

SOME INEQUALITIES WITH FIBONACCI NUMBERS VIA POWER MEANS

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ABSTRACT. In this paper, we give refinements of upper bounds for ratio and difference of power means and apply those results to Fibonacci power means, which inherit the properties of power means. This allows us to construct various inequalities involving Fibonacci numbers by utilizing many different known identities concerning Fibonacci, Lucas and Catalan numbers through the selection of the weights.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$ be n -tuples of strictly positive real numbers with $W_n = \sum_{i=1}^n w_i$. The class of power means is defined as

$$(1.1) \quad M_{[r]}(\mathbf{x}, \mathbf{w}) = \begin{cases} \left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i^r \right)^{1/r}, & r \neq 0, \\ \prod_{i=1}^n x_i^{w_i/W_n}, & r = 0, \\ \min_{1 \leq i \leq n} x_i, & r = -\infty, \\ \max_{1 \leq i \leq n} x_i, & r = +\infty. \end{cases}$$

The power means form a natural extension of the elementary means (arithmetic for $r = 1$, geometric for $r = 0$ and harmonic for $r = -1$). The case $r = 2$ is often called the quadratic mean with weight $\mathbf{w}$.

A strong and useful property of power means is their comparability

$$(1.2) \quad M_{[r]}(\mathbf{x}, \mathbf{w}) \leq M_{[s]}(\mathbf{x}, \mathbf{w}), \quad \text{for all } -\infty \leq r < s \leq +\infty,$$

which automatically gives us the order between elementary means.

Key words and phrases. Fibonacci numbers, Lucas numbers, power means.

2020 Mathematics Subject Classification. Primary: 11B39. Secondary: 26D15, 26E60.

DOI

Received: April 03, 2025.

Accepted: January 18, 2026.

In [5] the authors gave the following refinement of Cargo and Shisha's [1] bound for the ratio of power means.

Theorem 1.1. *Let w_i, x_i be positive numbers such that $0 < m \leq x_n \leq \dots \leq x_1 \leq M$ for some real numbers $m < M$ and let $W_k = (\sum_{i=1}^k w_i) / (\sum_{i=1}^n w_i)$, $-\infty < r \leq s < +\infty$, $\beta = M/m$. Then, the following inequalities are valid*

$$(1.3) \quad \frac{M_{[s]}(\mathbf{x}, \mathbf{w})}{M_{[r]}(\mathbf{x}, \mathbf{w})} \leq \max_{1 \leq k \leq n} \Phi_k \leq \Gamma_{r,s}(\beta),$$

where

$$(1.4) \quad \Phi_k = \begin{cases} (\beta^s W_k + 1 - W_k)^{\frac{1}{s}} (\beta^r W_k + 1 - W_k)^{-\frac{1}{r}}, & rs \neq 0, \\ (\beta^s W_k + 1 - W_k)^{\frac{1}{s}} \beta^{-W_k}, & r = 0, \\ (\beta^r W_k + 1 - W_k)^{-\frac{1}{r}} \beta^{W_k}, & s = 0, \end{cases}$$

and

$$(1.5) \quad \Gamma_{r,s}(\beta) = \begin{cases} \left(\frac{s-r}{\beta^s - \beta^r} \right)^{\frac{1}{r} - \frac{1}{s}} \left(\frac{\beta^s - 1}{s} \right)^{\frac{1}{r}} \left(\frac{r}{\beta^r - 1} \right)^{\frac{1}{s}}, & rs \neq 0, \\ \left(\frac{\beta^s / (\beta^s - 1)}{e \log(\beta^s / (\beta^s - 1))} \right)^{\frac{1}{s}}, & r = 0, \\ 1 / \Gamma_{0,r}(\beta), & s = 0. \end{cases}$$

Motivated by their result, we will give a refinement of Mond-Shisha's bound [6] for the difference of power means.

