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STRUCTURE RELATION AND SECOND-ORDER DIFFERENTIAL EQUATION OF SEMICLASSICAL ORTHOGONAL POLYNOMIALS OBTAINED VIA CUBIC DECOMPOSITION

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Abstract. In this work, we develop a generating method to derive the structure relation and the second-order differential equation of semiclassical orthogonal polynomials obtained via cubic decomposition. Specifically, we study two monic orthogonal polynomial sequences, $\{W_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$, linked through the relation $W_{3n}(x) = P_n\big(\varpi(x)\big)$, $n \geq 0$, where $\varpi(x)$ denotes a monic cubic polynomial. Building on the properties of $\{P_n\}_{n\geq 0}$, we provide explicit expressions for the structure relation and the corresponding second-order differential equation satisfied by $\{W_n\}_{n\geq 0}$. As an application, several illustrative examples of semiclassical orthogonal polynomials of classes 1 and 2 are presented and analyzed.

1. Introduction

Orthogonal polynomials (OPs) have attracted significant attention over the past 150 years. In this context, orthogonality is defined with respect to a linear functional (regular form) [5, 16], rather than solely through an inner product. Since 1939, a natural extension of the classical orthogonal polynomials has been the semiclassical class, first introduced by Shohat [19]. From 1985 onwards, this theory has been further developed from both algebraic and distributional perspectives by Maroni and has been the subject of extensive research by Maroni and collaborators over the last decade [16]. This approach has inspired numerous studies aimed at a thorough understanding of the semiclassical character of orthogonal polynomials.

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In addition to the fundamental properties of semiclassical orthogonal polynomial sequences (OPS) presented in [16], we recall that these families are characterized by a Pearson equation satisfied by the corresponding linear functional, a differential equation satisfied by the associated Stieltjes function, a structure relation, and a second-order linear differential equation. Let $\{W_n\}_{n\geq 0}$ be a monic semiclassical OPS. This means that $\{W_n\}_{n\geq 0}$ is orthogonal with respect to a linear functional $w : \mathcal{P} \to \mathbb{C}$ (where $\mathcal{P}$ denotes the space of all polynomials with complex coefficients) satisfying a Pearson-type distributional differential equation

$$(\phi w)' + \psi w = 0,$$

where ϕ and ψ are nonzero polynomials with $\deg \psi \geq 1$. Equivalently, the associated formal Stieltjes function $S(w)(z) := -\sum_{n\geq 0} \langle w, x^n \rangle / z^{n+1}$ satisfies a non-homogeneous first-order linear differential equation with polynomial coefficients

$$\phi(z)\, S'(w)(z) = C_0(z)\, S(w)(z) + D_0(z),$$

where

$$C_0 = -\phi' - \psi, \quad D_0 = -(w\theta_0\phi)' - (w\theta_0\psi).$$

Furthermore, if the polynomials ϕ, C_0, and D_0 are co-prime, then the class of w is given by

$$s = \max\{\deg C_0 - 1, \deg D_0\}.$$

According to the theory developed by Maroni [16], the sequence $\{W_n\}_{n\geq 0}$ satisfies the structure relation

$$\phi(x)W'_{n+1}(x) = \frac{1}{2}(C_{n+1}(x) - C_0(x))W_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)W_n(x), \quad n \geq 0,$$

where C_n and D_n are polynomials whose degrees are uniformly bounded independently of n. These polynomials can be computed recursively as

$$C_{n+1}(x) = -C_n(x) + 2(x - \beta_n)D_n(x), \quad n \geq 0,$$
$$\gamma_{n+1}D_{n+1}(x) = -\phi(x) + \gamma_n D_{n-1}(x) + (x - \beta_n)^2 D_n(x) - (x - \beta_n)C_n(x), \quad n \geq 0.$$

Here, $\phi(x)$, $C_0(x)$, and $D_0(x)$ denote the same polynomials as those defined above, whereas the sequences $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ are complex, with $\gamma_{n+1} \neq 0$ for all n, corresponding to the coefficients of the three-term recurrence relation

$$W_{n+2}(x) = (x - \beta_{n+1})\, W_{n+1}(x) - \gamma_{n+1}\, W_n(x), \quad n \geq 0,$$

with the initial conditions $W_0(x) = 1$ and $W_1(x) = x - \beta_0$.

Another characterization of semiclassical OPS is given by the second-order linear differential equation

$$J(x;n)W''_n(x) + K_n(x;n)W'_n(x) + L_n(x;n)W_n(x) = 0, \quad n \geq 0,$$

where $J(\cdot;n)$, $K(\cdot;n)$ and $L(\cdot;n)$ are polynomials whose degrees are uniformly bounded independently of n. If $\{W_n\}_{n\geq 0}$ satisfies the structure relations, then these polynomials are given by

$$J(x;n) = \phi(x)D_{n+1}(x), \quad n \geq 0,$$
$$K(x;n) = C_0(x)D_{n+1}(x) - \mathcal{W}(\phi, D_{n+1})(x), \quad n \geq 0,$$
$$L(x;n) = \mathcal{W}\Big(\frac{1}{2}(C_{n+1} - C_0), D_{n+1}\Big)(x) - D_{n+1}(x)\sum_{\nu=0}^{n} D_\nu(x), \quad n \geq 0,$$

with $\mathcal{W}(f,g) := fg' - f'g$. It is worth recalling that, as established in [7], the minimal order of a linear differential equation satisfied by orthogonal polynomial sequences can only be either two or four. The first case corresponds to semiclassical polynomials, whereas the so-called Laguerre-Hahn polynomials satisfy a fourth-order differential equation.

On the other hand, the study of the cubic decomposition (CD) of orthogonal polynomial sequences (OPS) has attracted considerable attention over the past decades. Interest in this topic notably increased following a question posed by Chihara regarding the existence of two OPS related by a cubic transformation. This question was addressed by Barrucand and Dickinson [1], who established necessary and sufficient conditions for the orthogonality of a symmetric OPS $\{W_n\}_{n\geq 0}$ satisfying $W_{3n}(x) = P_n(x^3 + \beta x)$, $n \geq 0$, where $\{P_n\}_{n\geq 0}$ denotes a symmetric sequence of monic orthogonal polynomials. A similar analysis was later carried out by Marcellán and Sansigre [15], extending the study to orthogonal sequences that are not necessarily symmetric or positive definite, and focusing on the cubic transformation x^3. This problem has also been investigated from various perspectives, including polynomial transformations of measures, sieved polynomials, polynomial mappings, and positive definite linear functionals [3, 6, 8]. In particular, the restrictive assumptions of symmetry and specific cubic forms were lifted in [14], where the authors examined the orthogonality of sequences $\{W_n\}_{n\geq 0}$ of the type $W_{3n+m}(x) = \theta_m(x)\,P_n(\pi_3(x))$, with $m \in \{0,1,2\}$, where $\pi_3(x)$ is a fixed cubic polynomial and $\theta_m(x)$ is a fixed polynomial of degree m. The general theory of cubic decompositions of orthogonal polynomials was subsequently developed in [17], and a symbolic approach to this topic was proposed in [18]. It is worth noting that cubic decompositions in the context of semiclassical orthogonal polynomial sequences had not been explored in the literature until the recent contributions [2, 9–13, 20, 21], which focused on specific cases of semiclassical orthogonal polynomials of class one and two. The analysis of the principal components arising in these decompositions constitutes the central objective of these studies.

Our work focuses on presenting a generator method for obtaining the structure relation and the second-order differential equation of semiclassical orthogonal polynomials derived via cubic decomposition.

Consider two sequences of monic orthogonal polynomials $\{W_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$. Let w and u be, respectively, the corresponding regular linear functionals such that

$$W_{3n}(x) = P_n(\varpi(x)), \quad n \geq 0,$$

where $\varpi(x)$ is a monic cubic polynomial.

In this context, we mention the work in [2], where the authors studied the structure relation and the second-order differential equation of semiclassical orthogonal polynomials for a particular case of the cubic decomposition defined above.

The structure of the paper is as follows. In Section 2, we introduce the notations and definitions that will be used throughout the manuscript. In particular, we recall some fundamental concepts related to semiclassical forms, as well as essential results concerning the cubic decomposition (CD) of monic orthogonal polynomial sequences (MOPS). Section 3 (resp. Section 4) focuses on explicitly deriving the structure relation (resp. the second-order linear differential equation) for the sequence $\{W_n\}_{n\geq 0}$, based on the corresponding properties of $\{P_n\}_{n\geq 0}$. As an application, several illustrative examples of semiclassical orthogonal polynomials of classes 1 and 2 are presented and thoroughly analyzed.

2. Notation and Basic Background

In this section we present some basic definitions, notations and results which are used throughout this paper.

2.1. **Basic tools.** Let $\mathcal{P}$ denote the vector space of polynomials with complex coefficients, and let $\mathcal{P}'$ be its algebraic dual space. The elements of $\mathcal{P}'$ will be referred to as *forms* (or linear functionals). The pairing between $\mathcal{P}$ and $\mathcal{P}'$ is expressed through the duality brackets $\langle \cdot, \cdot \rangle$.

For a form $w \in \mathcal{P}'$, the sequence of complex numbers $(w)_n$, $n \geq 0$, is called the *moment sequence* of w relative to the monomial basis $\{x^n\}_{n\geq 0}$. In particular, the n-th moment is given by $(w)_n := \langle w, x^n \rangle$, so that w is uniquely determined by the collection of its moments.

We now introduce some operations on $\mathcal{P}'$. For $c \in \mathbb{C}$, $f, p \in \mathcal{P}$, and $w \in \mathcal{P}'$, we define:

$$\langle fw, p \rangle = \langle w, fp \rangle, \quad \langle w', p \rangle = -\langle w, p' \rangle,$$
$$\langle (x-c)^{-1} w, p \rangle = \langle w, \theta_c p \rangle = \left\langle w, \frac{p(x)-p(c)}{x-c} \right\rangle.$$

Given $f \in \mathcal{P}$ and $w \in \mathcal{P}'$, the product wf is defined by $(wf)(x) := \left\langle w, \frac{xf(x)-\zeta f(\zeta)}{x-\zeta} \right\rangle$ [16].

This definition allows us to introduce the *Cauchy product* of two forms $v, w \in \mathcal{P}'$ by

$$\langle vw, f \rangle := \langle v, wf \rangle, \quad f \in \mathcal{P}.$$

The Cauchy product is commutative, associative, and distributive with respect to the addition of forms.

We introduce the operator $\sigma_\varpi : \mathcal{P}' \to \mathcal{P}'$ through

$$\langle \sigma_\varpi(w), f \rangle := \langle w, \sigma_\varpi(f) \rangle, \quad w \in \mathcal{P}', f \in \mathcal{P},$$

where $\sigma_{\varpi} : \mathcal{P} \to \mathcal{P}$ acts on $f \in \mathcal{P}$ as $\sigma_{\varpi}(f)(x) := f(x^3)$. Consequently, the moments of $\sigma_{\varpi}(w)$ satisfy

$$(\sigma_{\varpi}(w))_n = (w)_{3n}, \quad n \geq 0.$$

In addition, we make use of the formal Stieltjes function associated with $w \in \mathcal{P}'$, defined by [5, 16]

$$S(w)(z) := -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}},$$

which provides an alternative representation of the moment sequence $\{(w)_n\}_{n\geq 0}$. Since the moments uniquely determine w, the function $S(w)(z)$ does so as well.

A form w is said to be *regular* (or *quasi-definite*) if there exists a unique monic polynomial sequence $\{W_n\}_{n\geq 0}$ with $\deg W_n = n$ such that

$$\langle w, W_n W_m \rangle = r_n \delta_{n,m}, \quad n, m \geq 0,$$

where $\{r_n\}_{n\geq 0}$ is a sequence of nonzero complex numbers and $\delta_{n,m}$ denotes the Kronecker delta. The sequence $\{W_n\}_{n\geq 0}$ is then referred to as a *monic orthogonal polynomial sequence* (MOPS) with respect to w and is characterized by the three-term recurrence relation

$$\begin{aligned} &W_0(x) = 1, \quad W_1(x) = x - \beta_0, \\ &W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x), \quad n \geq 0, \end{aligned} \tag{2.1}$$

where $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ are complex sequences with $\gamma_{n+1} \neq 0$ for all n. This is the classical *Favard's theorem* (see [5, 16]). A form w is called *normalized* if its zeroth moment satisfies $(w)_0 = 1$. In what follows, we shall only work with normalized forms.

