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$A^{\mathcal{I}}$-STATISTICAL APPROXIMATION OF CONTINUOUS FUNCTIONS BY SEQUENCE OF CONVOLUTION OPERATORS

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ABSTRACT. In this paper, following the concept of $A^{\mathcal{I}}$-statistical convergence for real sequences introduced by Savas et al. [22], we deal with Korovkin type approximation theory for a sequence of positive convolution operators defined on $C[a,b]$, the space of all real valued continuous functions on $[a,b]$, in the line of Duman [6]. In the Section 3, we study the rate of $A^{\mathcal{I}}$-statistical convergence.

1. Introduction and Background

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers and $C[a,b]$ denotes the space of all real valued continuous functions defined on $[a,b]$, endowed with the supremum norm $||f|| = \sup_{x\in[a,b]} |f(x)|$ for $f \in C[a,b]$. For a sequence $\{T_n\}_{n\in\mathbb{N}}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [14] first established the necessary and sufficient conditions for the uniform convergence of $\{T_n(f)\}_{n\in\mathbb{N}}$ to a function f by using the test functions $e_1 = 1,\ e_2 = x,\ e_3 = x^2$ (see [1]). The study of the Korovkin type approximation theory has a long history and is a well-established area of research (see [4, 5, 7–11]).

Our primary interest, in this paper is to obtain a general Korovkin type approximation theorem for a sequence of positive convolution operators defined on $C[a,b]$, in $A^{\mathcal{I}}$-statistical sense. In the section 3, we study the rate of $A^{\mathcal{I}}$-statistical convergence.

The concept of statistical convergence of a sequence of real numbers was first introduced by Fast [12]. This is a generalization of usual convergence. Further investigations started in this area after the works of Šalát [19] and Fridy [13]. Consequently,

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the notion of $\mathcal{I}$-convergence of real sequences was introduced by Kostyrko et al. [17]. On the other hand statistical convergence was generalized to A-statistical convergence by Kolk ([15, 16]). Later a lot of works have been done on matrix summability and A-statistical convergence (see [2, 3, 15, 16, 18, 20]). In particular, in [21, 22] the very general notion of $A^{\mathcal{I}}$-statistical convergence was introduced.

Recall that a family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; (ii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If $\mathcal{I}$ is a non-trivial proper ideal in Y (i.e., $Y \notin \mathcal{I}$, $\mathcal{I} \neq \{\emptyset\}$) then the family of sets $F(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$. The real number sequence $\{x_k\}_{k\in\mathbb{N}}$ is said to be $\mathcal{I}$-convergent to L provided that for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}$.

If $\{x_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers and $A = (a_{nk})$ is an infinite matrix, then Ax is the sequence whose n-th term is given by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

We say that x is A-summable to L if $\lim_{n\to\infty} A_n(x) = L$. A matrix A is called regular if $A \in (c, c)$ and $\lim_{k\to\infty} A_k(x) = \lim_{k\to\infty} x_k$ for all $x = \{x_k\}_{k\in\mathbb{N}} \in c$, when c, as usual, stands for the set of all convergent sequences. It is well-known that the necessary and sufficient conditions for A to be regular are

I) $||A|| = \sup\limits_n \sum\limits_k |a_{nk}| < \infty$;

II) $\lim\limits_n a_{nk} = 0$, for each k;

III) $\lim\limits_n \sum\limits_k a_{nk} = 1$.

For a non-negative regular matrix $A = (a_{nk})$ following [15], a set K is said to have A-density if $\delta_A(K) = \lim_n \sum_{k\in K} a_{nk}$ exists.

The real number sequence $\{x_k\}_{k\in\mathbb{N}}$ is A-statistically convergent to L provided that for every $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has A-density zero (see [15]). Throughout the paper $\mathcal{I}$ will denote the non-trivial admissible ideal on $\mathbb{N}$.

2. $A^{\mathcal{I}}$-Statistical Approximation for a Sequence of Convolution Operators

We first recall the definition.

Definition 2.1 ([21, 22])**.** Let $A = (a_{nk})$ be a non-negative regular matrix. For an ideal $\mathcal{I}$ of $\mathbb{N}$, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L if for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{ n \in \mathbb{N} : \sum_{k\in K(\varepsilon)} a_{nk} \geq \delta \right\} \in \mathcal{I}$$

where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. In this case we write $A^{\mathcal{I}}\text{-st-}\lim_n x_n = L$.

Note that for $\mathcal{I} = \mathcal{I}_{fin}$, the ideal of all finite subsets of $\mathbb{N}$, $A^{\mathcal{I}}$-statistical convergence becomes A-statistical convergence [15].

