2)\\ xu\notin E_1, yv\in E_2}} \frac{1}{D+1-d_{G_1}(x,u)} &\leq 2m_2\left(RCW(G_1)-\frac{m_1}{D_1}\right). \end{aligned}$$

Also, we have

$$\begin{aligned} \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\boxtimes G_2)\\ xu\notin E_1, yv\notin E_2}} \frac{1}{D+1-\underbrace{\max\{d_{G_1}(x,u),d_{G_2}(y,v)\}}_{\leq D}} &\leq \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\boxtimes G_2)\\ xu\notin E_1, yv\notin E_2}} 1 \\ &= 2\left[\binom{n_1}{2}-m_1\right]\left[\binom{n_2}{2}-m_2\right]. \end{aligned}$$

On the other hand, it is easy to see that

$$\sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\in E_2}} \frac{1}{D} = \frac{n_1 m_2}{D}, \qquad \sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\boxtimes G_2)\\ xu\in E_1}} \frac{1}{D} = \frac{n_2 m_1}{D},$$

$$\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\boxtimes G_2)\\ xu\in E_1, yv\in E_2}} \frac{1}{D} = \frac{2m_1 m_2}{D}.$$

Therefore,

$$RCW(G_1 \boxtimes G_2) \leq \frac{1}{D}(2m_1m_2 + n_1m_2 + n_2m_1) + (n_1 + 2m_1)\left(RCW(G_2) - \frac{m_2}{D_2}\right) + (n_2 + 2m_2)\left(RCW(G_1) - \frac{m_1}{D_1}\right) + 2\left[\binom{n_1}{2} - m_1\right]\left[\binom{n_2}{2} - m_2\right].$$

□

Using similar arguments as in the proof of Theorem 2.5, one can prove the following result.

Lemma 2.2. *Let K_r be a complete graph on r vertices and G be a graph with n vertices, m edges and diameter d. Then*

$$RCW(G \boxtimes K_r) = \frac{1}{d}\left[rm + dr^2\left(RCW(G) - \frac{m}{d}\right) + (n + 2m)\binom{r}{2}\right].$$

Example 2.4. By the definition of the composition and strong product of two graphs one can see that, $G[K_n] = G \boxtimes K_n$. The open fence graph is the composition (or strong product) of path P_n and K_2. So, from Theorem 2.3 (i) (or Lemma 2.2), we have

$$RCW(P_n[K_2]) = RCW(P_n \boxtimes K_2) = \frac{n}{n-1} + 4n - 4, \quad n \geq 3.$$

As an application, in the following result, we obtain the reciprocal complementary Wiener number of the closed fence graph $C_n \boxtimes K_2$.

Lemma 2.3. *Let C_n be a cycle graph on n vertices. Then*

$$RCW(C_n \boxtimes K_2) = \begin{cases} 4n \sum_{k=1}^{\frac{n}{2}} \frac{1}{k} - 2n + 8, & 2 \mid n, \\ 4n \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{k} + \frac{2n}{n-1}, & 2 \nmid n. \end{cases}$$

Proof. We first obtain the reciprocal complementary Wiener number of a cycle graph C_n on n vertices. Regarding the structure of the cycle C_n, it can easily be concluded that if n is even then $D(C_n, k) = n$, $k = 1, 2, \ldots, \frac{n}{2} - 1$ and $D(C_n, \frac{n}{2}) = \frac{n}{2}$. On the other hand, if n is odd then $D(C_n, k) = n$, $k = 1, 2, \ldots, \frac{n-1}{2}$. Hence, by applying relation (1.3), we have

$$RCW(C_n) = \begin{cases} -\frac{n}{2} + n \sum_{k=1}^{\frac{n}{2}} \frac{1}{k}, & 2 \mid n, \\ n \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{k}, & 2 \nmid n. \end{cases}$$

Finally, the proof is completed using Lemma 2.2. □

2.6. **Symmetric difference.** The *symmetric difference* $G_1 \oplus G_2$ of graphs G_1 and G_2 is the graph with vertex set $V_1 \times V_2$ and (x, y) is adjacent with (u, v) whenever $xu \in E_1$ or $yv \in E_2$ but not both. Note that if $n_i = 1$ for some $i \in \{1, 2\}$, then $G_1 \oplus G_2 \cong G_{n_i'}$, where $n_i' = 3 - i$.

In the following lemma, we compute the symmetric difference of two graphs with respect to their numbers of vertices and edges.