Theorem 1.2. *Let r, s, m, M be given real numbers such that $r < s$, $0 < m < M$. Let I denote n -dimensional cube $\{\mathbf{x} = (x_1, \dots, x_n) : m \leq x_i \leq M, i = 1, \dots, n\} = [m, M]^n$, let $\mathbf{w} = (w_1, \dots, w_n)$ be a strictly positive n -tuple and let $W_J = (\sum_{i \in J} w_i) / (\sum_{i=1}^n w_i)$, $J \subset J_n = \{1, \dots, n\}$. Then, we have*

$$(1.6) \quad M_{[s]}(\mathbf{x}, \mathbf{w}) - M_{[r]}(\mathbf{x}, \mathbf{w}) \leq \max_{J \subset J_n} \Psi(J) \leq \Delta,$$

where

$$(1.7) \quad \Psi(J) = \begin{cases} (M^s W_J + (1 - W_J)m^s)^{\frac{1}{s}} - (M^r W_J + (1 - W_J)m^r)^{\frac{1}{r}}, & rs \neq 0, \\ (M^s W_J + (1 - W_J)m^3)^{\frac{1}{s}} - m^{1-W_J} M^{W_J}, & r = 0, \\ m^{1-W_J} M^{W_J} - (M^r W_J + (1 - W_J)m^r)^{-\frac{1}{r}}, & s = 0, \end{cases}$$

and

$$(1.8) \quad \Delta = \begin{cases} (\Theta M^s + (1 - \Theta)m^s)^{\frac{1}{s}} - (\Theta M^r + (1 - \Theta)m^r)^{\frac{1}{r}}, & rs \neq 0, \\ (\Theta M^s + (1 - \Theta)m^s)^{\frac{1}{s}} - M^\Theta m^{1-\Theta}, & r = 0, \\ M^\Theta m^{1-\Theta} - (\Theta M^r + (1 - \Theta)m^r)^{\frac{1}{r}}, & s = 0, \end{cases}$$

$$\Theta = \begin{cases} \frac{x^* - m^s}{M^s - m^2}, & s \neq 0, \\ \frac{x^* - m^r}{M^r - m^r}, & s = 0. \end{cases}$$

Here, x^* is defined as follows. Let J denote the open interval joining m^s to M^s if $s \neq 0$, and let $J = (M^r, m^r)$ if $s = 0$. Let

$$(1.9) \quad h(x) = \begin{cases} x^{\frac{1}{s}} - (ax + b)^{\frac{1}{r}}, & rs \neq 0, \\ x^{\frac{1}{s}} - m\beta^{\frac{x-m^s}{M^s-m^s}}, & r = 0, \\ -x^{\frac{1}{r}} + m\beta^{\frac{x-m^r}{M^r-m^r}}, & s = 0, \end{cases}$$

with $\beta = M/m$, $a = (M^r - m^r)/(M^s - m^s)$ and $b = (M^s m^r - M^r m^s)/(M^s - m^s)$. There is an $x^* \in J$ such that $h(x) < h(x^*)$ for every $x \in J$. Equality in (1.6) for a point $\mathbf{x} \in I$ holds if and only if there exists a subsequence $(k_1, k_2, \dots, k_p)$ of $(1, 2, \dots, n)$ such that $\sum_{m=1}^p w_{k_m} = \Theta$, $x_{k_m} = M$, and $x_k = m$ for all other k . Finally, if $s \geq 1$, then x^* is the unique solution of $h'(x) = 0$ in J .

Proof. Assume $rs \neq 0$ (cases $r = 0$ and $s = 0$ are proved analogously, so we will omit it) and let us define a function $F: I \rightarrow \mathbb{R}$ with

$$F(\mathbf{x}) = M_{[s]}(\mathbf{x}, \mathbf{w}) - M_{[r]}(\mathbf{x}, \mathbf{w}).$$

Let $\mathbf{x}^*$ be such that $F(\mathbf{x}^*) = \max\{F(\mathbf{x}) : \mathbf{x} \in I\}$. First we need to show that $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ is a vertex of I .

Suppose the opposite, i.e., suppose that there is j such that $m < x_j^* < M$. Let's take a look at the function

$$f(x) = F(x_1^*, \dots, x_{j-1}^*, x, x_{j+1}^*, \dots, x_n^*) = F(\bar{\mathbf{x}}).$$

Clearly, $f(x_j^*) = \max\{f(x) : x \in [m, M]\}$, so $f'(x_j^*) = 0$. After some straightforward calculations we get that

$$f'(x) = \frac{w_j x^{r-1}}{M_{[s]}^{s-1}(\bar{\mathbf{x}}, \mathbf{w}) M_{[r]}^{r-1}(\bar{\mathbf{x}}, \mathbf{w})} \left(x^{s-r} M_{[r]}^{r-1}(\bar{\mathbf{x}}, \mathbf{w}) - M_{[s]}^{s-1}(\bar{\mathbf{x}}, \mathbf{w}) \right).$$

The first expression in the product is strictly positive.

