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ERROR ANALYSIS OF THE SEMI-DISCRETIZED DOUBLY NONLINEAR NON-LOCAL THERMISTOR PROBLEM

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Abstract. In this paper, we study a doubly nonlinear parabolic equation obtained from the reduction of the wellknown nonlocal thermistor problem. Therefore, we focus our study on proving the existence of the solution to the semi-discrete problem. We also establish the stability and error estimates for a family of time discretization schemes. We investigate a time discretization of the continuous problem by the backward Euler scheme.

1. Intorduction

Thermistors are a type of resistor that can be found in a variety of goods and applications in our modern society. They have been around since the 1830s, when the industrial revolution came to an end. Back then, they were used to measure temperature sensing, as self-resetting over current protection and to limit inrush current, among other things. Thermistors were first discovered by Michael Faraday, a British scientist and chemist, who is credited with inventing the first NTC thermistor. Faraday is also known for his contributions to electrochemistry and electromagnetic induction. In 1833, Faraday wrote a treatise on the behavior (semiconducting) of Ag2S (silver sulfide), which is said to be the first thermistor ever.

There are two types of thermistors PTC and NTC, which have Positive and Negative Temperature Coefficient, respectively. In the former, the electrical conductivity decreases with increasing temperature, while in the latter it increases with increasing temperature. PTC thermistors can be found in a variety of applications, including

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switches and electric surge protection devices. The PTC electric surge device operates in the following manner: upon a sudden increase in current in the circuit, there is a rise in temperature that leads to a significant reduction in the electrical conductivity of the device. As a result, the circuit is effectively switched off. Once the current surge subsides, the device cools down, its electrical conductivity increases and the circuit resumes normal operation, as stated by Bartosz in [3]. However, it has been found that the rapid rise in temperature can generate large thermal stresses that can compromise the integrity of the device [19, 23] which can lead to breakage and failure of the device. Since the nonlinearities of the system were in the electrical conductivity, Joule heating and viscous heating terms, it was assumed that the constitutive behavior of the material was linear. The "Thermistor Problem" can be roughly defined as a non-linear parabolic equation describing the temperature coupled with an elliptic equation describing the quasi-static evolution of the electric potential. The non-linearity of the problem is primarily due to the coupling between these equations, which is partly due to the significant influence of temperature on electrical conductivity. Mathematically, the problem can be expressed as follows:

$$
\begin{aligned}
v_t &= \nabla \cdot (\kappa(v)\nabla v) + \rho(v)|\nabla\psi|^2, \\
\nabla \cdot (\rho(v)\nabla\psi) &= 0.
\end{aligned} \tag{1.1}
$$

In the above model, κ denotes the thermal conductivity and $\rho(v)$ the electrical conductivity, which is normally a positive function of real values. Furthermore, the temperature of the conductor is represented by v, while the electric potential is represented by ψ. The first equation describes heat diffusion, while the second governs charge conservation. Here, $\kappa(v)$ and σ represent the thermal and electrical conductivity respectively, with their specific functional forms determined by the physical properties of the system. Under suitable conditions, this coupled system can be reformulated as a parabolic problem if it is supplemented by suitable boundary and initial conditions. For detailed treatments of such systems we refer to[7, 20, 26]. In this paper, we will deal with the following non-local model

$$
\begin{cases}
\dfrac{\partial b(v)}{\partial t} - \Delta v = \dfrac{\lambda g(v)}{\left(\int_\Omega g(v)dx\right)^2}, & \text{in } Q, \\
v(x,0) = v_0, & \text{in } \Omega, \\
v = 0, & \text{on } \Gamma \times]0, M[,
\end{cases} \tag{1.2}
$$

which is considered as a generalization of the problem from the work of A. A. Lacey [23], where $g(v)$ is the electrical resistance of the conductor and $\frac{g(v)}{\left(\int_\Omega g(v)dx\right)^2}$ represents the non-local term of (1.2). Whereas$Q$ is defined as follows $Q := \Omega \times [0, M]$ where Ω is an open bounded subset of $\mathbb{R}^m$, $m \geq 2$, and T is a positive constant. The literature on the problems (1.1) and (1.2) is extensive (see [1, 5, 9–12, 15–18, 21, 24, 25, 27, 30, 32]). Our motivation is stimulated by various applications. A thermistor has been widely used in electronic circuits to protect, control and compensate temperature. On the other hand, b is a nonlinear function that can grow faster than any function at infinity

$b(v) = e^{(v+1)^2 e^v}$, for example). Furthermore, it does not have to be strictly growing on any part of $\mathbb{R}$. Thus, the evolution equation can become stationary in a subdomain of $\Omega\times]0, M[$, where $]0, M[$ denotes the time horizon. Our aim is to prove the existence of a solution to the problem (1.2). We will also prove some stability results and provide error estimates.