2.2. **Semiclassical form.** In this subsection, we provide a brief overview of the notion of semiclassical forms.

Definition 2.1. A form $w \in \mathcal{P}'$ is called *semiclassical* if it is regular and if there exist two nonzero polynomials ϕ and ψ, with ϕ monic and $\deg \psi \geq 1$, such that w satisfies the generalized Pearson-type distributional equation

$$(\phi w)' + \psi w = 0. \tag{2.2}$$

The *class* of a semiclassical form w is the smallest nonnegative integer s given by

$$s := \min_{(\phi,\psi)\in\mathcal{A}} \max\{\deg \phi - 2, \deg \psi - 1\},$$

where $\mathcal{A}$ denotes the set of all polynomial pairs (ϕ, ψ) with $\deg \psi \geq 1$ satisfying (2.2). The monic orthogonal polynomial sequence (OPS) $\{W_n\}_{n\geq 0}$ associated with w is then called a *semiclassical OPS* of class s.

Remark 2.1. When $s = 0$, w is said to be a *classical* form. In this case, the OPS $\{W_n\}_{n\geq 0}$ is called a classical OPS, and-up to an affine change of variable-one recovers the well-known families: Hermite forms $\mathcal{H}$ for $\phi \equiv$ const, Laguerre forms $\mathcal{L}(\alpha)$ for $\deg \phi = 1$, Jacobi forms $\mathcal{J}(\alpha, \beta)$ for $\deg \phi = 2$ with distinct roots, and Bessel forms $\mathcal{B}(\alpha)$ for $\deg \phi = 2$ with a double root.

There are several equivalent characterizations of semiclassical forms (see [16]). A particularly useful one relies on the formal Stieltjes function $S(w)(z)$ associated with w. Specifically, w is semiclassical if and only if there exist polynomials ϕ, C_0, and D_0 such that $S(w)(z)$ satisfies the first-order non-homogeneous differential equation

$$\phi(z)S'(w)(z) = C_0(z)S(w)(z) + D_0(z). \tag{2.3}$$

If ϕ, C_0, and D_0 are coprime, the class of w is given by

$$s = \max\{\deg C_0 - 1, \deg D_0\}.$$

Remark 2.2. It is well known that if $w \in \mathcal{P}'$ is semiclassical and satisfies (2.2), then its Stieltjes function $S(w)$ necessarily satisfies (2.3) with the same polynomial ϕ as in (2.2), and where

$$C_0 = -\phi' - \psi, \quad D_0 = -(w\theta_0\phi)' - (w\theta_0\psi). \tag{2.4}$$

2.3. **Cubic decomposition.** In what follows, we focus on the cubic decomposition (abbreviated as CD) introduced in [17]. Choosing a cubic polynomial

$$\varpi(x) = x^3 + px^2 + qx + r, \quad (p, q, r) \in \mathbb{C}^3.$$

Given any monic polynomial sequence (MPS) $\{W_n\}_{n\geq 0}$, there exist three associated MPSs,

$$\{P_n\}_{n\geq 0}, \quad \{Q_n\}_{n\geq 0}, \quad \{R_n\}_{n\geq 0},$$

such that

$$W_{3n}(x) = P_n(\varpi(x)) + xa^1_{n-1}(\varpi(x)) + x^2a^2_{n-1}(\varpi(x)), \quad n \geq 0, \tag{2.5}$$

$$W_{3n+1}(x) = b^1_n(\varpi(x)) + xQ_n(\varpi(x)) + x^2b^2_{n-1}(\varpi(x)), \quad n \geq 0, \tag{2.6}$$

$$W_{3n+2}(x) = c^1_n(\varpi(x)) + xc^2_n(\varpi(x)) + x^2R_n(\varpi(x)), \quad n \geq 0, \tag{2.7}$$

with $\deg a^1_{n-1} \leq n-1$, $\deg a^2_{n-1} \leq n-1$, $\deg b^1_n \leq n$, $\deg b^2_{n-1} \leq n-1$, $\deg c^1_n \leq n$, $\deg c^2_n \leq n$ and $a^1_{-1}(x) = a^2_{-1}(x) = b^2_{-1}(x) = 0$. In this cubic decomposition (2.5)-(2.7) of $\{W_n\}_{n\geq 0}$, the sequences: $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ are called the principal components; $\left\{a^1_{n-1}\right\}_{n\geq 0}$, $\left\{a^2_{n-1}\right\}_{n\geq 0}$, $\{b^1_n\}_{n\geq 0}$, $\left\{b^2_{n-1}\right\}_{n\geq 0}$, $\{c^1_n\}_{n\geq 0}$, $\{c^2_n\}_{n\geq 0}$ are called the secondary components, since they are sequences of polynomials, although, not necessarily bases for the vector space of polynomials $\mathcal{P}$.

In [17], the authors studied the CD of a MOPS. They characterize the orthogonality case such that $W_{3n}(x) = P_n(\varpi(x))$, $n \geq 0$, i.e., the two secondary components $\{a^1_n\}_{n\geq 0}$ and $\{a^2_n\}_{n\geq 0}$ vanish.

Proposition 2.1 ([17]). *Let $\{W_n\}_{n\geq 0}$ be a MOPS with respect to the form w defined by* (2.5)–(2.7). *The following statements are equivalent.*

(a) $a^1_n = a^2_n = 0,\ n \geq 0.$

(b) *The recurrence coefficients of* $\{W_n\}_{n\geq 0}$ *satisfy*

$$\beta_{3n} = \beta_0, \quad n \geq 0,$$
$$\beta_{3n+1} + \beta_{3n+2} = -(p+\beta_0), \quad n \geq 0,$$
$$\gamma_{3n+2} = -\gamma_1 - \beta_0(\beta_{3n+1}+\beta_0+p) - \beta_{3n+1}(\beta_{3n+1}+p) - q, \quad n \geq 0,$$
$$\gamma_{3n}(1-\delta_{n,0}) + \gamma_{3n+1} = \gamma_1, \quad n \geq 0.$$

Moreover, the sequence $\{P_n\}_{n\geq 0}$ *is orthogonal with respect to the form* $u = \sigma_\varpi(w)$ *and satisfies the following three-term recurrence relation*

$$P_{n+2}(x) = (x - \beta^P_{n+1})P_{n+1}(x) - \gamma^P_{n+1}P_n(x), \quad n \geq 0,$$

where

$$\begin{aligned}\beta^P_{n+1} =& r + \gamma_{3n+3}(\beta_{3n+2}+2\beta_0+p)\\ &+ \gamma_{3n+4}(2\beta_0+\beta_{3n+4}+p) + \beta_0(\beta_0^2+p\beta_0+q), \quad n \geq 0,\end{aligned} \tag{2.8}$$

$$\gamma^P_{n+1} = \gamma_{3n+1}\,\gamma_{3n+2}\,\gamma_{3n+3}, \quad n \geq 0. \tag{2.9}$$

Let us conclude this subsection with the following result, which will be needed in the sequel. Specifically, we present the relation between the Stieltjes formal functions of the regular forms corresponding to the polynomial sequences arising from the cubic decomposition (CD): $W_{3n}(x) = P_n(\varpi(x))$.

Proposition 2.2 ([14])**.** *Let* $\{W_n\}_{n\geq 0}$ *be a MOPS with respect to* w *fulfilling* (2.5)-(2.7) *with* $a^1_n = a^2_n = 0$, $n \geq 0$. *The formal Stieltjes functions* $S(w)$ *and* $S(\sigma_\varpi(w))$ *associated with the forms* w *and* $u = \sigma_\varpi(w)$ *with respect to* $\{W_n\}_{n\geq 0}$ *and* $\{P_n\}_{n\geq 0}$, *respectively, are related by*

$$S(w)(z) = \rho(z)S(\sigma_\varpi(w))(\varpi(z)), \tag{2.10}$$

where $\rho(z) = (z+\beta_0)(z+p) + \beta_0^2 + \gamma_1 + q$.

Remark 2.3. It is easy to see that (see [14])

$$\rho(z) = (z-\beta_{3n+1})(z-\beta_{3n+2}) - \gamma_{3n+2}, \quad n \geq 0. \tag{2.11}$$

The following lemma plays an important role in proving our result.

Proposition 2.3 ([14])**.** *If* $u = \sigma_\varpi(w)$ *is semiclassical and* $S(u)$ *satisfies*

$$\widetilde{\phi}(z)\, S'(u)(z) = \widetilde{C}_0(z)\, S(u)(z) + \widetilde{D}_0(z),$$

where $\widetilde{\phi}$, $\widetilde{C}_0$ *and* $\widetilde{D}_0$ *are polynomials, then the form* w *is also semiclassical, and* $S(w)$ *satisfies*

$$\phi(z)\, S'(w)(z) = C_0(z)\, S(w)(z) + D_0(z),$$

where

$$\phi(z) = \rho(z)\, \widetilde{\phi}(\varpi(z)), \tag{2.12}$$

$$C_0(z) = \rho'(z)\, \widetilde{\phi}(\varpi(z)) + \rho(z)\, \varpi'(z)\, \widetilde{C}_0(\varpi(z)), \tag{2.13}$$

$$D_0(z) = \rho^2(z)\, \varpi'(z)\, \widetilde{D}_0(\varpi(z)). \tag{2.14}$$

3. Structure Relation

3.1. **Main Result.** It is well known that if a form u is semiclassical, then its MOPS $\{P_n\}_{n\geq 0}$ satisfies the following compact structure relation (see [16])

$$\widetilde{\phi}(x)P'_{n+1}(x) = \frac{1}{2}(\widetilde{C}_{n+1}(x) - \widetilde{C}_0(x))P_{n+1}(x) - \widetilde{\gamma}_{n+1}\widetilde{D}_{n+1}(x)P_n(x), \quad n \geq 0, \tag{3.1}$$

with

$$\begin{aligned}\widetilde{C}_{n+1}(x) &= -\widetilde{C}_n(x) + 2(x - \widetilde{\beta}_n)\widetilde{D}_n(x), \quad n \geq 0,\\ \widetilde{\gamma}_{n+1}\widetilde{D}_{n+1}(x) &= -\widetilde{\phi}(x) + \widetilde{\gamma}_n\widetilde{D}_{n-1}(x) - (x - \widetilde{\beta}_n)\widetilde{C}_n(x) + (x - \widetilde{\beta}_n)^2\widetilde{D}_n(x), \quad n \geq 0,\end{aligned}$$

where $\widetilde{\phi}$, $\widetilde{C}_0(x)$ and $\widetilde{D}_0(x)$ are given by (2.12), (2.13) and (2.14), respectively, and $\widetilde{D}_{-1}(x) = 0$.

According to Proposition 2.3, the form w is itself semiclassical, and its associated MOPS $\{W_n\}_{n\geq 0}$ satisfies a structure relation. More generally, the sequence $\{W_n\}_{n\geq 0}$ satisfies

$$\phi(x)W'_{n+1}(x) = \frac{1}{2}(C_{n+1}(x) - C_0(x))W_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)W_n(x), \quad n \geq 0, \tag{3.2}$$

where

$$C_{n+1}(x) = -C_n(x) + 2(x - \beta_n)D_n(x), \quad n \geq 0, \tag{3.3}$$

$$\gamma_{n+1}D_{n+1}(x) = -\phi(x) + \gamma_n D_{n-1}(x) + (x - \beta_n)^2 D_n(x) - (x - \beta_n)C_n(x), \quad n \geq 0. \tag{3.4}$$

Here, ϕ, $C_0(x)$ and $D_0(x)$ denote the same polynomials defined in (2.3) and (2.4), while γ_n and β_n correspond to the coefficients of the three-term recurrence relation (2.1). It is worth noting that $D_{-1}(x) = 0$, that $\deg C_n \leq s + 1$ and $\deg D_n \leq s$ for all $n \geq 0$ [16].

In general, deriving explicit expressions for the sequences $\{C_n\}_{n\geq 0}$ and $\{D_n\}_{n\geq 0}$ from the recurrence relations (3.2)–(3.4) is challenging, but this becomes possible through the cubic decomposition.

We aim to express C_n and D_n for $n \geq 0$ in terms of $\widetilde{C}_n$ and $\widetilde{D}_n$, $n \geq 0$, associated with the sequence $\{P_n\}_{n\geq 0}$.

We now state the main result of this section.