- For $x = x_j^*$ the expression in the brackets is equal to zero.
- For $x \in [m, x_j^*)$ the expression in the brackets is negative.
- For $x \in (x_j^*, M]$ the expression in the brackets is positive.

This indicates that the function f has minimum in x_j^* , which is a contradiction with the definition of $\mathbf{x}^*$.

Now let's look at the function

$$H(t) = (tM^s + (1-t)m^s)^{\frac{1}{s}} - (tM^r + (1-t)m^r)^{\frac{1}{r}}.$$

For every $\mathbf{y}^*$ vertex of I there is $t_y^* \in [0, 1]$ such that $F(\mathbf{y}^*) = H(t_y^*)$. Thus, $F(\mathbf{x}) \leq H(t_x^*)$ for every $\mathbf{x} \in I$, and the inequality is strict if $\mathbf{x}$ is not a vertex.

Since $H(0) = H(1) = 0$, and since $H(t) > 0$ for $t \in (0, 1)$, by Rolle's theorem it follows that there is a $\Theta \in (0, 1)$ such that

$$F(\mathbf{x}) \leq \max_{t \in [0, 1]} \left\{ (tM^s + (1-t)m^s)^{\frac{1}{s}} - (tM^r + (1-t)m^r)^{\frac{1}{r}} \right\}$$

$$= (\Theta M^s + (1 - \Theta)m^s)^{\frac{1}{s}} - (\Theta M^r + (1 - \Theta)m^r)^{\frac{1}{r}}.$$

The rest was proved by Mond and Shisha in [6]. $\square$

In case of a monotone sequence, we have the following result.

Theorem 1.3. *Let r, s, m, M be given real numbers such that $r < s$, $0 < m < M$. Let I denote n -dimensional cube $\{\mathbf{x} = (x_1, \dots, x_n) : m \leq x_i \leq M, i = 1, \dots, n\} = [m, M]^n$, let $\mathbf{w} = (w_1, \dots, w_n)$ be a strictly positive and nondecreasing, and let $W_k = (\sum_{i=1}^k w_i) / (\sum_{i=1}^n w_i)$. Then, we have*

$$(1.10) \quad M_{[s]}(\mathbf{x}, \mathbf{w}) - M_{[r]}(\mathbf{x}, \mathbf{w}) \leq \max_{1 \leq k \leq n} \Psi_k \leq \Delta,$$

where Δ is defined in (1.8), $W_k = (\sum_{i=1}^k w_i) / (\sum_{i=1}^n w_i)$, and

$$(1.11) \quad \Psi_k = \begin{cases} (M^s W_k + (1 - W_k)m^s)^{\frac{1}{s}} - (M^r W_k + (1 - W_k)m^r)^{\frac{1}{r}}, & rs \neq 0, \\ (M^s W_k + (1 - W_k)m^s)^{\frac{1}{s}} - m^{1-W_k} M^{W_k}, & r = 0, \\ m^{1-W_k} M^{W_k} - (M^r W_k + (1 - W_k)m^r)^{-\frac{1}{r}}, & s = 0. \end{cases}$$

2. FIBONACCI MEANS

The Fibonacci sequence is one of the most iconic and widely studied concepts in mathematics. Fibonacci numbers are defined by recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n > 1, \quad \text{and} \quad F_0 := 0, \quad F_1 = 1.$$

Under some older definitions, the value $F_0 = 0$ is omitted, so that the sequence starts with $F_1 = F_2 = 1$ and the recurrence $F_n = F_{n-1} + F_{n-2}$ is valid for $n > 2$.

The research of Fibonacci numbers has spurred advancements in number theory, combinatorics, and even cryptography. Its recursive structure and predictable growth make it a fertile ground for mathematical exploration.

Let $n \in \mathbb{N}$, $u \in \mathbb{R}$ and let $\mathbf{w} = (w_1, \dots, w_n)$ be an n -tuple of strictly positive real numbers. The Fibonacci power mean is defined by

$$(2.1) \quad F_n^{[r]}(\mathbf{w}, u) = \begin{cases} \left(\frac{1}{W_n} \sum_{i=1}^n w_i F_i^{ru} \right)^{1/r}, & r \neq 0, \\ \prod_{i=1}^n F_i^{uw_i/W_n}, & r = 0, \\ \min\{F_1^u, F_n^u\}, & r = -\infty, \\ \max\{F_1^u, F_n^u\}, & r = +\infty, \end{cases}$$

where F_i denotes the i^{th} Fibonacci number, and $W_n = \sum_{i=1}^n w_i$. This definition was first used in paper [2]. Motivated by the method used in that paper, we will use an analog technique to give further improvements of their results.