Theorem 2.6. *Let G_1 and G_2 be two graphs on $n_1 > 1$ and $n_2 > 1$ vertices, respectively. Then*

$$RCW(G_1 \oplus G_2) = \frac{1}{2}\Big(4m_1m_2 + n_1^2n_2^2 - m_1n_2^2 - m_2n_1^2 - n_1n_2\Big).$$

Proof. By [8, Lemma 4], we have

$$d_{G_1\oplus G_2}\Big((x, y), (u, v)\Big) = \begin{cases} 0, & x = u \text{ and } y = v, \\ 1, & xu \in E_1 \text{ or } yv \in E_2, \text{ but not both,} \\ 2, & \text{otherwise.} \end{cases}$$

Hence, by applying these relations, we get $D = D(G_1 \oplus G_2) = 2$. So,

$$\begin{aligned} RCW(G_1 \oplus G_2) &= \sum_{\{(x,y),(u,v)\}\subseteq V(G_1\oplus G_2)} \frac{1}{D + 1 - d_{G_1\oplus G_2}\Big((x, y), (u, v)\Big)} \\ &= \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\oplus G_2)\\ yv\in E_2}} \frac{1}{2} + \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\oplus G_2)\\ yv\notin E_2}} 1 \\ &+ \sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\oplus G_2)\\ xu\in E_1}} \frac{1}{2} + \sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\oplus G_2)\\ xu\notin E_1}} 1 \\ &+ \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\oplus G_2)\\ xu\in E_1, yv\notin E_2}} \frac{1}{2} + \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\oplus G_2)\\ xu\in E_1, yv\in E_2}} 1 \\ &+ \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\oplus G_2)\\ xu\notin E_1, yv\in E_2}} \frac{1}{2} + \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\oplus G_2)\\ xu\notin E_1, yv\notin E_2}} 1 \\ &= \frac{1}{2}\Big(4m_1m_2 + n_1^2n_2^2 - m_1n_2^2 - m_2n_1^2 - n_1n_2\Big). \end{aligned}$$

□

2.7. **Corona product.** Let $V_1 = \{u_1, u_2, \dots, u_{n_1}\}$ and $V_2 = \{v_1, v_2, \dots, v_{n_2}\}$ be the vertex sets of given graphs G_1 and G_2, respectively. The *corona product* of G_1 and G_2 is denoted by $G_1 \circ G_2$ and defined as the graph obtained by taking n_1 copies of G_2 and joining each vertex of the i^{th} copy with vertex u_i of V_1, $i = 1, 2, \dots, n_1$. Denote by G_2^i the i^{th} copy of G_2 joined to the vertex u_i of G_1, and let $V_2^i = \{v_{i1}, v_{i2}, \dots, v_{in_2}\}$, $i = 1, 2, \dots, n_1$.

Theorem 2.7. *Let G_1 and G_2 be two graphs on $n_1 > 1$ and $n_2 \geq 1$ vertices, respectively. Then*

$$RCW(G_1 \circ G_2) < (n_2+1)^2 RCW(G_1) + \frac{n_1 n_2}{D_1+2} + \frac{n_1\left[\binom{n_2}{2} - m_2\right]}{D_1+1}.$$

Proof. From definition of the corona product of graphs, it is easy to check that

$$\begin{aligned}
d_{G_1 \circ G_2}(u_i, u_p) &= d_{G_1}(u_i, u_p),\\
d_{G_1 \circ G_2}(u_i, v_{pq}) &= d_{G_1}(u_i, u_p) + 1,\\
d_{G_1 \circ G_2}(v_{ij}, v_{pq}) &= \begin{cases} 0, & i = p \text{ and } j = q,\\ 1, & i = p \text{ and } v_j v_q \in E_2,\\ 2, & i = p \text{ and } v_j v_q \notin E_2,\\ d_{G_1}(u_i, u_p) + 2, & i \neq p. \end{cases}
\end{aligned}$$

So, we can see that $D = D(G_1 \circ G_2) = D_1 + 2$. Hence,

$$\begin{aligned}
RCW(G_1 \circ G_2) &= \sum_{\{x,y\} \subseteq V(G_1 \circ G_2)} \frac{1}{D + 1 - d_{G_1 \circ G_2}(x, y)}\\
&= \sum_{\{x,y\} \subseteq V_1} \frac{1}{D + 1 - d_{G_1 \circ G_2}(x, y)}\\
&\quad + \sum_{i=1}^{n_1} \sum_{\{v_{ij}, v_{iq}\} \subseteq V_2^i} \frac{1}{D + 1 - d_{G_1 \circ G_2}(v_{ij}, v_{iq})}\\
&\quad + \sum_{i=1}^{n_1} \sum_{p=1}^{n_1} \sum_{q=1}^{n_2} \frac{1}{D + 1 - d_{G_1 \circ G_2}(u_i, v_{pq})}\\
&\quad + \sum_{i=1}^{n_1-1} \sum_{p=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{q=1}^{n_2} \frac{1}{D + 1 - d_{G_1 \circ G_2}(v_{ij}, v_{pq})}.
\end{aligned}$$