Proposition 3.1. *The elements of the structure relation $\{C_n\}_{n\geq 0}$ and $\{D_n\}_{n\geq 0}$ of the MOPS $\{W_n\}_{n\geq 0}$ are*

$$\begin{aligned}C_1(x) = &-\rho'(x)\,\widetilde{\phi}(\varpi(x)) - \varpi'(x)\,\rho(x)\\ &\times\Big[\widetilde{C}_0(\varpi(x)) - 2(x - \beta_0)\rho(x)\widetilde{D}_0(\varpi(x))\Big],\end{aligned} \tag{3.5}$$

$$D_1(x) = -\widetilde{\phi}(\varpi(x)) + \varpi'(x)\Big\{\gamma_2\gamma_3\widetilde{D}_1(\varpi(x))$$

$$
\begin{aligned}
&- (x-\beta_2)\Big[-2(x-\beta_0)\rho(x)+\gamma_1(x-\beta_2)\Big]\widetilde{D}_0(\varpi(x))\Big\}, && (3.6)\\
C_2(x) =&(\rho'(x)-2(x-\beta_1))\,\widetilde{\phi}(\varpi(x))+\varpi'(x)\\
&\times\Big[(\rho(x)+2(x-\beta_1)(x-\beta_3))\,\widetilde{C}_0(\varpi(x))\\
&+\Big(-2(x-\beta_0)\rho(x)^2+4(x-\beta_1)(x-\beta_3)(x-\beta_0)\rho(x)\\
&-2(x-\beta_1)\gamma_1(x-\beta_3)^2\Big)\widetilde{D}_0(\varpi(x))+2(x-\beta_1)\gamma_2\gamma_3\widetilde{D}_1(\varpi(x))\Big], && (3.7)\\
D_2(x) =&\frac{1}{\gamma_2}\Big\{\Big[-\rho(x)-(x-\beta_1)^2+(x-\beta_1)\rho'(x)\Big]\widetilde{\phi}(\varpi(x))\\
&+\varpi'(x)\Big[(x-\beta_1)\rho(x)\,\widetilde{C}_0(\varpi(x))+(x-\beta_1)^2\gamma_2\gamma_3\,\widetilde{D}_1(\varpi(x))\\
&+\Big(\gamma_1\rho^2(x)+2(x-\beta_1)(x-\beta_0)\rho(x)-(x-\beta_1)^2(x-\beta_2)\\
&\times\Big(-2(x-\beta_0)\rho(x)+\gamma_1(x-\beta_2)\Big)\Big)\widetilde{D}_0(\varpi(x))\Big]\Big\}, && (3.8)\\
C_{3n+3}(x) =&\rho'(x)\,\widetilde{\phi}(\varpi(x))+\rho(x)\,\varpi'(x)\,\widetilde{C}_{n+1}(\varpi(x))\\
&+2\,\gamma_{3n+3}\,\rho(x)\,\varpi'(x)\,(x-\beta_{3n+1})\,\widetilde{D}_{n+1}(\varpi(x)),\quad n\ge 0, && (3.9)\\
D_{3n+3}(x) =&\varpi'(x)\rho^2(x)\widetilde{D}_{n+1}(\varpi(x)),\quad n\ge 0, && (3.10)\\
C_{3n+4}(x) =&-\rho'(x)\,\widetilde{\phi}(\varpi(x))-\varpi'(x)\,\rho(x)\Big[\widetilde{C}_{n+1}(\varpi(x))\\
&-2\Big((x-\beta_{3n+3})\rho(x)-\gamma_{3n+3}(x-\beta_{3n+1})\Big)\widetilde{D}_{n+1}(\varpi(x))\Big],\quad n\ge 0, && (3.11)\\
D_{3n+4}(x) =&-\widetilde{\phi}(\varpi(x))+\varpi'(x)\Big\{\gamma_{3n+5}\gamma_{3n+6}\,\widetilde{D}_{n+2}(\varpi(x))\\
&-(x-\beta_{3n+5})\Big(\widetilde{C}_{n+1}(\varpi(x))-2\widetilde{C}_0(\varpi(x))\Big)\\
&-(x-\beta_{3n+5})\Big[2\gamma_{3n+3}(x-\beta_{3n+1})-2(x-\beta_{3n+3})\rho(x)\\
&+\gamma_{3n+4}(x-\beta_{3n+5})\Big]\widetilde{D}_{n+1}(\varpi(x))\Big\},\quad n\ge 0, && (3.12)\\
C_{3n+5}(x) =&\rho'(x)\,\widetilde{\phi}(\varpi(x))-2(x-\beta_{3n+4})\widetilde{\phi}(\varpi(x))\\
&+\varpi'(x)\rho(x)\Big[\widetilde{C}_{n+1}(\varpi(x))-2\Big((x-\beta_{3n+3})\rho(x)\\
&-\gamma_{3n+3}(x-\beta_{3n+1})\Big)\widetilde{D}_{n+1}(\varpi(x))\Big]\\
&+2(x-\beta_{3n+4})\varpi'(x)\Big[\gamma_{3n+5}\gamma_{3n+6}\,\widetilde{D}_{n+2}(\varpi(x))\\
&-(x-\beta_{3n+5})\Big(\widetilde{C}_{n+1}(\varpi(x))-2\widetilde{C}_0(\varpi(x))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- (x-\beta_{3n+5})\Big[2\gamma_{3n+3}(x-\beta_{3n+1}) - 2(x-\beta_{3n+3})\rho(x)\\
&+ \gamma_{3n+4}(x-\beta_{3n+5})\Big]\widetilde{D}_{n+1}(\varpi(x))\Big], \quad n \geq 0, \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
D_{3n+5}(x) =& \frac{1}{\gamma_{3n+5}}\Big\{\Big[-\rho(x) - (x-\beta_{3n+4})^2 + (x-\beta_{3n+4})\rho'(x)\Big]\widetilde{\phi}(\varpi(x))\\
&+ \varpi'(x)\Big[(x-\beta_{3n+4})\rho(x)\widetilde{C}_{n+1}(\varpi(x))\\
&- (x-\beta_{3n+4})^2(x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x)) - \widetilde{C}_0(\varpi(x)))\\
&+ (x-\beta_{3n+4})^2\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x))\\
&+ \Big[\gamma_{3n+4}\rho^2(x) - (x-\beta_{3n+4})^2(x-\beta_{3n+5})\Big(2\gamma_{3n+3}(x-\beta_{3n+1})\\
&- 2(x-\beta_{3n+3})\rho(x) + \gamma_{3n+4}(x-\beta_{3n+5})\Big)\\
&+ 2(x-\beta_{3n+4})(\gamma_{3n+3}(x-\beta_{3n+1})\\
&- (x-\beta_{3n+3})\rho(x))\Big]\widetilde{D}_{n+1}(\varpi(x))\Big]\Big\}, \quad n \geq 0, \qquad (3.14)
\end{aligned}
$$

where $\widetilde{\phi}$, $\widetilde{C}_0$ *and* $\widetilde{D}_0$ *are defined by* (2.12), (2.13) *and* (2.14)*, respectively.*

Proof. First, we note that the sequences W_n and P_n are related by

$$W_{3n}(x) = P_n(\varpi(x)), \quad n \geq 0, \qquad (3.15)$$

which immediately implies

$$W_{3n+3}(x) = P_{n+1}(\varpi(x)), \quad n \geq 0. \qquad (3.16)$$

Differentiating the previous relation, we obtain

$$W'_{3n+3}(x) = \varpi'(x)P'_{n+1}(\varpi(x)), \quad n \geq 0. \qquad (3.17)$$

Substituting x with $\varpi(x)$ in (3.1) leads to

$$
\begin{aligned}
\widetilde{\phi}(\varpi(x))P'_{n+1}(\varpi(x)) =& \frac{1}{2}\Big(\widetilde{C}_{n+1}(\varpi(x)) - \widetilde{C}_0(\varpi(x))\Big)P_{n+1}(\varpi(x))\\
&- \widetilde{\gamma}_{n+1}\widetilde{D}_{n+1}(\varpi(x))P_n(\varpi(x)), \quad n \geq 0.
\end{aligned}
$$

By multiplying both sides by $\varpi'(x)$ and applying (3.15), (3.16) and (3.17), we arrive at

$$
\begin{aligned}
\widetilde{\phi}(\varpi(x))W'_{3n+3}(x) =& \frac{1}{2}\varpi'(x)\Big(\widetilde{C}_{n+1}(\varpi(x)) - \widetilde{C}_0(\varpi(x))\Big)W_{3n+3}(x)\\
&- \widetilde{\gamma}_{n+1}\varpi'(x)\widetilde{D}_{n+1}(\varpi(x))W_{3n}(x), \quad n \geq 0. \qquad (3.18)
\end{aligned}
$$

Shifting the indices in the three-term recurrence relation (2.1) by $n \leftarrow 3n$ and $n \leftarrow 3n+1$ yields, respectively,

$$W_{3n+2}(x) = (x-\beta_{3n+1})W_{3n+1}(x) - \gamma_{3n+1}W_{3n}(x), \quad n \geq 0, \qquad (3.19)$$

$$W_{3n+3}(x) = (x-\beta_{3n+2})W_{3n+2}(x) - \gamma_{3n+2}W_{3n+1}(x), \quad n \geq 0. \qquad (3.20)$$

By combining (3.19) and (3.20), we obtain

(3.21)

$$W_{3n}(x) = \frac{(x-\beta_{3n+1})(x-\beta_{3n+2})-\gamma_{3n+2}}{\gamma_{3n+1}\gamma_{3n+2}} W_{3n+2}(x) - \frac{x-\beta_{3n+1}}{\gamma_{3n+1}\gamma_{3n+2}} W_{3n+3}(x), \quad n \geq 0.$$

Moreover, from (2.11), it follows that (3.21) can be rewritten as

$$W_{3n}(x) = \frac{\rho(x)}{\gamma_{3n+1}\gamma_{3n+2}} W_{3n+2}(x) - \frac{x-\beta_{3n+1}}{\gamma_{3n+1}\gamma_{3n+2}} W_{3n+3}(x), \quad n \geq 0. \tag{3.22}$$

Substituting this result into (3.18) leads to

$$\widetilde{\phi}\big(\varpi(x)\big) W'_{3n+3}(x) = M(x;n) W_{3n+3}(x) + N(x;n) W_{3n+2}(x), \quad n \geq 0, \tag{3.23}$$

with

$$\begin{aligned} M(x;n) =& \frac{1}{2}\varpi'(x)\Big(\widetilde{C}_{n+1}(\varpi(x)) - \widetilde{C}_0(\varpi(x))\Big) \\ &+ \frac{\widetilde{\gamma}_{n+1}}{\gamma_{3n+1}\gamma_{3n+2}} \varpi'(x)(x-\beta_{3n+1})\widetilde{D}_{n+1}(\varpi(x)), \quad n \geq 0, \end{aligned} \tag{3.24}$$

and

$$N(x;n) = -\frac{\widetilde{\gamma}_{n+1}}{\gamma_{3n+1}\gamma_{3n+2}} \varpi'(x)\rho(x)\widetilde{D}_{n+1}(\varpi(x)), \quad n \geq 0. \tag{3.25}$$

On the other hand, by (3.2) we have

$$\begin{aligned} \phi(x) W'_{3n+3}(x) =& \frac{1}{2}(C_{3n+3}(x) - C_0(x)) W_{3n+3}(x) \\ & - \gamma_{3n+3} D_{3n+3}(x) W_{3n+2}(x), \quad n \geq 0. \end{aligned} \tag{3.26}$$

So, from (3.23) and (3.26) we have

$$\begin{aligned} &\Big(\rho(x) M(x;n) - \frac{1}{2}(C_{3n+3}(x) - C_0(x))\Big) W_{3n+3}(x) \\ &+ \Big(\rho(x) N(x;n) + \gamma_{3n+3} D_{3n+3}(x)\Big) W_{3n+2}(x) = 0, \quad n \geq 0. \end{aligned}$$

Since $W_{3n+2}(x)$ and $W_{3n+3}(x)$ have no common roots, $W_{2n+3}(x)$ divides

$$\rho(x) N(x;n) + \gamma_{3n+3} D_{3n+3}(x),$$

which is a polynomial of degree at most equal to $3s+4$. Therefore, we have necessarily $\rho(x)N(x;n)+\gamma_{3n+3}D_{3n+3}(x) = 0$ for $n \geq s+1$, and also $\rho(x)M(x;n) = \frac{1}{2}\Big(C_{3n+3}(x) - C_0(x)\Big)$, $n \geq s+1$. By virtue of the recurrence relation we can easily prove by induction that

$$\frac{1}{2}(C_{3n+3}(x) - C_0(x)) = \rho(x) M(x;n), \quad n \geq 0, \tag{3.27}$$

$$-\gamma_{3n+3} D_{3n+3}(x) = \rho(x) N(x;n), \quad n \geq 0. \tag{3.28}$$

Thus, equation (3.27) can be written as

$$C_{3n+3}(x) = C_0(x) + 2\rho(x) M(x;n), \quad n \geq 0.$$

This yields

$$C_{3n+3}(x) = C_0(x) + \rho(x)\varpi'(x)\Big(\widetilde{C}_{n+1}(\varpi(x)) - \widetilde{C}_0(\varpi(x))\Big) + \frac{2\widetilde{\gamma}_{n+1}}{\gamma_{3n+1}\gamma_{3n+2}}\rho(x)\varpi'(x)(x-\beta_{3n+1})\widetilde{D}_{n+1}(\varpi(x)), \quad n \geq 0.$$

Then, using (2.9), we obtain

$$C_{3n+3}(x) = C_0(x) + \rho(x)\varpi'(x)\Big(\widetilde{C}_{n+1}(\varpi(x)) - \widetilde{C}_0(\varpi(x))\Big) + 2\gamma_{3n+3}\rho(x)\varpi'(x)(x-\beta_{3n+1})\widetilde{D}_{n+1}(\varpi(x)), \quad n \geq 0. \tag{3.29}$$

Finally, using (2.13), we recover (3.9).