Since Fibonacci means inherit the properties of power means, the following theorem arises from Theorem 1.1 and it gives an estimation for the ratio of the Fibonacci means. It is a refinement of the result of the ratio type from [2].

Theorem 2.1. Let $n \in \mathbb{N}$, $u \in \mathbb{R}$, $m = \min\{F_1^u, F_n^u\}$, $M = \max\{F_1^u, F_n^u\}$ and denote $\beta = M/m$. Then, for any $-\infty < r < s < +\infty$ we have

$$(2.2) \quad 1 \leq \frac{F_n^{[s]}(\mathbf{w}, u)}{F_n^{[r]}(\mathbf{w}, u)} \leq \max_{1 \leq k \leq n} \Phi_k \leq \Gamma_{r,s}(\beta),$$

where $\Gamma_{r,s}(\beta)$ and Φ_k are defined in Theorem 1.1.

Our next result directly follows from Theorem 1.3 and presents an estimate for the difference of the Fibonacci means. It is also a refinement of the result of the difference type from [2].

Theorem 2.2. Let $n \in \mathbb{N}$, $u \in \mathbb{R}$, $m = \min\{F_1^u, F_n^u\}$, $M = \max\{F_1^u, F_n^u\}$ and let $\mathbf{w} = (w_1, \dots, w_n)$ be an n -tuple of strictly positive real numbers. For $-\infty < r < s < +\infty$ let us denote $I = [\min\{m^s, M^s\}, \max\{m^s, M^s\}]$ if $s \neq 0$ and $I = [\min\{m^r, M^r\}, \max\{m^r, M^r\}]$ if $s = 0$. Then, for $n > 1$ we have

$$(2.3) \quad 0 \leq F_n^{[s]}(\mathbf{w}, u) - F_n^{[r]}(\mathbf{w}, u) \leq \max_{1 \leq k \leq n} \Psi_k \leq \Delta,$$

where Δ is defined in Theorem 1.2 and Ψ_k is defined in (1.11).

Remark 2.1. First inequality in (2.2) and (2.3) is a direct consequence of the comparability property of Fibonacci means

$$F_n^{[r]}(\mathbf{w}, u) \leq F_n^{[s]}(\mathbf{w}, u), \quad \text{for } -\infty < r < s < +\infty$$

inherited from power means.

3. INEQUALITIES WITH FIBONACCI NUMBERS

In this section we will utilize Theorem 2.1 and Theorem 2.2 by selecting weights $\mathbf{w}$ which allow direct calculation of the sum $W_n = \sum_{i=1}^n w_i$. This approach was used in paper [2], and it will enable us to obtain different inequalities for Fibonacci number by using various identities for Fibonacci numbers.

The following identities were found in [3]:

$$\begin{aligned} w_i &= F_i^2, & W_n &= F_n F_{n+1}, \\ w_i &= F_i, & W_n &= F_{n+2} - 2, \\ w_i &= F_{2i-1}, & W_n &= F_{2n}, \\ w_i &= iF_i, & W_n &= F_{n+2} - F_{n+3} + 2, \\ w_i &= \binom{n}{i} F_i, & W_n &= F_{2n}. \end{aligned}$$

By applying Theorem 2.1 to the first identity, we obtain the following.

- If $u > 0$, then we have

$$\left(\sum_{i=1}^n F_i^{su+2} \right)^{\frac{1}{s}} \leq \max_{1 \leq k \leq n} \left\{ \frac{(F_n^{su} F_k F_{k+1} + F_n F_{n+1} - F_k F_{k+1})^{\frac{1}{s}}}{(F_n^{ru} F_k F_{k+1} + F_n F_{n+1} - F_k F_{k+1})^{\frac{1}{r}}} \right\} \left(\sum_{i=1}^n F_i^{ru+2} \right)^{\frac{1}{r}}$$

$$\leq \Gamma_{r,s}(F_n^u) \left(\sum_{i=1}^n F_i^{ru+2} \right)^{\frac{1}{r}}.$$

- If $u < 0$, then we have

$$\begin{aligned} \left(\sum_{i=1}^n F_i^{su+2} \right)^{\frac{1}{s}} &\leq \max_{1 \leq k \leq n} \left\{ \frac{F_k F_{k+1} + F_n^{su} (F_n F_{n+1} - F_k F_{k+1})}{F_k F_{k+1} + F_n^{ru} (F_n F_{n+1} - F_k F_{k+1})} \right\}^{\frac{1}{s}} \left(\sum_{i=1}^n F_i^{ru+2} \right)^{\frac{1}{r}} \\ &\leq \Gamma_{r,s}(F_n^u) \left(\sum_{i=1}^n F_i^{ru+2} \right)^{\frac{1}{r}}. \end{aligned}$$

Similarly, by applying Theorem 2.2 to the same identity, we obtain the following inequalities.