Consider now for convenience:

$$\begin{aligned}
S_1 &= \sum_{\{x,y\} \subseteq V_1} \frac{1}{D + 1 - d_{G_1 \circ G_2}(x, y)},\\
S_2 &= \sum_{i=1}^{n_1} \sum_{\{v_{ij}, v_{iq}\} \subseteq V_2^i} \frac{1}{D + 1 - d_{G_1 \circ G_2}(v_{ij}, v_{iq})},\\
S_3 &= \sum_{i=1}^{n_1} \sum_{p=1}^{n_1} \sum_{q=1}^{n_2} \frac{1}{D + 1 - d_{G_1 \circ G_2}(u_i, v_{pq})},\\
S_4 &= \sum_{i=1}^{n_1-1} \sum_{p=i+1}^{n_1} \sum_{j=1}^{n_2} \sum_{q=1}^{n_2} \frac{1}{D + 1 - d_{G_1 \circ G_2}(v_{ij}, v_{pq})}.
\end{aligned}$$

So, we have

$$
\begin{aligned}
S_1 &= \sum_{\{x,y\}\subseteq V_1} \frac{1}{D+1-d_{G_1\circ G_2}(x,y)} \\
&= \sum_{\{x,y\}\subseteq V_1} \frac{1}{D_1+3-d_{G_1}(x,y)} \\
&< RCW(G_1), \\
S_2 &= \sum_{i=1}^{n_1} \sum_{\{v_{ij},v_{iq}\}\subseteq V_2^i} \frac{1}{D+1-d_{G_1\circ G_2}(v_{ij},v_{iq})} \\
&= \sum_{i=1}^{n_1} \sum_{\substack{\{v_{ij},v_{iq}\}\subseteq V_2^i \\ v_jv_q\in E_2}} \frac{1}{D_1+2} + \sum_{i=1}^{n_1} \sum_{\substack{\{v_{ij},v_{iq}\}\subseteq V_2^i \\ v_jv_q\notin E_2}} \frac{1}{D_1+1} \\
&= \frac{n_1m_2}{D_1+2} + \frac{n_1\left[\binom{n_2}{2}-m_2\right]}{D_1+1}, \\
S_3 &= \sum_{i=1}^{n_1}\sum_{p=1}^{n_1}\sum_{q=1}^{n_2} \frac{1}{D+1-d_{G_1\circ G_2}(u_i,v_{pq})} \\
&= \sum_{i=1}^{n_1}\sum_{q=1}^{n_2} \frac{1}{D_1+2} + \sum_{i=1}^{n_1}\sum_{\substack{p=1 \\ p\neq i}}^{n_1}\sum_{q=1}^{n_2} \frac{1}{D_1+2-d_{G_1}(u_i,u_p)} \\
&< \frac{n_1n_2}{D_1+2} + 2n_2RCW(G_1), \\
S_4 &= \sum_{i=1}^{n_1-1}\sum_{p=i+1}^{n_1}\sum_{j=1}^{n_2}\sum_{q=1}^{n_2} \frac{1}{D+1-d_{G_1\circ G_2}(v_{ij},v_{pq})} \\
&= \sum_{i=1}^{n_1-1}\sum_{p=i+1}^{n_1}\sum_{j=1}^{n_2}\sum_{q=1}^{n_2} \frac{1}{D_1+1-d_{G_1}(u_i,u_p)} \\
&= n_2^2RCW(G_1).
\end{aligned}
$$

Therefore,

$$
RCW(G_1\circ G_2) < (n_2+1)^2RCW(G_1) + \frac{n_1n_2}{D_1+2} + \frac{n_1\left[\binom{n_2}{2}-m_2\right]}{D_1+1}. \qquad \square
$$

2.8. **Splice and link.** Let G_1 and G_2 be two connected graphs with disjoint vertex sets V_1 and V_2, respectively. For given vertices $u \in V_1$ and $v \in V_2$, a *splice* of G_1 and G_2 by vertices u and v is denoted by $(G_1.G_2)(u,v)$ and defined by identifying the vertices u and v in the union of G_1 and G_2. Also, a *link* of G_1 and G_2 by vertices u and v is denoted by $(G_1 \sim G_2)(u,v)$ and obtained by joining u and v by an edge in the union of these graphs.