Now, equation (3.28) can be expressed as

$$D_{3n+3}(x) = -\frac{1}{\gamma_{3n+3}}\rho(x)N(x;n), \quad n \geq 0,$$

which then becomes, using relation (3.25),

$$D_{3n+3}(x) = \frac{\widetilde{\gamma}_{n+1}}{\gamma_{3n+1}\gamma_{3n+2}\gamma_{3n+3}}\varpi'(x)\rho^2(x)\widetilde{D}_{n+1}(\varpi(x)), \quad n \geq 0.$$

Then, using (2.9), we get

$$D_{3n+3}(x) = \varpi'(x)\rho^2(x)\widetilde{D}_{n+1}(\varpi(x)), \quad n \geq 0. \tag{3.30}$$

This is exactly (3.10).

Hence, (3.11) is deduced by changing $n \leftarrow 3n+3$ in (3.3) and using (3.9)–(3.10). Indeed,

$$C_{3n+4}(x) = -C_{3n+3}(x) + 2(x-\beta_{3n+3})D_{3n+3}(x), \quad n \geq 0,$$

which, from (3.29) and (3.30), becomes

$$C_{3n+4}(x) = -C_0(x) - \varpi'(x)\,\rho(x)\Big\{\widetilde{C}_{n+1}(\varpi(x)) - \widetilde{C}_0(\varpi(x)) + 2[\gamma_{3n+3}(x-\beta_{3n+1}) - (x-\beta_{3n+3})\rho(x)]\widetilde{D}_{n+1}(\varpi(x))\Big\}, \quad n \geq 0,$$

which is exactly relation (3.11), taking (2.13) into account.

Now, making the changes $n \leftarrow 3n+4$, and $n \leftarrow 3n+5$ respectively in (3.4), and $n \leftarrow 3n+4$ in (3.3), successively, we get

$$\gamma_{3n+5}D_{3n+5}(x) = -\phi(x) + \gamma_{3n+4}D_{3n+3}(x) + (x-\beta_{3n+4})^2 D_{3n+4}(x) - (x-\beta_{3n+4})C_{3n+4}(x), \quad n \geq 0, \tag{3.31}$$

$$\gamma_{3n+6}D_{3n+6}(x) = -\phi(x) + \gamma_{3n+5}D_{3n+4}(x) + (x-\beta_{3n+5})^2 D_{3n+5}(x) - (x-\beta_{3n+5})C_{3n+5}(x), \quad n \geq 0, \tag{3.32}$$

$$C_{3n+5}(x) = -C_{3n+4}(x) + 2(x-\beta_{3n+4})D_{3n+4}(x), \quad n \geq 0. \tag{3.33}$$

Substituting (3.31) and (3.33) in (3.32), we obtain by taking (2.11) into account

$$\rho^2(x)D_{3n+4}(x) = E(x;n) + F(x;n), \quad n \geq 0, \tag{3.34}$$

with

$$\begin{aligned}E(x;n) =&\gamma_{3n+5}\gamma_{3n+6}D_{3n+6}(x)-\gamma_{3n+4}(x-\beta_{3n+5})^2D_{3n+3}(x), \quad n\geq 0,\\ F(x;n) =&\phi(x)((x-\beta_{3n+5})^2+\gamma_{3n+5})+\rho(x)(x-\beta_{3n+5})C_{3n+4}(x), \quad n\geq 0.\end{aligned}$$

Our goal now is to factor $E(x;n)$ and $F(x;n)$ by $\rho^2(x)$. Indeed, on one hand,

$$\begin{aligned}E(x;n) =&\varpi'(x)\rho^2(x)\Big[\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x))\\ &-\gamma_{3n+4}(x-\beta_{3n+5})^2\widetilde{D}_{n+1}(\varpi(x))\Big], \quad n\geq 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}F(x;n) =&\rho(x)\Big[\widetilde{\phi}(\varpi(x))((x-\beta_{3n+5})^2+\gamma_{3n+5})+(x-\beta_{3n+5})C_{3n+4}(x)\Big]\\ =&\rho^2(x)\Big[-\varpi'(x)(x-\beta_{3n+5})\Big(\widetilde{C}_{n+1}(\varpi(x))-\widetilde{C}_0(\varpi(x))\Big)\\ &-2\gamma_{3n+3}\varpi'(x)(x-\beta_{3n+1})(x-\beta_{3n+5})\widetilde{D}_{n+1}(\varpi(x))\\ &+2(x-\beta_{3n+3})(x-\beta_{3n+5})\varpi'(x)\rho(x)\widetilde{D}_{n+1}(\varpi(x))\Big]\\ &+\rho(x)\Big[\widetilde{\phi}(\varpi(x))((x-\beta_{3n+5})^2+\gamma_{3n+5})-(x-\beta_{3n+5})C_0(x)\Big]\\ =&\rho^2(x)\Big[-\varpi'(x)(x-\beta_{3n+5})\Big(\widetilde{C}_{n+1}(\varpi(x))-\widetilde{C}_0(\varpi(x))\Big)\\ &-2\gamma_{3n+3}\varpi'(x)(x-\beta_{3n+1})(x-\beta_{3n+5})\widetilde{D}_{n+1}(\varpi(x))\\ &+2(x-\beta_{3n+3})(x-\beta_{3n+5})\varpi'(x)\rho(x)\widetilde{D}_{n+1}(\varpi(x))\\ &-(x-\beta_{3n+5})\varpi'(x)\widetilde{C}_0\left(\varpi(x)\right)\Big]\\ &+\rho(x)\widetilde{\phi}(\varpi(x))\Big[(x-\beta_{3n+5})^2+\gamma_{3n+5}-(x-\beta_{3n+5})\rho'(z)\Big], \quad n\geq 0.\end{aligned}$$

Using the identity

$$(x-\beta_{3n+5})^2+\gamma_{3n+5}-(x-\beta_{3n+5})\rho'(x)=-\rho(x), \quad n\geq 0,$$

which can be straightforwardly verified using (2.11), the last equation becomes

$$\begin{aligned}F(x;n) =&-\rho^2(x)\Big\{\widetilde{\phi}(\varpi(x))+\varpi'(x)(x-\beta_{3n+5})\Big[\widetilde{C}_{n+1}(\varpi(x))-\widetilde{C}_0(\varpi(x))\\ &+2\gamma_{3n+3}(x-\beta_{3n+1})\widetilde{D}_{n+1}(\varpi(x))\\ &-2(x-\beta_{3n+3})\rho(x)\widetilde{D}_{n+1}(\varpi(x))-\widetilde{C}_0\left(\varpi(x)\right)\Big]\Big\}, \quad n\geq 0.\end{aligned}$$

By substituting the expressions for $E(x;n)$ and $F(x;n)$ obtained in (3.34) and simplifying both sides of the resulting equation by $\rho^2(x)$, we obtain

$$\begin{aligned}D_{3n+4}(x) =&-\widetilde{\phi}(\varpi(x))+\varpi'(x)\Big[\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x))\\ &-\gamma_{3n+4}(x-\beta_{3n+5})^2\widetilde{D}_{n+1}(\varpi(x))\Big]\\ &-\varpi'(x)(x-\beta_{3n+5})\Big[\widetilde{C}_{n+1}(\varpi(x))-\widetilde{C}_0(\varpi(x))\\ &+2\Big(\gamma_{3n+3}(x-\beta_{3n+1})-(x-\beta_{3n+3})\rho(x)\Big)\end{aligned}$$

$$\times \widetilde{D}_{n+1}(\varpi(x)) - \widetilde{C}_0(\varpi(x))\Big], \quad n \geq 0.$$

According to (3.33) and using the expression for C_{3n+4} and D_{3n+4} alreardy determined, we obtain the expression of C_{3n+2} given in (3.13). Finally, (3.14) is obtained by using (3.9), (3.13) and (3.33). In the same manner as above, one can recover the relations (3.5), (3.6), (3.7) and (3.8). Thus, the proof is complete. □

3.2. **Illustrative Examples.** In this section, we present several illustrative examples of Proposition 3.1 for semiclassical forms of class less than or equal to two, obtained via a cubic decomposition.

3.2.1. *Examples of class one.* Recently, all semiclassical forms of class one, whose MOPS $\{W_n\}_{n\geq 0}$ satisfy $W_{3n}(x) = P_n(x^3 + qx + r)$, $n \geq 0$, have been determined (see [4, 20]). In this subsection, we establish the structure relations for several of the MOPS families presented in [20].

Remark 3.1. In [20], the coefficients of the three-term recurrence relation satisfied by the above families are provided.

We now present the elements of the structure relation for the sequence $\{W_n\}_{n\geq 0}$ corresponding to the three families in [20, Proposition 4.2].

Proposition 3.2. *The elements of the structure relation,* $\{C_n\}_{n\geq 0}$ *and* $\{D_n\}_{n\geq 0}$*, for the sequence* $\{W_n\}_{n\geq 0}$ *of the first family in* [20, Proposition 4.2] *are*

$$\begin{aligned}
C_{3n}(x) =& \frac{1}{2}(12n + 6\alpha + 1)(x + \mu_q)x + q(2\alpha + 1), \quad n \geq 0,\\
D_{3n}(x) =& \frac{1}{2}(12n + 6\alpha + 3)(x + \mu_q), \quad n \geq 0,\\
C_{3n+1}(x) =& \frac{1}{2}(12n + 6\alpha + 5)(x + \mu_q)x - q(2\alpha + 1), \quad n \geq 0,\\
D_{3n+1}(x) =& \frac{1}{2}(12n + 6\alpha + 7)x + (6n + 2)\mu_q, \quad n \geq 0,\\
C_{3n+2}(x) =& \frac{1}{2}(12n + 6\alpha + 9)x^2 + \Big(\frac{1}{2}(12n - 6\alpha + 3)\\
&+ \frac{(2\alpha + 1)(12n + 6\alpha + 7)}{4n + 2\alpha + 3}\Big)\mu_q x + \frac{q(2\alpha + 1)(6\alpha + 5)}{3(4n + 2\alpha + 3)}, \quad n \geq 0,\\
D_{3n+2}(x) =& \frac{1}{2}(12n + 6\alpha + 11)x + (6n + 6\alpha + 7)\mu_q, \quad n \geq 0.
\end{aligned}$$

Remark 3.2. For this first family of [20, Proposition 4.2], the sequence $\{P_n\}_{n\geq 0}$ is classical and corresponds to a shifted Jacobi sequence. More precisely, we have

$$P_n(x) = \Big(\frac{2q}{3}\mu_q\Big)^n \widetilde{P}_n^{(\alpha,-1/2)}\Big(\Big(\frac{2q}{3}\mu_q\Big)^{-1}(x - r)\Big), \quad n \geq 0, \tag{3.35}$$

where $\{\widetilde{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ is the sequence of monic Jacobi polynomials, orthogonal with respect to $\widetilde{u}_0 = \mathcal{J}(\alpha, \beta)$.