- If $u > 0$, then we have

$$\begin{aligned} &\left(\sum_{i=1}^n F_i^{su+2} \right)^{\frac{1}{s}} \\ &\leq (F_n F_{n+1})^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n F_i^{ru+2} \right)^{\frac{1}{r}} \\ &\quad + \max_{1 \leq k \leq n} \left\{ (F_n^{us} F_k F_{k+1} + F_n F_{n+1} - F_k F_{k+1})^{\frac{1}{s}} - \frac{(F_n^{ur} (F_k F_{k+1} - 2) + F_n F_{n+1} - F_k F_{k+1})^{\frac{1}{r}}}{(F_n F_{n+1})^{\frac{1}{r} - \frac{1}{s}}} \right\} \\ &\leq (F_n F_{n+1})^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n F_i^{ru+2} \right)^{\frac{1}{r}} + (F_n F_{n+1})^{\frac{1}{s}} \Delta. \end{aligned}$$

- If $u < 0$, then we have

$$\begin{aligned} &\left(\sum_{i=1}^n F_i^{su+2} \right)^{\frac{1}{s}} \\ &\leq (F_n F_{n+1})^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n F_i^{ru+2} \right)^{\frac{1}{r}} \\ &\quad + \max_{1 \leq k \leq n} \left\{ (F_k F_{k+1} + F_n^{us} (F_n F_{n+1} - F_k F_{k+1}))^{\frac{1}{s}} - \frac{(F_k F_{k+1} + F_n^{ur} (F_n F_{n+1} - F_k F_{k+1}))^{\frac{1}{r}}}{(F_n F_{n+1})^{\frac{1}{r} - \frac{1}{s}}} \right\} \\ &\leq (F_n F_{n+1})^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n F_i^{ru+1} \right)^{\frac{1}{r}} + (F_n F_{n+1})^{\frac{1}{s}} \Delta. \end{aligned}$$

4. INEQUALITIES WITH LUCAS NUMBERS

Just like the Fibonacci numbers, each Lucas number is defined as the sum of its two previous terms. The first two Lucas numbers are $L_0 = 2$ and $L_1 = 1$, what distinguishes them from Fibonacci numbers. Although closely related in definition, Lucas and Fibonacci numbers have different properties.

The Lucas numbers are defined as follows:

$$L_n := \begin{cases} 2, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ L_{n-1} + L_{n-2}, & \text{if } n > 1. \end{cases}$$

The connection between Lucas numbers and Fibonacci numbers can be expressed as

$$L_n = F_{n-1} + F_{n+1}, \quad n \geq 1.$$

As in the previous section, here we will also use some known identities about Lucas numbers and Theorem 2.1 and Theorem 2.2 in order to construct different inequalities involving Lucas numbers.

The following identities were found in [3]. We will express them in a form from which it is immediately clear how we will use them in our results.

$$\begin{aligned} w_i &= L_{2i-1}, & W_n &= L_{2n} - 2, \\ w_i &= iL_i, & W_n &= nL_{n+2} - L_{n+3} + 4, \\ w_i &= L_i^2, & W_n &= L_n L_{n+1} - 2. \end{aligned}$$

Let $n \in \mathbb{N}$, $-\infty < r < s < +\infty$, $u \in \mathbb{R}$ and let $\Gamma_{r,s}(\cdot)$ and Δ be as defined in Section 2. First identity gives us the following inequalities.