We consider the Banach space $C[a, b]$ endowed with the supremum norm $||f|| = \sup_{x\in[a,b]} |f(x)|$ for $f \in C[a, b]$. Let L be a positive linear operator. Then $L(f) \geq 0$ for any positive function f. Also, we denote the value of $L(f)$ at a point $x \in [a, b]$ by $L(f; x)$.

Theorem 2.1. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. If $A^{\mathcal{I}}$-st-$\lim_n \|L_n(f_i) - f_i\| = 0$, with $f_i = t^i$, $i = 0, 1, 2$, then for all $f \in C[a, b]$ we have $A^{\mathcal{I}}$-st-$\lim_n \|L_n(f) - f\| = 0$.*

Proof. Our objective is to show that for given $\varepsilon > 0$ there exist constants C_0, C_1, C_2 (depending on $\varepsilon > 0$) such that

$$\|L_n(f) - f\| \leq \varepsilon + C_2\|L_n(f_2) - f_2\| + C_1\|L_n(f_1) - f_1\| + C_0\|L_n(f_0) - f_0\|.$$

If this is done then our hypothesis implies that for $\varepsilon > 0$, $\delta > 0$

$$\left\{n \in \mathbb{N} : \sum_{k\in K(\varepsilon)} a_{nk} \geq \delta\right\} \in \mathcal{I},$$

where $K(\varepsilon) = \{k \in N : \|L_k(f) - f\| \geq \varepsilon\}$.

To this end, start by observing that for each $x \in [a, b]$ the function $0 \leq \Psi \in C[a, b]$ defined by $\Psi(t) = (t - x)^2$. Since each L_n is positive, $L_n(\Psi; x)$ is a positive function. In particular, we have

$$\begin{aligned}0 \leq L_n(\Psi; x) =& L_n(t^2; x) - 2xL_n(t; x) + x^2L_n(1; x)\\ =& (L_n(t^2; x) - t^2(x)) - 2x(L_n(t; x) - t(x)) + x^2(L_n(1; x) - 1(x))\\ \leq& \|L_n(t^2) - t^2\| + 2b\|L_n(t) - t\| + b^2\|L_n(1) - 1\|,\end{aligned}$$

for each $x \in [a, b]$. Let $M = \|f\|$. Since f is bounded on the whole real axis, we can write

$$|f(t) - f(x)| < 2M, \quad -\infty < t, x < \infty.$$

Also, since f is continuous on $[a, b]$, we have

$$|f(t) - f(x)| < \varepsilon,$$

for all t, x satisfying $|t - x| \leq \delta$.

On the other hand, if $|t - x| \geq \delta$, then it follows that

$$-\frac{2M}{\delta^2}(t - x)^2 \leq -2M \leq f(t) - f(x) \leq 2M \leq \frac{2M}{\delta^2}(t - x)^2.$$

Therefore, for all $t \in (-\infty, \infty)$ and all $x \in [a, b]$ we get

$$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2}(t - x)^2,$$

where δ is a fixed real number.

Since each L_n is positive, we have

$$\begin{aligned}-\varepsilon L_n(f_0;x)-\frac{2M}{\delta^2}L_n(\Psi;x)\leq & L_n(f(t);x)-f(x)L_n(f_0;x)\\ \leq & \varepsilon L_n(f_0;x)+\frac{2M}{\delta^2}L_n(\Psi;x).\end{aligned}$$

Next, let $K=\frac{2M}{\delta^2}$ and we get

$$\begin{aligned}|L_n(f(t);x)-f(x)L_n(f_0;x)|\leq & \varepsilon L_n(f_0;x)+\frac{2M}{\delta^2}L_n(\Psi;x)\\ = & \varepsilon+\varepsilon[L_n(f_0;x)-f_0(x)]+KL_n(\Psi;x)\\ \leq & \varepsilon+\varepsilon|L_n(f_0;x)-f_0(x)|+KL_n(\Psi;x).\end{aligned}$$