Let us recall the structure relation for the Jacobi polynomials (see [16])

$$(x^2-1)\widetilde{P}^{(\alpha,\beta)\prime}_{n+1}(x) = \frac{1}{2}\Big(C_{n+1}(x;\alpha,\beta) - C_0(x;\alpha,\beta)\Big)\widetilde{P}^{(\alpha,\beta)}_{n+1}(x) - \gamma^{(\alpha,\beta)}_{n+1} D_{n+1}(x;\alpha,\beta)\widetilde{P}^{(\alpha,\beta)}_{n}(x), \quad n \geq 0,$$

where

$$C_n(x;\alpha,\beta) = (2n+\alpha+\beta)x - \frac{\alpha^2-\beta^2}{2n+\alpha+\beta}, \quad n \geq 0,$$
$$D_n(x;\alpha,\beta) = 2n+\alpha+\beta+1, \quad n \geq 0,$$
$$\gamma^{(\alpha,\beta)}_{n+1} = 4\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}, \quad n \geq 0.$$

Proposition 3.3. *The elements of the structure relation* $\{C_n\}_{n\geq 0}$ *and* $\{D_n\}_{n\geq 0}$ *of the MOPS* $\{W_n\}_{n\geq 0}$ *of the second family of* [20, Proposition 4.2], *are given by*

$$C_{3n}(x) = (6n+3\lambda+1)x^2 - \kappa x - (6n-1)\kappa^2, \quad n \geq 0,$$
$$D_{3n}(x) = (6n+3\lambda+2)(x+\kappa), \quad n \geq 0,$$
$$C_{3n+1}(x) = (6n+3\lambda+3)x^2 + \kappa x - (6n+6\lambda+5)\kappa^2, \quad n \geq 0,$$
$$D_{3n+1}(x) = (6n+3\lambda+4)x - (3n+3\lambda+2)\kappa, \quad n \geq 0,$$
$$C_{3n+2}(x) = (6n+3\lambda+5)x^2 - \frac{3\lambda+1}{6n+3\lambda+5}\kappa x + \left(3n+3\lambda+3 - \frac{(3\lambda+1)(3n+3\lambda+2)}{6n+3\lambda+5}\right)\kappa^2, \quad n \geq 0,$$
$$D_{3n+2}(x) = (6n+3\lambda+6)x - (3n+3)\kappa, \quad n \geq 0.$$

Remark 3.3. Notice that the above result appears in [2].

Remark 3.4. For the second family of [20, Proposition 4.2], the MOPS $\{P_n\}_{n\geq 0}$ is also classical and corresponds to a shifted Jacobi polynomial sequence. In particular, we have

$$P_n(x) = \left(\frac{\kappa^3}{2}\right)^n \widetilde{P}^{(\lambda,-1/3)}_n\left(\left(\frac{\kappa^3}{2}\right)^{-1}(x-r)+1\right), \quad n \geq 0.$$

Proposition 3.4. *The elements of the structure relation,* $\{C_n\}_{n\geq 0}$ *and* $\{D_n\}_{n\geq 0}$, *for the MOPS* $\{W_n\}_{n\geq 0}$ *of the third family in* [20, Proposition 4.2] *are given by*

$$C_{3n}(x) = \frac{1}{2}(12n+6\alpha+1)x^2 - \frac{1}{6}\tau(12n+6\alpha+7)x - \frac{1}{3}\tau^2(12n+2\alpha-1), \quad n \geq 0,$$
$$D_{3n}(x) = \frac{1}{2}(12n+6\alpha+3)\left(x+\frac{2}{3}\tau\right), \quad n \geq 0,$$
$$C_{3n+1}(x) = \frac{1}{2}(12n+6\alpha+5)x^2 - \frac{1}{6}\tau(12n+6\alpha-1)x - \frac{1}{3}\tau^2(12n+10\alpha+7), \quad n \geq 0,$$
$$D_{3n+1}(x) = \frac{1}{2}(12n+6\alpha+7)x - \frac{1}{3}\tau(15n+9\alpha+8), \quad n \geq 0,$$

$$
\begin{aligned}
C_{3n+2}(x) =& \frac{1}{2}(12n+6\alpha+9)x^2 - \Big)\frac{1}{2}(4n+2\alpha+1) + \frac{2}{3}\cdot\frac{(6n+4\alpha+5)}{(4n+2\alpha+3)}\Big)\tau x \\
&+ \frac{1}{3}\tau^2\Big)12n+10\alpha+7 - \frac{2(6n+4\alpha+5)(15n+9\alpha+8)}{3(4n+2\alpha+3)}\Big), \quad n \geq 0, \\
D_{3n+2}(x) =& \frac{1}{2}(12n+6\alpha+11)x - \frac{1}{6}\tau(30n+12\alpha+29), \quad n \geq 0.
\end{aligned}
$$

Remark 3.5. For the third family of [20, Proposition 4.2], the MOPS $\{P_n\}_{n\geq 0}$ remains classical but corresponds to another shifted Jacobi polynomial sequence, with a different set of parameters. More precisely, we have

$$
P_n(x) = \left(\frac{2\tau^3}{27}\right)^n \widetilde{P}_n^{(\alpha,-1/2)}\left(\left(\frac{2\tau^3}{27}\right)^{-1}(x-r)\right), \quad n \geq 0.
$$

3.2.2. *Examples of class two.* Recently, all semiclassical forms of class one that their corresponding MOPS $\{W_n\}_{n\geq 0}$, satisfies $W_{3n}(x) = P_n(x^3)$ have been determined (see [4, 21]).

Proposition 3.5. *The elements of the structure relation* $\{C_n\}_{n\geq 0}$ *and* $\{D_n\}_{n\geq 0}$ *of the MOPS* $\{W_n\}_{n\geq 0}$ *of the third family of* [21, Theorem 2.4] *are*

$$
\begin{aligned}
C_{3n}(x) =& (6n+3\alpha+3\beta+2)x^3 - \tau x^2 + \frac{1}{2}\tau^2\left(\frac{9n(n+\beta)}{\alpha+1}+1\right)x \\
&+ \frac{3}{8}\tau^3\left(\frac{6n(n+\beta)}{\alpha+1}+2n+\beta\right), \quad n \geq 0, \\
D_{3n}(x) =& 3(2n+\alpha+\beta+1)\left(x+\frac{1}{2}\tau\right)^2, \quad n \geq 0, \\
C_{3n+1}(x) =& (6n+3\alpha+3\beta+4)x^3 + \tau x^2 - \frac{1}{2}\tau^2\left(\frac{9n(n+\beta)}{\alpha+1}+18n+9\alpha+9\beta+10\right)x \\
&- \frac{3}{8}\tau^3\left(\frac{6n(n+\beta)}{\alpha+1}+10n+4\alpha+5\beta+4\right), \quad n \geq 0, \\
D_{3n+1}(x) =& (6n+3\alpha+3\beta+5)x^2 - \frac{1}{2}\tau(6n+4\alpha+3\beta+5)x \\
&- \frac{1}{4}\tau^2\left(\frac{9n(n+\beta)}{\alpha+1}+12n+4\alpha+6\beta+4\right), \quad n \geq 0, \\
C_{3n+2}(x) =& (6n+3\alpha+3\beta+6)x^3 - \frac{1}{3}\tau\frac{\alpha+1}{2n+\alpha+\beta+2}x^2 - \frac{1}{6}\tau^2\frac{\alpha^2-1}{2n+\alpha+\beta+2}x \\
&+ \frac{3}{8}(2n+\beta)\tau^3 - \frac{1}{12}\tau^3\left(\frac{9n(n+\beta)-2(\alpha+4)(\alpha+1)}{2n+\alpha+\beta+2}\right), \quad n \geq 0, \\
D_{3n+2}(x) =& (6n+3\alpha+3\beta+7)x^2 - \frac{1}{2}\tau(6n+2\alpha+3\beta+7)x
\end{aligned}
$$

$$+\frac{1}{4}\tau^2\left(\frac{9(n+1)(n+\beta+1)}{\alpha+1}+6n+\alpha+3\beta+7\right),\quad n\geq 0.$$

Proposition 3.6. *The elements of the structure relation* $\{C_n\}_{n\geq 0}$ *and* $\{D_n\}_{n\geq 0}$ *of the MOPS* $\{W_n\}_{n\geq 0}$ *of the fourth family of* [21, Theorem 2.4] *are*

$$\begin{aligned}
C_{3n}(x) =&(6n+3\alpha+3\beta+2)x^3-\tau x^2-\tau^2(6n-1)x+\tau^3(3\beta+1),\quad n\geq 0,\\
D_{3n}(x) =&3(2n+\alpha+\beta+1)x(x+\tau),\quad n\geq 0,\\
C_{3n+1}(x) =&(6n+3\alpha+3\beta+4)x^3+\tau x^2\\
&-\tau^2(6n+6\alpha+6\beta+7)x-\tau^3(3\beta+1),\quad n\geq 0,\\
D_{3n+1}(x) =&(6n+3\alpha+3\beta+5)x^2-\tau(3n+3\alpha+2)x-\tau^2(3\beta+1),\quad n\geq 0,\\
C_{3n+2}(x) =&(6n+3\alpha+3\beta+6)x^3+\tau\frac{\beta-\alpha}{2n+\alpha+\beta+2}x^2\\
&-\tau^2\left(\frac{2(n+\alpha+1)(3n+3\alpha+2)}{2n+\alpha+\beta+2}-(6n+6\alpha+5)\right)x\\
&+\tau^3\frac{(3\beta+1)(\beta-\alpha)}{2n+\alpha+\beta+2},\quad n\geq 0,\\
D_{3n+2}(x) =&(6n+3\alpha+3\beta+7)x^2-\tau(3n+3\beta+4)x+\tau^2(3\beta+1),\quad n\geq 0.
\end{aligned}$$

4. Differential Equation

In this section, we establish the second-order differential equation satisfied by each of the MOPS presented in [20].

It is well known that a semiclassical orthogonal polynomials sequence fulfills a second-order linear differential equation [16]. The objective of this section is to give explicitly the second-order linear differential equation fulfilled by $\{W_n\}_{n\geq 0}$. Let us recall that $\{W_n\}_{n\geq 0}$ satisfy

$$J(x;n)W''_{n+1}(x)+K(x;n)W'_{n+1}(x)+L(x;n)W_{n+1}(x)=0,\quad n\geq 0, \tag{4.1}$$

with

$$J(x;n)=\phi(x)D_{n+1}(x),\quad n\geq 0, \tag{4.2}$$

$$K(x;n)=C_0(x)D_{n+1}(x)-\mathcal{W}(\phi,D_{n+1})(x),\quad n\geq 0, \tag{4.3}$$

$$L(x;n)=\mathcal{W}\Big(\frac{1}{2}(C_{n+1}-C_0),D_{n+1}\Big)(x)-D_{n+1}(x)\sum_{\nu=0}^{n}D_\nu(x),\quad n\geq 0, \tag{4.4}$$

where $\mathcal{W}(f,g)=fg'-gf'$ is the Wronskian of f and g.

We proceed to state the main result of this section.