- If $u > 0$, then we have

$$\begin{aligned} \left(\sum_{i=1}^n L_{2i-1} F_i^{su} \right)^{\frac{1}{s}} &\leq \max_{1 \leq k \leq n} \left\{ \frac{(F_n^{su}(L_{2k} - 2) + L_{2n} - L_{2k})^{\frac{1}{s}}}{(F_n^{ru}(L_{2k} - 2) + L_{2n} - L_{2k})^{\frac{1}{r}}} \right\} \left(\sum_{i=1}^n L_{2i-1} F_i^{ru} \right)^{\frac{1}{r}} \\ &\leq \Gamma_{r,s}(F_n^u) \left(\sum_{i=1}^n L_{2i-1} F_i^{ru} \right)^{\frac{1}{r}} \end{aligned}$$

and

$$\begin{aligned} &\left(\sum_{i=1}^n L_{2i-1} F_i^{su} \right)^{\frac{1}{s}} \\ &\leq (L_{2n} - 2)^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n L_{2i-1} F_i^{ru} \right)^{\frac{1}{r}} \\ &\quad + \max_{1 \leq k \leq n} \left\{ (F_n^{su}(L_{2k} - 2) + L_{2n} - L_{2k})^{\frac{1}{s}} - \frac{(F_n^{ru}(L_{2k} - 2) + L_{2n} - L_{2k})^{\frac{1}{r}}}{(L_{2n} - 2)^{\frac{1}{r} - \frac{1}{s}}} \right\} \\ &\leq (L_{2n} - 2)^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n L_{2i-1} F_i^{ru} \right)^{\frac{1}{r}} + (L_{2n} - 2)^{\frac{1}{s}} \Delta. \end{aligned}$$

- If $u < 0$, then we have

$$\left(\sum_{i=1}^n L_{2i-1} F_i^{su} \right)^{\frac{1}{s}} \leq \max_{1 \leq k \leq n} \left\{ \frac{(L_{2k} - 2 + F_n^{su}(L_{2n} - L_{2k}))^{\frac{1}{s}}}{(L_{2k} - 2 + F_n^{ru}(L_{2n} - L_{2k}))^{\frac{1}{r}}} \right\} \left(\sum_{i=1}^n L_{2i-1} F_i^{ru} \right)^{\frac{1}{r}}$$

$$\leq \Gamma_{r,s} \left(F_n^{-u} \right) \left(\sum_{i=1}^n L_{2i-1} F_i^{ru} \right)^{\frac{1}{r}}$$

and

$$\begin{aligned} & \left(\sum_{i=1}^n L_{2i-1} F_i^{su} \right)^{\frac{1}{s}} \\ & \leq (L_{2n} - 2)^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n L_{2i-1} F_i^{ru} \right)^{\frac{1}{r}} \\ & \quad + \max_{1 \leq k \leq n} \left\{ (L_{2k} - 2 + F_n^{su} (L_{2n} - L_{2k}))^{\frac{1}{s}} - \frac{(L_{2k} - 2 + F_n^{ru} (L_{2n} - L_{2k}))^{\frac{1}{r}}}{(L_{2n} - 2)^{\frac{1}{r} - \frac{1}{s}}} \right\} \\ & \leq (L_{2n} - 2)^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n L_{2i-1} F_i^{ru} \right)^{\frac{1}{r}} + (L_{2n} - 2)^{\frac{1}{s}} \Delta. \end{aligned}$$

We can also use the following mixed identities with Lucas numbers and Fibonacci numbers as seen above.

$$\begin{aligned} w_i &= \binom{n}{i} F_i F_{n-i}, \quad W_n = \frac{1}{5} (2^n L_n - 2), \\ w_i &= \binom{n}{i} F_i L_{n-i}, \quad W_n = 2^n F_n. \end{aligned}$$

By utilizing second identity and our results, we get the following.

- If $u > 0$, then we have

$$\begin{aligned} \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{su+1} \right)^{\frac{1}{s}} & \leq \max_{1 \leq k \leq n} \left\{ \frac{(2^k F_n^{su} F_k + 2^n F_n - 2^k F_k)^{\frac{1}{s}}}{(2^k F_n^{ru} F_k + 2^n F_n - 2^k F_k)^{\frac{1}{r}}} \right\} \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{ru+1} \right)^{\frac{1}{r}} \\ & \leq \Gamma_{r,s} (F_n^u) \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{ru+1} \right)^{\frac{1}{r}} \end{aligned}$$

and

$$\begin{aligned} & \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{su+1} \right)^{\frac{1}{s}} \\ & \leq (2^n F_n)^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{ru+1} \right)^{\frac{1}{r}} \\ & \quad + \max_{1 \leq k \leq n} \left\{ (2^k F_n^{su} F_k + 2^n F_n - 2^k F_k)^{\frac{1}{s}} - \frac{(2^k F_n^{ru} F_k + 2^n F_n - 2^k F_k)^{\frac{1}{r}}}{(2^n F_n)^{\frac{1}{r} - \frac{1}{s}}} \right\} \\ & \leq (2^n F_n)^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{ru+1} \right)^{\frac{1}{r}} + (2^n F_n)^{\frac{1}{s}} \Delta. \end{aligned}$$