In particular,

$$\begin{aligned}|L_n(f(t);x)-f(x)|\leq & |L_n(f(t);x)-f(x)L_n(f_0;x)|+|f(x)||L_n(f_0;x)-f_0(x)|\\ \leq & \varepsilon+KL_n(\Psi;x)+(M+\varepsilon)|L_n(f_0;x)-f_0(x)|,\end{aligned}$$

which implies

$$\|L_n(f)-f\|\leq\varepsilon+C_2\|L_n(f_2)-f_2\|+C_1\|L_n(f_1)-f_1\|+C_0\|L_n(f_0)-f_0\|,$$

where $C_2=K$, $C_1=2bK$ and $C_0=(\varepsilon+b^2K+M)$, i.e.,

$$\|L_n(f)-f\|\leq\varepsilon+C\sum_{i=0}^{2}\|L_n(f_i)-f_i\|,\quad i=0,1,2,$$

where $C=\max\{C_0,C_1,C_2\}$. For a given $\varepsilon'>0$, choose $\varepsilon>0$ such that $\varepsilon<\varepsilon'$ and let us define the following sets

$$\begin{aligned}D= & \{n:\|L_n(f)-f\|\geq\varepsilon'\},\\ D_1= & \left\{n:\|L_n(f_0)-f_0\|\geq\frac{\varepsilon'-\varepsilon}{3C}\right\},\\ D_2= & \left\{n:\|L_n(f_1)-f_1\|\geq\frac{\varepsilon'-\varepsilon}{3C}\right\},\\ D_3= & \left\{n:\|L_n(f_2)-f_2\|\geq\frac{\varepsilon'-\varepsilon}{3C}\right\}.\end{aligned}$$

It follows that $D\subseteq D_1\cup D_2\cup D_3$ and consequently for all $n\in\mathbb{N}$

$$\sum_{k\in D}a_{nk}\leq\sum_{k\in D_1}a_{nk}+\sum_{k\in D_2}a_{nk}+\sum_{k\in D_3}a_{nk},$$

which implies that for any $\sigma>0$

$$\left\{n\in\mathbb{N}:\sum_{k\in D}a_{nk}\geq\sigma\right\}\subseteq\bigcup_{i=1}^{3}\left\{n\in\mathbb{N}:\sum_{k\in D_i}a_{nk}\geq\frac{\sigma}{3}\right\}.$$

Therefore, from hypothesis,

$$\left\{n \in \mathbb{N} : \sum_{k \in D} a_{nk} \geq \sigma\right\} \in \mathfrak{I}.$$

Hence, we have the proof. $\square$

We now consider the following convolution operators defined on $C[a,b]$ by

$$L_n(f;x) = \int_a^b f(y)K_n(y-x)dy, \quad n \in \mathbb{N},\ x \in [a,b] \text{ and } f \in C[a,b], \tag{2.1}$$

where a and b are two real numbers such that $a < b$. Throughout the paper we assume that K_n is a continuous function on $[a-b, b-a]$ and also that $K_n(u) \geq 0$ for all $n \in \mathbb{N}$ and for every $u \in [a-b, b-a]$. Consider the function Ψ on $[a,b]$ defined by $\Psi(y) = (y-x)^2$ for each $x \in [a,b]$.

Theorem 2.2. *Let $A = (a_{ij})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators from $C[a,b]$ into $C[a,b]$. If $A^{\mathfrak{I}}$-st-$\lim_n \|L_n(f_0) - f_0\| = 0$, with $f_0(y) = 1$ and $A^{\mathfrak{I}}$-st-$\lim_n \|L_n(\Psi)\| = 0$, then for all $f \in C[a,b]$ we have*

$$A^{\mathfrak{I}}\text{-}st\text{-}\lim_n \|L_n(f) - f\| = 0.$$

Proof. Let $\Psi(y) := (y-x)^2$ be a function on $[a,b]$, where $x \in [a,b]$ and $L_n(f;x) = \int_a^b f(y)K_n(y-x)dy$, $n \in \mathbb{N}$, $x \in [a,b]$ and $f \in C[a,b]$, where a, b are two real numbers such that $a < b$. Since L_n is a positive linear operator then $L_n(\Psi;x) \geq 0$.

Let $M = \|f\|$ and $\varepsilon > 0$. By the uniform continuity of $f \in C[a,b]$ and $x \in [a,b]$ there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \varepsilon, \quad \text{whenever } |y-x| \leq \delta.$$

Let $I_\delta = [x-\delta, x+\delta] \cap [a,b]$. So,

$$\begin{aligned}|f(y) - f(x)| =& |f(y) - f(x)|\Psi_{I_\delta}(y) + |f(y) - f(x)|\Psi_{[a,b]-I_\delta}(y)\\ \leq& \varepsilon + 2M\delta^{-2}(y-x)^2.\end{aligned}$$