Proposition 4.1. *The elements of the second-order linear differential equation,* $\{J(\cdot;n)\}_{n\geq 0}$, $\{K(\cdot;n)\}_{n\geq 0}$, *and* $\{L(\cdot;n)\}_{n\geq 0}$, *of the MOPS* $\{W_n\}_{n\geq 0}$ *are*

$$J(x;0)=\rho(x)\,\widetilde{\phi}(\varpi(x))\Big\{-\widetilde{\phi}(\varpi(x))+\varpi'(x)\Big[\gamma_2\gamma_3\,\widetilde{D}_1(\varpi(x))$$

$$
\begin{aligned}
&- (x-\beta_2)(-2(x-\beta_0)\rho(x)+\gamma_1(x-\beta_2))\,\widetilde{D}_0(\varpi(x))\Big]\Big\},\\
K(x;0) =&\Big(2\,\rho'(x)\,\widetilde{\phi}(\varpi(x))+\rho(x)\,\varpi'(x)\,\widetilde{\phi}'(\varpi(x))+\rho(x)\,\varpi'(x)\,\widetilde{C}_0(\varpi(x))\Big)\\
&\times\Big[-\,\widetilde{\phi}(\varpi(x))+\varpi'(x)\Big\{\gamma_2\gamma_3\widetilde{D}_1(\varpi(x))\\
&-(x-\beta_2)\Big[-2(x-\beta_0)\rho(x)+\gamma_1(x-\beta_2)\Big]\widetilde{D}_0(\varpi(x))\Big\}\Big]\\
&-\rho(x)\,\widetilde{\phi}(\varpi(x))\frac{d}{dx}\Big[-\,\widetilde{\phi}(\varpi(x))+\varpi'(x)\Big\{\gamma_2\gamma_3\widetilde{D}_1(\varpi(x))\\
&-(x-\beta_2)\Big[-2(x-\beta_0)\rho(x)+\gamma_1(x-\beta_2)\Big]\widetilde{D}_0(\varpi(x))\Big\}\Big],\\
L(x;0) =&\Big[-\rho'(x)\,\widetilde{\phi}(\varpi(x))-\rho(x)\,\varpi'(x)\,\widetilde{C}_0(\varpi(x))\\
&+(x-\beta_0)\,\rho^2(x)\,\varpi'(x)\,\widetilde{D}_0(\varpi(x))\Big]\\
&\times\frac{d}{dx}\Big[-\,\widetilde{\phi}(\varpi(x))+\varpi'(x)\Big\{\gamma_2\gamma_3\widetilde{D}_1(\varpi(x))-(x-\beta_2)(-2(x-\beta_0)\rho(x)\\
&+\gamma_1(x-\beta_2))\widetilde{D}_0(\varpi(x))\Big\}\Big]-\Big[-\,\widetilde{\phi}(\varpi(x))+\varpi'(x)\Big\{\gamma_2\gamma_3\widetilde{D}_1(\varpi(x))\\
&-(x-\beta_2)(-2(x-\beta_0)\rho(x)+\gamma_1(x-\beta_2))\widetilde{D}_0(\varpi(x))\Big\}\Big]\\
&\times\Big[\frac{d}{dx}\Big(-\rho'(x)\,\widetilde{\phi}(\varpi(x))-\rho(x)\,\varpi'(x)\,\widetilde{C}_0(\varpi(x))\\
&+(x-\beta_0)\,\rho^2(x)\,\varpi'(x)\,\widetilde{D}_0(\varpi(x))\Big)+S_0(x)\Big],\\
J(x;1) =&\frac{1}{\gamma_2}\rho(x)\,\widetilde{\phi}(\varpi(x))\Big\{\Big[-\rho(x)-(x-\beta_1)^2+(x-\beta_1)\rho'(x)\Big]\widetilde{\phi}(\varpi(x))\\
&+\varpi'(x)\Big[(x-\beta_1)\rho(x)\,\widetilde{C}_0(\varpi(x))+(x-\beta_1)^2\gamma_2\gamma_3\,\widetilde{D}_1(\varpi(x))\\
&+\Big(\gamma_1\rho^2(x)+2(x-\beta_1)(x-\beta_0)\rho(x)-(x-\beta_1)^2(x-\beta_2)\\
&\times\Big(-2(x-\beta_0)\rho(x)+\gamma_1(x-\beta_2)\Big)\Big)\widetilde{D}_0\Big(\varpi(x)\Big)\Big]\Big\},\\
K(x;1) =&\Big(\rho'(x)\,\widetilde{\phi}(\varpi(x))+\rho(x)\,\varpi'(x)\,\widetilde{\phi}'(\varpi(x))+\rho(x)\,\varpi'(x)\,\widetilde{C}_0(\varpi(x))\Big)\\
&\times\frac{1}{\gamma_2}\Big\{\Big[-\rho(x)-(x-\beta_1)^2+(x-\beta_1)\rho'(x)\Big]\widetilde{\phi}(\varpi(x))\\
&+\varpi'(x)\Big[(x-\beta_1)\rho(x)\,\widetilde{C}_0(\varpi(x))+(x-\beta_1)^2\gamma_2\gamma_3\widetilde{D}_1(\varpi(x))\\
&+\Big(\gamma_1\rho^2(x)+2(x-\beta_1)(x-\beta_0)\rho(x)\\
&-(x-\beta_1)^2(x-\beta_2)\Big(-2(x-\beta_0)\rho(x)+\gamma_1(x-\beta_2)\Big)\Big)\widetilde{D}_0(\varpi(x))\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&- \rho(x)\, \widetilde{\phi}(\varpi(x))\, \frac{d}{dx}\Big[\frac{1}{\gamma_2}\Big\{\Big[-\rho(x)-(x-\beta_1)^2+(x-\beta_1)\rho'(x)\Big]\widetilde{\phi} \\
&\times (\varpi(x)) + \varpi'(x)\Big[(x-\beta_1)\rho(x)\,\widetilde{C}_0(\varpi(x)) + (x-\beta_1)^2\gamma_2\gamma_3\widetilde{D}_1(\varpi(x)) \\
&+ \Big(\gamma_1\rho^2(x) + 2(x-\beta_1)(x-\beta_0)\rho(x) \\
&- (x-\beta_1)^2(x-\beta_2)\Big(-2(x-\beta_0)\rho(x)+\gamma_1(x-\beta_2)\Big)\Big)\widetilde{D}_0(\varpi(x))\Big]\Big\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
L(x;1) =&\Big[-(x-\beta_1)\,\widetilde{\phi}(\varpi(x)) + \varpi'(x)\Big((x-\beta_1)(x-\beta_3)\,\widetilde{C}_0(\varpi(x)) \\
&+ \Big(-(x-\beta_0)\rho(x)^2 + 2(x-\beta_1)(x-\beta_3)(x-\beta_0)\rho(x) \\
&- (x-\beta_1)\gamma_1(x-\beta_3)^2\Big)\widetilde{D}_0(\varpi(x)) + (x-\beta_1)\gamma_2\gamma_3\,\widetilde{D}_1(\varpi(x))\Big)\Big] \\
&\times \frac{d}{dx}\Big\{\frac{1}{\gamma_2}\Big[[-\rho(x)-(x-\beta_1)^2+(x-\beta_1)\rho'(x)]\widetilde{\phi}(\varpi(x)) \\
&+ \varpi'(x)\Big[(x-\beta_1)\rho(x)\widetilde{C}_0(\varpi(x)) + (x-\beta_1)^2\gamma_2\gamma_3\widetilde{D}_1(\varpi(x)) + (\gamma_1\rho^2(x) \\
&+ 2(x-\beta_1)(x-\beta_0)\rho(x) - (x-\beta_1)^2(x-\beta_2)(-2(x-\beta_0)\rho(x) \\
&+ \gamma_1(x-\beta_2)))\widetilde{D}_0(\varpi(x))\Big]\Big]\Big\} \\
&- \frac{1}{\gamma_2}\Big[[-\rho(x)-(x-\beta_1)^2+(x-\beta_1)\rho'(x)]\widetilde{\phi}(\varpi(x)) \\
&+ \varpi'(x)\Big[(x-\beta_1)\rho(x)\widetilde{C}_0(\varpi(x)) + (x-\beta_1)^2\gamma_2\gamma_3\widetilde{D}_1(\varpi(x)) + (\gamma_1\rho^2(x) \\
&+ 2(x-\beta_1)(x-\beta_0)\rho(x) - (x-\beta_1)^2(x-\beta_2)(-2(x-\beta_0)\rho(x) \\
&+ \gamma_1(x-\beta_2)))\widetilde{D}_0(\varpi(x))\Big]\Big] \\
&\times \Big[\frac{d}{dx}\Big(-(x-\beta_1)\,\widetilde{\phi}(\varpi(x)) + \varpi'(x) \times \Big((x-\beta_1)(x-\beta_3)\,\widetilde{C}_0(\varpi(x)) \\
&+ \Big(-(x-\beta_0)\rho(x)^2 + 2(x-\beta_1)(x-\beta_3)(x-\beta_0)\rho(x) \\
&- (x-\beta_1)\gamma_1(x-\beta_3)^2\Big)\widetilde{D}_0(\varpi(x)) \\
&+ (x-\beta_1)\gamma_2\gamma_3\,\widetilde{D}_1(\varpi(x))\Big)\Big) + S_1(x)\Big],
\end{aligned}
$$

$$
J(x;2) = \varpi'(x)\,\rho^3(x)\,\widetilde{\phi}(\varpi(x))\,\widetilde{D}_1(\varpi(x)),
$$

$$
\begin{aligned}
K(x;2) =&\Big(\rho'(x)\,\widetilde{\phi}(\varpi(x)) + \rho(x)\,\varpi'(x)\,\widetilde{\phi}'(\varpi(x)) + \rho(x)\,\varpi'(x)\,\widetilde{C}_0(\varpi(x))\Big) \\
&\times \varpi'(x)\,\rho^2(x)\,\widetilde{D}_1(\varpi(x)) - \rho(x)\,\widetilde{\phi}(\varpi(x))\,\frac{d}{dx}\Big[\varpi'(x)\,\rho^2(x)\,\widetilde{D}_1(\varpi(x))\Big],
\end{aligned}
$$

$$
\begin{aligned}
L(x;2) =&\Big[\frac{1}{2}\rho(x)\,\varpi'(x)\,\Big(\widetilde{C}_1(\varpi(x)) - \widetilde{C}_0(\varpi(x))\Big)\\
&+ \gamma_3\,\rho(x)\,\varpi'(x)\,(x-\beta_1)\,\widetilde{D}_1(\varpi(x))\Big]\\
&\times \frac{d}{dx}\Big[\varpi'(x)\rho^2(x)\widetilde{D}_1(\varpi(x))\Big] - \Big[\varpi'(x)\rho^2(x)\widetilde{D}_1(\varpi(x))\Big]\\
&\times \Big[\frac{d}{dx}\Big(\frac{1}{2}\rho(x)\,\varpi'(x)\,\Big(\widetilde{C}_1(\varpi(x)) - \widetilde{C}_0(\varpi(x))\Big)\\
&+ \gamma_3\,\rho(x)\,\varpi'(x)\,(x-\beta_1)\,\widetilde{D}_1(\varpi(x))\Big) + S_2(x)\Big],\\
J(x;3n+3) =&\rho(x)\,\widetilde{\phi}\Big(\varpi(x)\Big)\Big\{-\,\widetilde{\phi}\Big(\varpi(x)\Big) + \varpi'(x)\Big[\gamma_{3n+5}\gamma_{3n+6}\,\widetilde{D}_{n+2}\Big(\varpi(x)\Big)\\
&- (x-\beta_{3n+5})\Big(\widetilde{C}_{n+1}\Big(\varpi(x)\Big) - 2\widetilde{C}_0\Big(\varpi(x)\Big)\Big)\\
&- (x-\beta_{3n+5})\Big(2\gamma_{3n+3}(x-\beta_{3n+1})\\
&- 2(x-\beta_{3n+3})\rho(x) + \gamma_{3n+4}(x-\beta_{3n+5})\Big)\widetilde{D}_{n+1}\Big(\varpi(x)\Big)\Big]\Big\}, \quad n\ge 0,\\
K(x;3n+3) =&\Big(\rho'(x)\widetilde{\phi}(\varpi(x)) + \rho(x)\varpi'(x)\widetilde{\phi}'(\varpi(x)) + \rho(x)\varpi'(x)\widetilde{C}_0(\varpi(x))\Big)\\
&\times\Big[-\widetilde{\phi}(\varpi(x)) + \varpi'(x)\{\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x))\\
&- (x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x)) - 2\widetilde{C}_0(\varpi(x)))\\
&- (x-\beta_{3n+5})(2\gamma_{3n+3}(x-\beta_{3n+1})\\
&- 2(x-\beta_{3n+3})\rho(x) + \gamma_{3n+4}(x-\beta_{3n+5}))\widetilde{D}_{n+1}(\varpi(x))\}\Big]\\
&- \rho(x)\widetilde{\phi}(\varpi(x))\frac{d}{dx}\Big[-\widetilde{\phi}(\varpi(x)) + \varpi'(x)\Big\{\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x))\\
&- (x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x)) - 2\widetilde{C}_0(\varpi(x)))\\
&- (x-\beta_{3n+5})(2\gamma_{3n+3}(x-\beta_{3n+1})\\
&- 2(x-\beta_{3n+3})\rho(x) + \gamma_{3n+4}(x-\beta_{3n+5}))\widetilde{D}_{n+1}(\varpi(x))\Big\}\Big], \quad n\ge 0,\\
L(x;3n+3) =&\Big[-\rho'(x)\,\widetilde{\phi}(\varpi(x)) - \frac{1}{2}\,\rho(x)\,\varpi'(x)\Big(\widetilde{C}_{n+1}(\varpi(x)) + \widetilde{C}_0(\varpi(x))\Big)\\
&+ \rho(x)\,\varpi'(x)\Big((x-\beta_{3n+3})\rho(x) - \gamma_{3n+3}(x-\beta_{3n+1})\Big)\,\widetilde{D}_{n+1}(\varpi(x))\Big]\\
&\times\frac{d}{dx}\Big[-\widetilde{\phi}(\varpi(x)) + \varpi'(x)\{\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x))\\
&- (x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x)) - 2\widetilde{C}_0(\varpi(x)))\\
&- (x-\beta_{3n+5})(2\gamma_{3n+3}(x-\beta_{3n+1})
\end{aligned}
$$