- If $u < 0$, then we have

$$\begin{aligned} \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{su+1} \right)^{\frac{1}{s}} &\leq \max_{1 \leq k \leq n} \left\{ \frac{(2^k F_k + F_n^{su} (2^n F_n - 2^k F_k))^{\frac{1}{s}}}{(2^k F_k + F_n^{ru} (2^n F_n - 2^k F_k))^{\frac{1}{r}}} \right\} \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{ru+1} \right)^{\frac{1}{r}} \\ &\leq \Gamma_{r,s} (F_n^{-u}) \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{ru+1} \right)^{\frac{1}{r}} \end{aligned}$$

and

$$\begin{aligned} &\left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{su+1} \right)^{\frac{1}{s}} \\ &\leq (2^n F_n)^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{ru+1} \right)^{\frac{1}{r}} \\ &\quad + \max_{1 \leq k \leq n} \left\{ (2^k F_k + 2^n F_n^{su+1} - 2^k F_n^{su} F_k)^{\frac{1}{s}} - \frac{(2^k F_k + 2^n F_n^{ru+1} - 2^k F_n^{ru} F_k)^{\frac{1}{r}}}{(2^n F_n)^{\frac{1}{r} - \frac{1}{s}}} \right\} \\ &\leq (2^n F_n)^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n \binom{n}{i} L_{n-i} F_i^{ru+1} \right)^{\frac{1}{r}} + (2^n F_n)^{\frac{1}{s}} \Delta. \end{aligned}$$

All other mentioned identities when utilized in the same way as seen above give different inequalities concerning Fibonacci and Lucas numbers.

Remark 4.1. Analogous to the definition of Fibonacci means, Lucas means can be defined by using Lucas numbers in the following manner. Let $n \in \mathbb{N}$, $u \in \mathbb{R}$ and let $\mathbf{w} = (w_1, \dots, w_n)$ be an n -tuple of strictly positive real numbers. The Lucas power mean is defined by

$$(4.1) \quad L_n^{[r]}(\mathbf{w}, u) = \begin{cases} \left(\frac{1}{W_n} \sum_{i=1}^n w_i L_i^{ru} \right)^{1/r}, & r \neq 0, \\ \prod_{i=1}^n L_i^{uw_i/W_n}, & r = 0, \\ \min\{L_1^u, L_n^u\}, & r = -\infty, \\ \max\{L_1^u, L_n^u\}, & r = +\infty, \end{cases}$$

where L_i denotes the i^{th} Lucas number, and $W_n = \sum_{i=1}^n w_i$. Theorem 2.1 and Theorem 2.2 are valid for Lucas means as well, so by utilizing identities stated above, they can also be used for constructing various inequalities involving both Lucas number and Fibonacci numbers in the way described so far.

5. INEQUALITIES VIA CATALAN AND NARAYANA NUMBERS

Catalan numbers are a solution to many counting problems in combinatorics. For $n \in \mathbb{N}_0$, Catalan numbers are defined as

$$C_n = \frac{1}{n+1} \binom{2n}{n}, \quad C_0 = 0.$$

They satisfy the recurrence relation

$$C_0 = 1, \quad C_n = \sum_{i=1}^n C_{i-1} C_{n-i}, \quad \text{for } n > 0,$$

which we can use in our results by setting $w_i = C_{i-1} C_{n-i}$ and $W_n = C_n$. Let $-\infty < r < s < +\infty$, $u \in \mathbb{R}$ and let $\Gamma_{r,s}(\cdot)$ and Δ be as defined in Section 2. We get the following.

- If $u > 0$, then we have

$$\begin{aligned} \left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{su} \right)^{\frac{1}{s}} &\leq \max_{1 \leq k \leq n} \left\{ \frac{((F_n^{su} - 1)C_k + C_n)^{\frac{1}{s}}}{((F_n^{ru} - 1)C_k + C_n)^{\frac{1}{r}}} \right\} \left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{ru} \right)^{\frac{1}{r}} \\ &\leq \Gamma_{r,s}(F_n^u) \left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{ru} \right)^{\frac{1}{r}} \end{aligned}$$

and

$$\begin{aligned} &\left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{su} \right)^{\frac{1}{s}} \\ &\leq C_n^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{ru} \right)^{\frac{1}{r}} \\ &\quad + \max_{1 \leq k \leq n} \left\{ ((F_n^{su} - 1)C_k + C_n)^{\frac{1}{s}} - C_n^{\frac{1}{s} - \frac{1}{r}} ((F_n^{ru} - 1)C_k + C_n)^{\frac{1}{r}} \right\} \\ &\leq C_n^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{ru} \right)^{\frac{1}{r}} + C_n^{\frac{1}{s}} \Delta. \end{aligned}$$