Since L_n's are positive and linear so we have,

$$\begin{aligned}|L_n(f;x) - f(x)| &= \left|\int_a^b f(y)K_n(y-x)dy - f(x)\right|\\ &= \left|\int_a^b (f(y)-f(x))K_n(y-x)dy + f(x)\int_a^b K_n(y-x)dy - f(x)\right|\\ &\leq \left|\int_a^b (f(y)-f(x))K_n(y-x)dy\right| + |f(x)| \cdot \left|\int_a^b K_n(y-x)dy - 1\right|\\ &\leq \int_a^b |f(y)-f(x)| \cdot |K_n(y-x)dy| + |f(x)| \cdot |L_n(f_0;x) - f_0(x)|\\ &\leq \int_a^b (\varepsilon + 2M\delta^{-2}(y-x)^2)K_n(y-x)dy + M|L_n(f_0;x) - f_0(x)|\end{aligned}$$

$$=\varepsilon + (\varepsilon + M)|L_n(f_0;x) - f_0(x)| + 2M\delta^{-2} \mid L_n(\Psi;x) \mid$$
$$\leq \varepsilon + \alpha\{|L_n(f_0;x) - f_0(x)| + |L_n(\Psi;x)|\},$$

where $\alpha = \max\{\varepsilon + M, \frac{2M}{\delta^2}\}$. Therefore,

$$\|L_n(f) - f\| \leq \varepsilon + \alpha\{\|L_n(f_0) - f_0\| + \|L_n(\Psi)\|\}.$$

For given $r > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < r$ and define the following sets

$$D = \{n : \|L_n(f) - f\| \geq r\},$$
$$D_1 = \left\{n : \|L_n(f_0) - f_0\| \geq \frac{r - \varepsilon}{2\alpha}\right\},$$
$$D_2 = \left\{n : \|L_n(\Psi)\| \geq \frac{r - \varepsilon}{2\alpha}\right\}.$$

It follows that $D \subseteq D_1 \cup D_2$ and consequently for all $n \in \mathbb{N}$

$$\sum_{k\in D} a_{nk} \leq \sum_{k\in D_1} a_{nk} + \sum_{k\in D_2} a_{nk},$$

which implies that for any $\sigma > 0$

$$\left\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\right\} \subseteq \bigcup_{i=1}^{2} \left\{n \in \mathbb{N} : \sum_{k\in D_i} a_{nk} \geq \frac{\sigma}{2}\right\}.$$

Therefore, from hypothesis

$$\left\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\right\} \in \mathfrak{I}.$$

Hence, we have the proof. □

Let δ be a positive real number so that $\delta < \frac{b-a}{2}$ and let $\|f\|_\delta = \sup_{a+\delta\leq x\leq b-\delta} |f(x)|$, $f \in C[a,b]$.

In order to give our main result we need the following lemmas.

Lemma 2.1. *Let $A = (a_{ij})$ be a non negative regular summability matrix. Assume that δ is a fixed positive number such that $\delta < \frac{b-a}{2}$. If the conditions*

$$A^{\mathfrak{I}}\text{-}st\text{-}\lim_n \int_{-\delta}^{\delta} K_n(y)dy = 1, \tag{2.2}$$

$$A^{\mathfrak{I}}\text{-}st\text{-}\lim_n (\sup_{|y|\geq\delta} K_n(y)) = 0 \tag{2.3}$$

hold, then for the operators L_n, where $L_n(f;x) = \int_a^b f(y)K_n(y-x)dy$, $n \in \mathbb{N}$, $x \in [a,b]$, $f \in C[a,b]$ and a,b are real numbers $a < b$, we have

$$A^{\mathfrak{I}}\text{-}st\text{-}\lim_n \|L_n(f_0) - f_0\|_\delta = 0, \quad \textit{with } f_0(y) = 1.$$

Proof. Let $0 < \delta < \frac{b-a}{2}$ and let $x \in [a+\delta, b-\delta]$. Then

$$\delta \leq x - a \leq b - a \Rightarrow -(b-a) \leq a - x \leq -\delta$$

and

$$\delta \leq b - x \leq b - a.$$

Now $L_n(f_0; x) = \int_a^b K_n(y-x)dy = \int_{a-x}^{b-x} K_n(y)dy$. Then we have

$$\int_{-\delta}^{\delta} K_n(y)dy \leq L_n(f_0; x) \leq \int_{-(b-a)}^{b-a} K_n(y)dy.$$

Therefore,

$$\|L_n(f_0) - f_0\|_\delta \leq u_n,$$

where $u_n = \max\left\{\left|\int_{-\delta}^{\delta} K_n(y)dy - 1\right|, \left|\int_{-(b-a)}^{b-a} K_n(y)dy - 1\right|\right\}.$