$$
\begin{aligned}
&- 2(x-\beta_{3n+3})\rho(x) + \gamma_{3n+4}(x-\beta_{3n+5}))\widetilde{D}_{n+1}(\varpi(x))\}\Big] \\
&- \Big[- \widetilde{\phi}(\varpi(x)) + \varpi'(x)\{\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x)) \\
&- (x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x)) - 2\widetilde{C}_0(\varpi(x))) \\
&- (x-\beta_{3n+5})(2\gamma_{3n+3}(x-\beta_{3n+1}) \\
&- 2(x-\beta_{3n+3})\rho(x) + \gamma_{3n+4}(x-\beta_{3n+5}))\widetilde{D}_{n+1}(\varpi(x))\}\Big] \\
&\times \Big[\frac{d}{dx}\Big(- \rho'(x)\,\widetilde{\phi}(\varpi(x)) - \frac{1}{2}\,\rho(x)\,\varpi'(x)\Big(\widetilde{C}_{n+1}(\varpi(x)) + \widetilde{C}_0(\varpi(x))\Big) \\
&+ \rho(x)\varpi'(x)\Big((x-\beta_{3n+3})\rho(x) - \gamma_{3n+3}(x-\beta_{3n+1})\Big)\,\widetilde{D}_{n+1}(\varpi(x))\Big) \\
&+ S_{3n+3}(x)\Big], \quad n \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
J(x;3n+4) =& \frac{1}{\gamma_{3n+5}}\rho(x)\,\widetilde{\phi}\Big(\varpi(x)\Big)\Big\{\Big[- \rho(x) - (x-\beta_{3n+4})^2 \\
&+ (x-\beta_{3n+4})\rho'(x)\Big]\widetilde{\phi}(\varpi(x)) + \varpi'(x)\Big[(x-\beta_{3n+4})\rho(x)\widetilde{C}_{n+1}(\varpi(x)) \\
&- (x-\beta_{3n+4})^2(x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x)) - \widetilde{C}_0(\varpi(x))) \\
&+ (x-\beta_{3n+4})^2\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x)) \\
&+ \Big[\gamma_{3n+4}\rho^2(x) - (x-\beta_{3n+4})^2(x-\beta_{3n+5})\Big(2\gamma_{3n+3}(x-\beta_{3n+1}) \\
&- 2(x-\beta_{3n+3})\rho(x) + \gamma_{3n+4}(x-\beta_{3n+5})\Big) \\
&+ 2(x-\beta_{3n+4})(\gamma_{3n+3}(x-\beta_{3n+1}) \\
&- (x-\beta_{3n+3})\rho(x))\Big]\widetilde{D}_{n+1}(\varpi(x))\Big]\Big\}, \quad n \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
K(x;3n+4) =& \Big(\rho'(x)\widetilde{\phi}(\varpi(x)) + \rho(x)\varpi'(x)\widetilde{\phi}'(\varpi(x)) + \rho(x)\varpi'(x)\widetilde{C}_0(\varpi(x))\Big) \\
&\times \frac{1}{\gamma_{3n+5}}\Big\{\Big[- \rho(x) - (x-\beta_{3n+4})^2 + (x-\beta_{3n+4})\rho'(x)\Big]\widetilde{\phi}(\varpi(x)) \\
&+ \varpi'(x)\Big[(x-\beta_{3n+4})\rho(x)\widetilde{C}_{n+1}(\varpi(x)) \\
&- (x-\beta_{3n+4})^2(x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x)) - \widetilde{C}_0(\varpi(x))) \\
&+ (x-\beta_{3n+4})^2\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x)) \\
&+ \Big[\gamma_{3n+4}\rho^2(x) - (x-\beta_{3n+4})^2(x-\beta_{3n+5})\Big(2\gamma_{3n+3}(x-\beta_{3n+1}) \\
&- 2(x-\beta_{3n+3})\rho(x) + \gamma_{3n+4}(x-\beta_{3n+5})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+2(x-\beta_{3n+4})(\gamma_{3n+3}(x-\beta_{3n+1})-(x-\beta_{3n+3})\rho(x))\Big]\widetilde{D}_{n+1}(\varpi(x))\Big]\Big\}\\
&-\rho(x)\widetilde{\phi}(\varpi(x))\frac{d}{dx}\Big[\frac{1}{\gamma_{3n+5}}\Big\{\Big[-\rho(x)-(x-\beta_{3n+4})^2\\
&+(x-\beta_{3n+4})\rho'(x)\Big]\widetilde{\phi}(\varpi(x))+\varpi'(x)\Big[(x-\beta_{3n+4})\rho(x)\widetilde{C}_{n+1}(\varpi(x))\\
&-(x-\beta_{3n+4})^2(x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x))-\widetilde{C}_0(\varpi(x)))\\
&+(x-\beta_{3n+4})^2\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x))\\
&+\Big[\gamma_{3n+4}\rho^2(x)-(x-\beta_{3n+4})^2(x-\beta_{3n+5})\Big(2\gamma_{3n+3}(x-\beta_{3n+1})\\
&-2(x-\beta_{3n+3})\rho(x)+\gamma_{3n+4}(x-\beta_{3n+5})\Big)\\
&+2(x-\beta_{3n+4})(\gamma_{3n+3}(x-\beta_{3n+1})\\
&-(x-\beta_{3n+3})\rho(x))\Big]\widetilde{D}_{n+1}(\varpi(x))\Big]\Big\}\Big],\quad n\geq 0,
\end{aligned}
$$

$$
\begin{aligned}
L(x;3n+4)=&\Big[-(x-\beta_{3n+4})\widetilde{\phi}(\varpi(x))+\frac{1}{2}\varpi'(x)\rho(x)(\widetilde{C}_{n+1}(\varpi(x))-\widetilde{C}_0(\varpi(x)))\\
&-\varpi'(x)\rho(x)((x-\beta_{3n+3})\rho(x)-\gamma_{3n+3}(x-\beta_{3n+1}))\widetilde{D}_{n+1}(\varpi(x))\\
&+(x-\beta_{3n+4})\varpi'(x)\Big(\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x))\\
&-(x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x))-2\widetilde{C}_0(\varpi(x)))\\
&-(x-\beta_{3n+5})(2\gamma_{3n+3}(x-\beta_{3n+1})\\
&-2(x-\beta_{3n+3})\rho(x)+\gamma_{3n+4}(x-\beta_{3n+5}))\widetilde{D}_{n+1}(\varpi(x))\Big)\Big]\\
&\times\frac{d}{dx}\Big[\frac{1}{\gamma_{3n+5}}\Big\{[-\rho(x)-(x-\beta_{3n+4})^2+(x-\beta_{3n+4})\rho'(x)]\widetilde{\phi}(\varpi(x))\\
&+\varpi'(x)\Big((x-\beta_{3n+4})\rho(x)\widetilde{C}_{n+1}(\varpi(x))\\
&-(x-\beta_{3n+4})^2(x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x))-\widetilde{C}_0(\varpi(x)))\\
&+(x-\beta_{3n+4})^2\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x))\\
&+[\gamma_{3n+4}\rho^2(x)-(x-\beta_{3n+4})^2(x-\beta_{3n+5})\\
&\times(2\gamma_{3n+3}(x-\beta_{3n+1})-2(x-\beta_{3n+3})\rho(x)+\gamma_{3n+4}(x-\beta_{3n+5}))\\
&+2(x-\beta_{3n+4})(\gamma_{3n+3}(x-\beta_{3n+1})-(x-\beta_{3n+3})\rho(x))]\widetilde{D}_{n+1}(\varpi(x))\Big)\Big\}\Big]\\
&-\frac{1}{\gamma_{3n+5}}\Big\{[-\rho(x)-(x-\beta_{3n+4})^2+(x-\beta_{3n+4})\rho'(x)]\widetilde{\phi}(\varpi(x))\\
&+\varpi'(x)\Big((x-\beta_{3n+4})\rho(x)\widetilde{C}_{n+1}(\varpi(x))\\
&-(x-\beta_{3n+4})^2(x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x))-\widetilde{C}_0(\varpi(x)))
\end{aligned}
$$

$$
\begin{aligned}
&+ (x-\beta_{3n+4})^2\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x))\\
&+ [\gamma_{3n+4}\rho^2(x) - (x-\beta_{3n+4})^2(x-\beta_{3n+5})\\
&\times (2\gamma_{3n+3}(x-\beta_{3n+1}) - 2(x-\beta_{3n+3})\rho(x) + \gamma_{3n+4}(x-\beta_{3n+5}))\\
&+ 2(x-\beta_{3n+4})(\gamma_{3n+3}(x-\beta_{3n+1}) - (x-\beta_{3n+3})\rho(x))]\widetilde{D}_{n+1}(\varpi(x))\Big)\Big\}\\
&\times \Big[\frac{d}{dx}\Big(-(x-\beta_{3n+4})\widetilde{\phi}(\varpi(x)) + \frac{1}{2}\varpi'(x)\rho(x)(\widetilde{C}_{n+1}(\varpi(x))\\
&- \widetilde{C}_0(\varpi(x))) - \varpi'(x)\rho(x)((x-\beta_{3n+3})\rho(x)\\
&- \gamma_{3n+3}(x-\beta_{3n+1}))\widetilde{D}_{n+1}(\varpi(x)) + (x-\beta_{3n+4})\\
&\times \varpi'(x)\Big(\gamma_{3n+5}\gamma_{3n+6}\widetilde{D}_{n+2}(\varpi(x)) - (x-\beta_{3n+5})(\widetilde{C}_{n+1}(\varpi(x))\\
&- 2\widetilde{C}_0(\varpi(x))) - (x-\beta_{3n+5})(2\gamma_{3n+3}(x-\beta_{3n+1}) - 2(x-\beta_{3n+3})\rho(x)\\
&+ \gamma_{3n+4}(x-\beta_{3n+5}))\widetilde{D}_{n+1}(\varpi(x))\Big)\Big) + S_{3n+4}(x)\Big], \quad n \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
J(x;3n+5) =& \varpi'(x)\,\rho^3(x)\,\widetilde{\phi}\big(\varpi(x)\big)\,\widetilde{D}_{n+2}(\varpi(x)), \quad n \geq 0,\\
K(x;3n+5) =& \Big(\rho'(x)\widetilde{\phi}(\varpi(x)) + \rho(x)\varpi'(x)\widetilde{\phi}'(\varpi(x)) + \rho(x)\varpi'(x)\widetilde{C}_0(\varpi(x))\Big)\\
&\times \varpi'(x)\rho^2(x)\widetilde{D}_{n+2}(\varpi(x))\\
&- \rho(x)\widetilde{\phi}(\varpi(x))\,\frac{d}{dx}\Big[\varpi'(x)\rho^2(x)\widetilde{D}_{n+2}(\varpi(x))\Big], \quad n \geq 0,\\
L(x;3n+5) =& \Big[\frac{1}{2}\,\rho(x)\,\varpi'(x)\,\Big(\widetilde{C}_{n+2}(\varpi(x)) - \widetilde{C}_0(\varpi(x))\Big)\\
&+ \gamma_{3n+6}\,\rho(x)\,\varpi'(x)\,(x-\beta_{3n+4})\,\widetilde{D}_{n+2}(\varpi(x))\Big]\\
&\times \frac{d}{dx}\Big[\varpi'(x)\rho^2(x)\widetilde{D}_{n+2}(\varpi(x))\Big] - \varpi'(x)\rho^2(x)\widetilde{D}_{n+2}(\varpi(x))\\
&\times \Big[\frac{d}{dx}\Big(\frac{1}{2}\,\rho(x)\,\varpi'(x)\,\Big(\widetilde{C}_{n+2}(\varpi(x)) - \widetilde{C}_0(\varpi(x))\Big)\\
&+ \gamma_{3n+6}\,\rho(x)\,\varpi'(x)\,(x-\beta_{3n+4})\,\widetilde{D}_{n+2}(\varpi(x))\Big) + S_{3n+5}(x)\Big], \quad n \geq 0,
\end{aligned}
$$

with $S_n(x) := \sum_{\nu=0}^{n} D_\nu(x),\ n \geq 0$, *where* $D_\nu(x)$ *are given in Proposition* 3.1, *and* $\widetilde{\phi}$, $\widetilde{C}_0$, *and* $\widetilde{D}_0$ *are defined by* (2.12), (2.13), *and* (2.14), *respectively.*

Proof. From (4.2)–(4.4), we observe that

$$
\begin{aligned}
J(x;n) &= \phi(x)\,D_{n+1}(x), \quad n \geq 0,\\
K(x;n) &= \Big(\phi'(x) + C_0(x)\Big)D_{n+1}(x) - \phi(x)D'_{n+1}(x), \quad n \geq 0,\\
L(x;n) &= \frac{1}{2}\Big(C_{n+1}(x) - C_0(x)\Big)D'_{n+1}(x)
\end{aligned}
$$

$$- D_{n+1}(x)\left(\frac{1}{2}\Big(C_{n+1}(x) - C_0(x)\Big)' + S_n(x)\right), \quad n \geq 0,$$

and the result follows directly by taking into account Proposition 3.1. □

We now derive, based on this result and the previous section, the second-order linear differential equation satisfied by certain semiclassical MOPS obtained through cubic decomposition.