Theorem 2.8. *Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, respectively. Then*

(i) $RCW\Big((G_1.G_2)(u,v)\Big) \le (n_1-1)(n_2-1) + RCW(G_1) + RCW(G_2)$;

(ii) $RCW\Big((G_1 \sim G_2)(u,v)\Big) \le n_1 n_2 + RCW(G_1) + RCW(G_2)$.

Equality in (i) *holds if and only if one of the following cases occurs:*

(i_1) $n_i = 1$, *for some* $i \in \{1,2\}$;

(i_2) G_1 *and* G_2 *are non-complete graphs and* $\varepsilon_{G_1}(u) = \varepsilon_{G_2}(v) = 1$.

Moreover, equality in (ii) *holds if and only if* $n_1 = n_2 = 1$.

Proof. Suppose $\dot{D}$ and $\tilde{D}$ are the diameter of the splice and link of graphs G_1 and G_2 by vertices u and v, respectively. By above definitions of the splice and link of graphs, one can easily see that

$$d_{(G_1.G_2)(u,v)}(x,y) = \begin{cases} d_{G_1}(x,y), & x,y \in V_1, \\ d_{G_2}(x,y), & x,y \in V_2, \\ d_{G_1}(x,u) + d_{G_2}(y,v), & x \in V_1 \text{ and } y \in V_2, \end{cases}$$

and also,

$$d_{(G_1 \sim G_2)(u,v)}(x,y) = \begin{cases} d_{G_1}(x,y), & x,y \in V_1, \\ d_{G_2}(x,y), & x,y \in V_2, \\ d_{G_1}(x,u) + d_{G_2}(y,v) + 1, & x \in V_1 \text{ and } y \in V_2. \end{cases}$$

Hence, in graph $(G_1.G_2)(u,v)$, if the endpoints of a diametral path (i.e. a shortest path between two vertices whose distance is equal to the diameter of the graph) are in the graph G_1 (or G_2) then $\dot{D}$=D_1 (or D_2), otherwise if one of these endpoints belongs to V_1 and the other endpoint belongs to V_2, then $\dot{D}$=$\varepsilon_{G_1}(u) + \varepsilon_{G_2}(v)$. Thus, $\dot{D}$=max$\{D_1, D_2, \varepsilon_{G_1}(u) + \varepsilon_{G_2}(v)\}$. Similarly, $\tilde{D}$=max$\{D_1, D_2, \varepsilon_{G_1}(u) + \varepsilon_{G_2}(v) + 1\}$. By applying the above obtained relationships and also definitions of the splice and link of graphs, it is obvious that if $n_1 = 1$ or $n_2 = 1$, then the equality in (i) holds. Assume that $n_1, n_2 \ge 2$, then

$$\begin{aligned} RCW\Big((G_1.G_2)(u,v)\Big) &= \sum_{\{x,y\} \subseteq V\left((G_1.G_2)(u,v)\right)} \frac{1}{\dot{D} + 1 - d_{(G_1.G_2)(u,v)}(x,y)} \\ &= \sum_{\{x,y\} \subseteq V_1} \frac{1}{\dot{D} + 1 - d_{G_1}(x,y)} + \sum_{\{x,y\} \subseteq V_2} \frac{1}{\dot{D} + 1 - d_{G_2}(x,y)} \\ &\quad + \sum_{\substack{x \in V_1 \setminus \{u\} \\ y \in V_2 \setminus \{v\}}} \frac{1}{\dot{D} + 1 - d_{G_1}(x,u) - d_{G_2}(y,v)} \\ &\le (n_1-1)(n_2-1) + RCW(G_1) + RCW(G_2), \end{aligned}$$

and equality holds when $\dot{D} = D_1 = D_2 = d_{G_1}(x,u) + d_{G_2}(y,v)$, for all $x \in V_1 \setminus \{u\}$ and $y \in V_2 \setminus \{v\}$. On the other hand, since G_1 and G_2 are connected graphs, we conclude that equality holds if and only if $d_{G_1}(x,u) = d_{G_2}(y,v) = 1$ and $D_1 = D_2 = 2$, for all $x \in V_1 \setminus \{u\}$ and $y \in V_2 \setminus \{v\}$. This means that G_1 and G_2 are non-complete graphs and $\varepsilon_{G_1}(u) = \varepsilon_{G_2}(v) = 1$, which completes the proof of part (i). The proof of part (ii) can be completed by using the similar arguments as in the proof of part (i). $\square$

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Department of Mathematics,
Imam Hossein Comprehensive University,
Tehran, Iran
Email address: R.Nasiri.82@gmail.com
Email address: kpnakhaei@ihu.ac.ir
Email address: ashojaeifard@ihu.ac.ir