Therefore, $A^{\mathcal{I}}$-st-$\lim_n u_n = 0$ for all $\delta > 0$ such that $\delta < \frac{b-a}{2}$. Now for given $\varepsilon > 0$ define the following sets

$$D := \{n \in \mathbb{N} : \|L_n(f_0) - f_0\|_\delta \geq \varepsilon\},$$
$$D' := \{n \in \mathbb{N} : u_n \geq \varepsilon\}.$$

So $D \subseteq D'$. Then for all $n \in \mathbb{N}$ we have,

$$\sum_{k \in D} a_{nk} \leq \sum_{k \in D'} a_{nk}.$$

Then for any $\sigma > 0$

$$\left\{n \in \mathbb{N} : \sum_{k \in D} a_{nk} \geq \sigma\right\} \subseteq \left\{n \in \mathbb{N} : \sum_{k \in D'} a_{nk} \geq \sigma\right\}.$$

From hypothesis

$$\left\{n \in \mathbb{N} : \sum_{k \in D'} a_{nk} \geq \sigma\right\} \in \mathcal{I}.$$

Hence,

$$\left\{n \in \mathbb{N} : \sum_{k \in D} a_{nk} \geq \sigma\right\} \in \mathcal{I}.$$

So , we have the proof. □

Lemma 2.2. *Let $A = (a_{ij})$ be a non negative regular summability matrix. If conditions (2.2) and (2.3) hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all convolution operators L_n defined by $L_n(f; x) = \int_a^b f(y)K_n(y-x)dy$, $n \in \mathbb{N}$, $x \in [a,b]$ and $f \in C[a,b]$, where a, b are two real numbers such that $a < b$, we have*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(\Psi)\|_\delta = 0, \quad \textit{with } \Psi(y) = (y-x)^2.$$

Proof. For a fixed $0 < \delta < \frac{b-a}{2}$, let $x \in [a+\delta, b-\delta]$. Since $\Psi(y) = y^2 - 2xy + x^2$, then $\Psi \in C[a,b]$ for all $x \in [a+\delta, b-\delta]$. Now $L_n(\Psi; x) = L_n(f_2; x) - 2xL_n(f_1; x) + x^2 L_n(f_0; x)$, with $f_i(y) = y^i$, $i = 0, 1, 2$. Then for all $n \in \mathbb{N}$

$$L_n(\Psi; x) = \int_a^b (y-x)^2 K_n(y-x)dy = \int_{a-x}^{b-x} y^2 K_n(y)dy \leq \int_{-(b-a)}^{b-a} y^2 K_n(y)dy.$$

Since the function f_2 is continuous at $y = 0$ for given $\varepsilon > 0$ exists $\eta > 0$ such that $y^2 < \varepsilon$ for all y satisfying $|y| \leq \eta$. We have two cases such that $\eta \geq b-a$ or $\eta < b-a$.

Case 1. Let $\eta \geq b-a$. Therefore, $0 \leq L_n(\Psi; x) \leq \varepsilon \int_{-(b-a)}^{b-a} K_n(y)dy$. By condition (2.3), $0 \leq L_n(\Psi; x) \leq \varepsilon$ and $A^{\mathcal{I}}\text{-st-}\lim_n \|L_n(\Psi)\|_\delta = 0$ for $\eta \geq b-a$.

Case 2: Let $\eta < b-a$. Therefore, $L_n(\Psi; x) \leq \int_{|y|\geq\eta} y^2 K_n(y)dy + \int_{|y|\leq\eta} y^2 K_n(y)dy$ and hence we obtain

$$\|L_n(\Psi; x)\|_\delta \leq a_n \int_\eta^{b-a} y^2 dy + \varepsilon \int_{|y|\leq\eta} K_n(y)dy = a_n \frac{(b-a)^3 - \eta^3}{3} + \varepsilon b_n,$$

where $a_n = \sup_{|y|\geq\eta} K_n(y)$ and $b_n = \int_{|y|\leq\eta} K_n(y)dy$. Also we have from hypotheses

$$A^{\mathcal{I}}\text{-st-}\lim_n a_n = 0$$

and

$$A^{\mathcal{I}}\text{-st-}\lim_n b_n = 1.$$

Taking, $M = \max\left\{\frac{(b-a)^3-\eta^3}{3}, \varepsilon\right\}$ we have for all $n \in \mathbb{N}$

$$\|L_n(\Psi)\|_\delta \leq \varepsilon + M(a_n + |b_n - 1|).$$

For given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Let

$$D = \{n \in \mathbb{N} : \|L_n(\Psi)\|_\delta \geq r\},$$
$$D_1 = \left\{n \in \mathbb{N} : a_n \geq \frac{r-\varepsilon}{2M}\right\},$$
$$D_2 = \left\{n \in \mathbb{N} : |b_n - 1| \geq \frac{r-\varepsilon}{2M}\right\}.$$