Proposition 4.2. *The MOPS* $\{W_n\}_{n\geq 0}$ *of the first family in* [20, Proposition 4.2] *satisfies* (4.1), *where the characteristic elements* $J(x;n)$, $K(x;n)$ *and* $L(x;n)$ *are*

$$\begin{aligned}
J(x;3n) =& \Big(x^2 + \frac{4q}{3}\Big)(x+\mu_q)\Big(\frac{1}{2}(12n+6\alpha+7)x + (6n+2)\mu_q\Big), \quad n \geq 0,\\
K(x;3n) =& \frac{1}{4}(12n+6\alpha+7)(6\alpha+5)x^3\\
&- \frac{1}{4}\Big((12\alpha+4)(12\alpha+10) + (6\alpha+3)^2\Big)\mu_q x^2\\
&+ \frac{q}{6}\Big((6\alpha+3)(6n+6\alpha+5) - 4(3n+1)\Big)x\\
&+ 6q\mu_q(2\alpha+1)n, \quad n \geq 0,\\
L(x;3n)(x) =& -\frac{1}{4}(3n+1)(12n+6\alpha+7)(6n+6\alpha+5)x^2\\
&- (3n+1)\Big(\frac{3}{4}(2n+2\alpha+1)(12n+6\alpha+7) + (3n+1)(6n+6\alpha+7)\Big)\\
&- (3n+1)^2(6n+6\alpha+5)\mu_q^2 x - \frac{1}{2}(12n+6\alpha+7)(2\alpha+1)q, \quad n \geq 0,\\
J(x;3n+1) =& \Big(x^2 + \frac{4q}{3}\Big)(x+\mu_q)\Big(\frac{1}{2}(12n+6\alpha+11)x + (6n+6\alpha+7)\mu_q\Big), \quad n \geq 0,\\
K(x;3n+1) =& \frac{1}{4}(12n+6\alpha+11)(6\alpha+5)x^3\\
&+ \Big((6n+6\alpha+7)(6\alpha+5) - \frac{1}{4}(6\alpha+3)^2\Big)\mu_q x^2\\
&+ \frac{q}{3}\Big((6\alpha+1)(6n+6\alpha+7) - (6\alpha+3)^2\Big)x\\
&+ 3q\mu_q(2\alpha+1)(2n+2\alpha+3), \quad n \geq 0,\\
L(x;3n+1) =& -\frac{1}{4}(6n+6\alpha+7)\Big[(3n+2)(12n+6\alpha+11)x^2\\
&+ \Big(6(3n+2)(2n+2\alpha+3) + (3n+1)(12n+6\alpha+11)\Big)\mu_q x\Big]\\
&+ \frac{q}{6}\Big[(6n+6\alpha+7)\Big(6n-6\alpha+1 + (3n+1)(6n+6\alpha+7)\Big)\\
&+ 9(2\alpha+1)^2\Big], \quad n \geq 0,
\end{aligned}$$

$$\begin{aligned}
J(x;3n+2) =& \frac{3}{2}(4n+2\alpha+5)\Big(x^2+\frac{4q}{3}\Big)(x+\mu_q)^2x, \quad n\geq 0,\\
K(x;3n+2) =& \frac{3}{4}(4n+2\alpha+5)(x+\mu_q)\Big((6\alpha+5)(x+\mu_q)x+2q(2\alpha+1)\Big), \quad n\geq 0,\\
L(x;3n+2) =& -\frac{27}{4}(n+1)(2n+2\alpha+3)(4n+2\alpha+5)(x+\mu_q)^2, \quad n\geq 0.
\end{aligned}$$

Proposition 4.3. *The MOPS* $\{W_n\}_{n\geq 0}$ *of the second family of* [20, Proposition 4.2] *fulfils* (4.1)*, where* $J(x;n)$*,* $K(x;n)$ *and* $L(x;n)$ *are*

$$\begin{aligned}
J(x;3n) =& (x^3+\kappa^3)\Big((6n+3\lambda+4)x-(3n+3\lambda+2)\kappa\Big), \quad n\geq 0,\\
K(x;3n) =& (6n+3\lambda+4)(3\lambda+3)x^3-\Big((3\lambda+4)(3n+3\lambda+3)+6n\Big)\kappa x^2\\
&+(9n+6\lambda+6)\kappa^2x-(9n+6\lambda+6)\kappa^3, \quad n\geq 0,\\
L(x;3n) =& -(3n+3\lambda+3)\Big[(3n+1)(6n+3\lambda+4)x^2\\
&-3n(3n+3\lambda+2)\kappa x+(3n+2)\kappa^2\Big], \quad n\geq 0,\\
J(x;3n+1) =& (x^3+\kappa^3)\Big((6n+3\lambda+6)x-(3n+3)\kappa\Big), \quad n\geq 0,\\
K(x;3n+1) =& (6n+3\lambda+6)(3\lambda+3)x^3-\Big((3n+3)(3\lambda+6)+3\lambda\Big)\kappa x^2\\
&+(9n+3\lambda+9)\kappa^2x-(9n+3\lambda+9)\kappa^3, \quad n\geq 0,\\
L(x;3n+1) =& (3n+2)\Big[-(6n+3\lambda+6)(3n+3\lambda+4)x^2\\
&+(3n+3)(3n+3\lambda+5)\kappa x\\
&+\Big(\frac{(3n+3\lambda+3)(6n+3\lambda+6)+3n+3}{6n+3\lambda+5}\Big)\kappa^2\Big], \quad n\geq 0,\\
J(x;3n+2) =& (6n+3\lambda+8)(x+\kappa)(x^3+\kappa^3), \quad n\geq 0,\\
K(x;3n+2) =& (6n+3\lambda+8)(3\lambda+3)(x+\kappa)x^2, \quad n\geq 0,\\
L(x;3n+2) =& -(3n+3)(6n+3\lambda+8)(3n+3\lambda+5)(x+\kappa)x, \quad n\geq 0.
\end{aligned}$$

Proposition 4.4. *The MOPS* $\{W_n\}_{n\geq 0}$ *of the third family in* [20, Proposition 4.2] *satisfies* (4.1)*, where* $J(x;n)$*,* $K(x;n)$ *and* $L(x;n)$ *are*

$$\begin{aligned}
J(x,3n) =& \Big(x-\frac{1}{3}\tau\Big)\Big(x-\frac{2}{3}\tau\Big)\Big(x+\frac{2}{3}\tau\Big)\\
&\times\Big(\frac{1}{2}(12n+6\alpha+7)x-\frac{1}{3}\tau(15n+9\alpha+8)\Big), \quad n\geq 0,\\
K(x,3n) =& \frac{1}{4}(12n+6\alpha+7)(6\alpha+5)x^3\\
&-\frac{2}{27}\tau^3\Big[12n+6\alpha+7-(15n+9\alpha+8)\Big(3\alpha+\frac{1}{2}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{6}\tau\Big[(15n+9\alpha+8)(6\alpha+7)+3(12n+6\alpha+7)\Big(\alpha+\frac{3}{2}\Big)\Big]x^2\\
&+ \frac{1}{9}\tau^2\Big[(15n+9\alpha+8)\Big(3\alpha+\frac{11}{2}\Big)\\
&- 3(12n+6\alpha+7)\Big(\alpha-\frac{1}{2}\Big)\Big]x, \quad n\geq 0,\\
L(x,3n) =& -\frac{1}{4}(3n+1)(12n+6\alpha+7)(6n+6\alpha+5)x^2\\
&+ \Big[\frac{1}{4}(3n-2)(2n+2\alpha+1)(12n+6\alpha+7)\\
&+ \frac{1}{6}(3n+1)(15n+9\alpha+8)(6n+6\alpha+7)\Big]\tau x\\
&- \frac{1}{3}\tau^2\Big[(3n+2\alpha+2)(12n+6\alpha+7)\\
&- \frac{1}{2}(3n-2)(15n+9\alpha+8)(6n+6\alpha+5)\Big], \quad n\geq 0,\\
J(x,3n+1) =& \Big(x-\frac{1}{3}\tau\Big)\Big(x-\frac{2}{3}\tau\Big)\Big(x+\frac{2}{3}\tau\Big)\\
&\times \Big(\frac{1}{2}(12n+6\alpha+11)x-\frac{1}{6}\tau(30n+12\alpha+29)\Big), n\geq 0,\\
K(x,3n+1) =& \frac{1}{4}(12n+6\alpha+11)(6\alpha+5)x^3\\
&- \frac{1}{6}\tau\Big[(30n+12\alpha+29)\Big(3\alpha+\frac{7}{2}\Big)+(12n+6\alpha+11)\Big(3\alpha+\frac{9}{2}\Big)\Big]x^2\\
&+ \frac{1}{18}\tau^2\Big[(30n+12\alpha+29)\Big(3\alpha+\frac{11}{2}\Big)-(12n+6\alpha+11)(6\alpha-3)\Big]x\\
&- \frac{1}{27}\tau^3\Big[2(12n+6\alpha+11)-(30n+12\alpha+29)\Big(3\alpha+\frac{1}{2}\Big)\Big], \quad n\geq 0,\\
L(x,3n+1) =& (3n+2)\Big[-(6n+3\alpha+6)(3n+3\alpha+4)x^2+(3n+3)(3n+3\alpha+5)\tau x\\
&+ \Big(\frac{(3n+3\alpha+3)(6n+3\alpha+6)+3n+3}{6n+3\alpha+5}\Big)\tau^2\Big], \quad n\geq 0,\\
J(x;3n+2) =& \frac{3}{2}(4n+2\alpha+5)\Big(x-\frac{1}{3}\tau\Big)\Big(x-\frac{2}{3}\tau\Big)\Big(x+\frac{2}{3}\tau\Big)^2, \quad n\geq 0,\\
K(x;3n+2) =& \frac{3}{4}(4n+2\alpha+5)\Big(x+\frac{2}{3}\tau\Big)\\
&\times \Big((6\alpha+5)x^2-\frac{1}{3}\tau(6\alpha+5)x-\frac{2}{3}\tau^2(2\alpha+1)\Big), \quad n\geq 0,\\
L(x;3n+2) =& -\frac{27}{4}(n+1)(2n+2\alpha+3)(4n+2\alpha+5)\Big(x-\frac{1}{3}\tau\Big)\Big(x-\frac{2}{3}\tau\Big), \quad n\geq 0.
\end{aligned}
$$

Remark 4.1. It is worth noting that the second-order linear differential equations satisfied by semiclassical orthogonal polynomials play an important role in providing an electrostatic interpretation of their zeros, which can be regarded as equilibrium positions of repelling unit charges under a logarithmic potential, confined by an external field determined by the corresponding weight function.

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