- If $u < 0$, then we have

$$\begin{aligned} \left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{su} \right)^{\frac{1}{s}} &\leq \max_{1 \leq k \leq n} \left\{ \frac{(C_k + (C_n - C_k)F_n^{su})^{\frac{1}{s}}}{(C_k + (C_n - C_k)F_n^{ru})^{\frac{1}{r}}} \right\} \left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{ru} \right)^{\frac{1}{r}} \\ &\leq \Gamma_{r,s}(F_n^{-u}) \left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{ru} \right)^{\frac{1}{r}} \end{aligned}$$

and

$$\begin{aligned} &\left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{su} \right)^{\frac{1}{s}} \\ &\leq C_n^{\frac{1}{s} - \frac{1}{r}} \left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{ru} \right)^{\frac{1}{r}} \\ &\quad + \max_{1 \leq k \leq n} \left\{ (C_k + (C_n - C_k)F_n^{su})^{\frac{1}{s}} - C_n^{\frac{1}{s} - \frac{1}{r}} ((C_k + (C_n - C_k)F_n^{ru})^{\frac{1}{r}}) \right\} \end{aligned}$$

$$\leq C_n^{\frac{1}{s}-\frac{1}{r}} \left(\sum_{i=1}^n C_{i-1} C_{n-i} F_i^{ru} \right)^{\frac{1}{r}} + C_n^{\frac{1}{s}} \Delta.$$

The Narayana polynomials are defined by

$$N_n(q) = \sum_{i=1}^n \frac{1}{n} \binom{n}{i-1} \binom{n}{i} q^i, \quad \text{for } n \geq 1$$

with the initial values $N_0(q) = 1$. The coefficients

$$N_{n,i} = \frac{1}{n} \binom{n}{i-1} \binom{n}{i},$$

with $N_{0,0} = 1$ and $N_{0,i} = 0$ are called the Narayana numbers, and there is a well known connection between Catalan numbers and Narayana polynomials and Narayana numbers:

$$C_n = N_n(1) = \sum_{i=1}^n N_{n,i} = \sum_{i=1}^n \frac{1}{n} \binom{n}{i-1} \binom{n}{i}.$$

Narayana polynomials and Narayana numbers are also connected to the large Schröder numbers, which describe the number of lattice paths from the southwest corner $(0,0)$ of an $n \times n$ grid to the northeast corner (n,n) using only single sub-diagonal steps:

$$S_n = N_n(2) = \sum_{i=1}^n N_{n,i} 2^i = \sum_{i=1}^n \frac{1}{n} \binom{n}{i-1} \binom{n}{i} 2^i.$$

We can utilize these connections by setting

$$w_i = \frac{1}{n} \binom{n}{i-1} \binom{n}{i} = N_{n,i}, \quad W_n = C_n$$

$$w_i = \frac{1}{n} \binom{n}{i-1} \binom{n}{i} 2^i, \quad W_n = S_n$$

in our results, and in that way we can obtain different inequalities for Fibonacci numbers via Catalan, Narayana and Schröder numbers.

6. CONCLUSION

In this paper, a unified and refined framework for inequalities involving power means has been developed, and its broad applicability to several important combinatorial sequences has been demonstrated.

Classical bounds for the ratio and difference of power means have been sharpened, and these improvements have been extended to Fibonacci and Lucas power means, thereby strengthening the theoretical foundations of mean inequalities and widening their analytical scope.

By systematically employing weighted identities connected to Fibonacci, Lucas, Catalan, Narayana, and Schröder numbers, a general method has been established that yields numerous new explicit inequalities. In this way, a meaningful connection has

been created between analytic inequality theory and combinatorial number sequences, providing a versatile platform for further investigation.

Since the method used in this paper works through identities and weights, it is anticipated that future research may extend this framework to other recursively or combinatorially defined sequences and various weighted mean constructions, develop q -analogues of the results, or explore applications in probabilistic and stochastic versions of mean inequalities, thereby further broadening the impact of the methods presented here.

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