Therefore, $D \subseteq D_1 \cup D_2$. Hence, for all $n \in \mathbb{N}$ we have,

$$\sum_{k\in D} a_{nk} \leq \sum_{k\in D_1} a_{nk} + \sum_{k\in D_2} a_{nk},$$

which implies that for any $\sigma > 0$

$$\left\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\right\} \subseteq \bigcup_{i=1}^{2} \left\{n \in \mathbb{N} : \sum_{k\in D_i} a_{nk} \geq \frac{\sigma}{2}\right\}.$$

Therefore, from the hypothesis

$$\left\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\right\} \in \mathcal{I}.$$

Hence, we have the proof. □

Now the following main result follows from Theorem 2.2 and Lemma 2.1, 2.2.

Theorem 2.3. *Let $A = (a_{ij})$ be a non negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators on $C[a,b]$ given by (2.1). If conditions (2.2) and (2.3) hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all $f \in C[a,b]$ we have*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f) - f\|_\delta = 0.$$

If we take $\mathcal{I} = \mathcal{I}_{fin}$, the ideal of all finite subsets of $\mathbb{N}$, we get the following result.

Corollary 2.1. ([6, Corollary 2.5]). *Let $A = (a_{ij})$ be a non negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators on $C[a,b]$ given by*

$$L_n(f;x) = \int_a^b f(y)K_n(y-x)dy,$$

$n \in \mathbb{N}$, $x \in [a,b]$ and $f \in C[a,b]$, where a and b are two real numbers such that $a < b$. If conditions

$$st_A - \lim_n \int_{-\delta}^{\delta} K_n(y)dy = 1$$

and

$$st_A - \lim_n \sup_{|y|\geq\delta} K_n(y) = 0$$

hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all $f \in C[a,b]$ we have

$$st_A - \lim_n \|L_n(f) - f\|_\delta = 0.$$

Remark 2.1. We now exhibit a sequence of positive convolution operators for which Corollary 2.1 does not apply but Theorem 2.3 does. Let

$$u_n = \begin{cases} 1, & \text{for } n \text{ even,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{I}$ be a non-trivial admissible ideal of $\mathbb{N}$. Choose an infinite subset $C = \{p_1 < p_2 < p_3 \cdots\}$ from $\mathcal{I} \setminus \mathcal{I}_d$, where $\mathcal{I}_d$ denotes the set of all subsets of $\mathbb{N}$ with natural density zero.

Let $A = (a_{nk})$ be given by

$$a_{nk} = \begin{cases} 1, & \text{if } n = p_i,\ k = 2p_i \text{ for some } i \in \mathbb{N}, \\ 1, & \text{if } n \neq p_i \text{ for any } i, k = 2n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Now for $0 < \varepsilon < 1$, $K(\varepsilon) = \{k \in \mathbb{N} : |u_k - 0| \geq \varepsilon\}$ is the set of all even integers. Observe that

$$\sum_{k\in K(\varepsilon)} a_{nk} = \begin{cases} 1, & \text{if } n = p_i \text{ for some } i \in \mathbb{N}, \\ 0, & \text{if } n \neq p_i \text{ for any } i \in \mathbb{N}. \end{cases}$$

Thus, for any $\delta > 0$, $\left\{n \in \mathbb{N} : \sum_{k \in K(\varepsilon)} a_{nk} \geq \delta\right\} = C \in \mathcal{I} \setminus \mathcal{I}_d$ which shows that $\{u_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to 0 though x is not A-statistically convergent.

Now let the operators L_n on $C[a,b]$ be defined by

$$L_n(f;x) = \frac{n(1+u_n)}{\sqrt{\pi}} \int_a^b f(y) e^{-n^2(y-x)^2} dy.$$

If we choose $K_n(y) = \frac{n(1+u_n)}{\sqrt{\pi}} e^{-n^2y^2}$, then

$$L_n(f;x) = \frac{n(1+u_n)}{\sqrt{\pi}} \int_a^b f(y) K_n(y-x) dy.$$

Now for every $\delta > 0$ such that $\delta < \frac{b-a}{2}$ we have

$$\begin{aligned}\int_{-\delta}^{\delta} K_n(y)dy =& \frac{n(1+u_n)}{\sqrt{\pi}} \left(\int_{-\infty}^{\infty} e^{-n^2y^2} dy - \int_{|y|\geq\delta} e^{-n^2y^2} dy \right) \\ =& \frac{2(1+u_n)}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-y^2} dy - \int_{\delta.n}^{\infty} e^{-y^2} dy \right).\end{aligned}$$

Since $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} < \infty$, it is clear that $\lim_n \int_{\delta.n}^{\infty} e^{-y^2} dy = 0$. Also since $A^{\mathcal{I}}\text{-st-}\lim_n (1+u_n) = 1$, we immediately get

$$A^{\mathcal{I}}\text{-st-}\lim_n \int_{-\delta}^{\delta} K_n(y)dy = 1.$$

On the other hand, we have

$$\sup_{|y|\geq\delta} K_n(y) = \frac{n(1+u_n)}{\sqrt{\pi}} \sup_{|y|\geq\delta} e^{-n^2y^2} \leq \frac{n(1+u_n)}{e^{n^2\delta^2}}.$$

Since $\lim_n \frac{n}{e^{n^2\delta^2}} = 0$ and $A^{\mathcal{I}}\text{-st-}\lim_n (1+u_n) = 1$, we conclude that

$$A^{\mathcal{I}}\text{-st-}\lim_n \sup_{|y|\geq\delta} K_n(y) = 0.$$

Therefore, from Theorem 2.3,

$$A^{\mathcal{I}}\text{-st-}\lim_n \|L_n(f) - f\|_\delta = 0, \quad \text{for all } f \in C[a,b].$$

However note that, as $\{u_k\}_{k\in\mathbb{N}}$ is not A-statistically convergent to zero so K_n do not satisfy the hypotheses of Corollary 2.1.

3. Rate of $A^{\mathcal{I}}$-Statistical Convergence

In this section we study the rates of $A^{\mathcal{I}}$-statistical convergence in Theorem 2.3 using the modulus of continuity. Let $f \in C[a,b]$. The modulus of continuity denoted by $\omega(f,\alpha)$ is defined to be

$$\omega(f,\alpha) = \sup_{|y-x|\leq\alpha} |f(y) - f(x)|.$$

The modulus of continuity of the function f in $C[a,b]$ gives the maximum oscillation of f in any interval of length not exceeding $\alpha > 0$. It is well-known that if $f \in C[a,b]$, then

$$\lim_{\alpha \to 0} \omega(f,\alpha) = \omega(f,0) = 0,$$

and that for any constants $c > 0$, $\alpha > 0$,

$$\omega(f, c\alpha) \leq (1 + [c])\omega(f,\alpha),$$

where $[c]$ is the greatest integer less than or equal to c.

Next we introduce the following definition.

Definition 3.1. Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{c_n\}_{n\in\mathbb{N}}$ be a positive non-increasing sequence of real numbers. Then a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to a number L with the rate of $o(c_n)$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\{ j \in \mathbb{N} : \frac{1}{c_j} \sum_{\{n : |x_n - L| \geq \varepsilon\}} a_{jn} \geq \delta \right\} \in \mathcal{I}.$$

In this case we write $A^{\mathcal{I}}$-st-$o(c_n)$- $\lim_n x_n = L$.

We establish the following theorem.

Theorem 3.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators given by* (2.1)*. Assume further that $\{c_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ are two positive non-increasing sequences. If for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$*

$$A^{\mathcal{I}}\text{-}st\text{-}o(c_n)\text{-}\lim_n \|L_n(f_0) - f_0\|_\delta = 0$$

and

$$A^{\mathcal{I}}\text{-}st\text{-}o(d_n)\text{-}\lim_n \omega(f,\alpha_n) = 0,$$

where $\alpha_n := \sqrt{\|L_n(\Psi)\|_\delta}$, then for all $f \in C[a,b]$ we have

$$A^{\mathcal{I}}\text{-}st\text{-}o(p_n)\text{-}\lim_n \|L_n(f) - f\|_\delta = 0,$$

where $p_n := \max\{c_n, d_n\}$.

Proof. Let $0 < \delta < \frac{b-a}{2}$, $f \in C[a,b]$ and $x \in [a+\delta, b-\delta]$. By positivity and linearity of the operators L_n and using the inequalities for any $\alpha > 0$ we get

$$\begin{aligned}
|L_n(f;x) - f(x)| \leq & L_n(|f(y) - f(x)|;x) + |f(x)| \cdot |L_n(f_0;x) - f_0(x)| \\
\leq & L_n\left(\omega\left(f, \alpha\frac{|y-x|}{\alpha}\right);x\right) + |f(x)| \cdot |L_n(f_0;x) - f_0(x)| \\
\leq & \omega(f,\alpha) L_n\left(1 + \left[\frac{|y-x|}{\alpha}\right];x\right) + |f(x)| \cdot |L_n(f_0;x) - f_0(x)| \\
\leq & \omega(f,\alpha)\left\{L_n(f_0;x) + \frac{1}{\alpha^2}L_n(\psi;x)\right\} + |f(x)| \cdot |L_n(f_0;x) - f_0(x)|.
\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$

$$\|L_n(f) - f\|_\delta \leq \omega(f,\alpha)\left\{\|L_n(f_0)\|_\delta + \frac{1}{\alpha^2}\|L_n(\Psi)\|_\delta\right\} + M_1\|L_n(f_0) - f_0\|_\delta,$$

where $M_1 := \|f\|_\delta$. Now let $\alpha := \alpha_n = \sqrt{\|L_n(\Psi)\|_\delta}$. Then we have

$$\begin{aligned}\|L_n(f) - f\|_\delta \leq & \omega(f,\alpha_n)\{\|L_n(f_0)\|_\delta + 1\} + M_1\|L_n(f_0) - f_0\|_\delta \\ \leq & 2\omega(f,\alpha_n) + \omega(f,\alpha_n)\|L_n(f_0) - f_0\|_\delta + M_1\|L_n(f_0) - f_0\|_\delta.\end{aligned}$$

Let $M = \max\{2, M_1\}$. Then we can write for all $n \in \mathbb{N}$ that

$$\|L_n(f) - f\|_\delta \leq M\{\omega(f,\alpha_n) + \|L_n(f_0) - f_0\|_\delta\} + \omega(f,\alpha_n)\|L_n(f_0) - f_0\|_\delta.$$

Given $\varepsilon > 0$, define the following sets:

$$\begin{aligned}D &:= \{n : \|L_n(f) - f\|_\delta \geq \varepsilon\},\\ D_1 &:= \left\{n : \omega(f,\alpha_n) \geq \frac{\varepsilon}{3M}\right\},\\ D_2 &:= \left\{n : \omega(f,\alpha_n)\|L_n(f_0) - f_0\|_\delta \geq \frac{\varepsilon}{3}\right\},\\ D_3 &:= \left\{n : \|L_n(f_0) - f_0\|_\delta \geq \frac{\varepsilon}{3M}\right\}.\end{aligned}$$

Then $D \subseteq D_1 \cup D_2 \cup D_3$. Also, we define

$$\begin{aligned}D_2' &= \left\{n : \omega(f,\alpha_n) \geq \sqrt{\frac{\varepsilon}{3}}\right\},\\ D_2'' &= \left\{n : \|L_n(f_0) - f_0\|_\delta \geq \sqrt{\frac{\varepsilon}{3}}\right\}.\end{aligned}$$

Therefore, $D_2 \subseteq D_2' \cup D_2''$. Hence, we get $D \subseteq D_1 \cup D_2' \cup D_2'' \cup D_3$. Since $p_n = \max\{c_n, d_n\}$ we obtain for all $j \in \mathbb{N}$ that

$$\frac{1}{p_j}\sum_{n\in D} a_{jn} \leq \frac{1}{d_j}\sum_{n\in D_1} a_{jn} + \frac{1}{d_j}\sum_{n\in D_2'} a_{jn} + \frac{1}{c_j}\sum_{n\in D_2''} a_{jn} + \frac{1}{c_j}\sum_{n\in D_3} a_{jn}.$$

As

$$A^{\mathcal{I}}\text{-st-}o(c_n)\text{-}\lim_n \|L_n(f_0) - f_0\|_\delta = 0$$

and

$$A^{\mathcal{I}}\text{-st-}o(d_n)\text{-}\lim_n \omega(f,\alpha_n) = 0.$$

Therefore,

$$\left\{j \in \mathbb{N} : \frac{1}{p_j}\sum_{n\in D} a_{jn} \geq \delta\right\} \in \mathcal{I},$$

i.e.,

$$A^{\mathcal{I}}\text{-st-}o(p_n)\text{-}\lim_n \|L_n(f) - f\|_\delta = 0, \quad \text{for all } f \in C[a,b],$$

where $p_n := \max\{c_n, d_n\}$. Hence, the result follows. □

4. Conclusions

Following the concept of $A^{\mathcal{I}}$-statistical convergence for real sequences, we have encountered a Korovkin type approximation theory (Theorem 2.3) for a sequence of positive convolution operators defined on $C[a,b]$. We have exhibited an example which shows that Theorem 2.3 is stronger than its A-statistical version [6, Corollary 2.5]. The third section states about the rates of the $A^{\mathcal{I}}$-statistical convergence.

We are very much interested whether the results of this paper are valid for the function f with two variables. Again we are interested whether the results are relevant on infinite interval.

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