The rest of the paper is organised as follows. Section 2 starts with a brief reminder of some notations and hypotheses that we need in this paper. In Section 3 we show the existence and uniqueness of a solution to a semi-discrete problem. In Section 4 we prove some stability results. Section 5 is dedicated to error analysis. Finally, in Section 6 we present conclusions and perspectives for future research.

2. Notations and Hypotheses

Throughout this paper, we assume the following.

(H1) $g : \mathbb{R} \to \mathbb{R}$ is a L_1-Lipschitzian function.

(H2) There exists a positive constant σ such that for all $s \in \mathbb{R}$, we have

$$\sigma \leq g(s). \tag{2.1}$$

(H3) b is an increasing Lipchitz function with $b(0) = 0$.

(H4) We assume that exists $C_\sigma \in \left]0, \frac{(\sigma\cdot\text{meas}(\Omega))^2}{\tau\lambda}\right[$ such that

$$|g(x) - g(y)| \leq C_\sigma |b(x) - b(y)|. \tag{2.2}$$

Throughout this paper, we will choose λ large enough to get that

$$\frac{(\sigma \cdot \text{meas}(\Omega))^2}{\tau\lambda} < 1.$$

(H5) $v_0 \in L^\infty(\Omega)$. We define for $r \in \mathbb{R}$

$$\Psi(r) = \int_0^r b(t)dt.$$

Then, the Legendre transform Ψ^* of Ψ is defined by

$$\Psi^*(r) = \sup_{s\in\mathbb{R}}\{rs - \Psi(s)\}.$$

Note that $\Psi^*(b(r)) + \Psi(r) = rb(r)$. Ω will denote an open bounded set of $\mathbb{R}^m$, $m \geq 2$ with smooth boundary and $\langle\cdot,\cdot\rangle$ denotes the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Throughout this paper, C_i and C will denote various positive constants.

Remark 2.1. Under hypotheses (H1), we can see that $g(s) \leq C_1|s| + C_2$. Indeed,

$$g(s) \leq |g(s) - g(0)| + g(0) \leq L_1|s| + g(0) \leq C_1|s| + C_2, \tag{2.3}$$

where $L_1 = C_1$ and $g(0) = C_2$.

3. Existence of Solution for a Semi-discrete Problem

For the time discretization of the problem (3.1) we will use the following notations. We denote the time step $\tau = T/N, t^n = n\tau$, and $J_n := (t^{n-1}, t^n)$ for $n = 1, \dots, N$. We consider the corresponding discrete scheme (backward Euler scheme) related to (1.2), which is represented by

$$\begin{cases} b(v_n) - \tau\Delta v_n = b(v_{n-1}) + \dfrac{\tau\lambda g(v_n)}{(\int_\Omega g(v_n)dx)^2}, & \text{in } Q, \\ v_n(x,0) = v_{0,n}, & \text{in } \Omega, \\ v_n = 0, & \text{on } \Gamma\times]0,M[. \end{cases} \tag{3.1}$$

Theorem 3.1. *Suppose that hypotheses* (H1)-(H4) *hold. Then, for each integer n, there exists a solution v_n to* (3.1).

Proof. To show the existence of a solution, Schauder fixed point theorem will be applied. For simplicity, putting $v = v_n$ substituting in the first equation of the system (3.1), we get that for a fixed w in $H_0^1(\Omega)$

$$b(v) - \tau\Delta v = b(v_{n-1}) + \frac{\tau\lambda g(w)}{(\int_\Omega g(w)dx)^2}.$$

Taking the inner product with the function v to get

$$\langle b(v), v\rangle - \tau\langle \Delta v, v\rangle = \mu\langle h(w), v\rangle, \tag{3.2}$$

where

$$h(w) = b(v_{n-1}) + \frac{\tau\lambda g(w)}{(\int_\Omega g(w)dx)^2} \quad \text{and} \quad \mu \in [0,1].$$

Let us consider the operator $A(\mu, w) = v,$ for all $\mu \in [0,1]$, being the solution to the following problem

$$\begin{cases} b(v) - \tau\Delta v = \mu h(w), & \text{in } Q, \\ v(x,0) = v_0, & \text{in } \Omega, \\ v = 0, & \text{on } \Gamma\times]0,M[. \end{cases} \tag{3.3}$$

If $v \geq b(v)$, by using (3.2), we obtain that

$$\int_\Omega (b(v))^2 \,\mathrm{dx} + \tau\int_\Omega |\nabla v|^2 \,\mathrm{dx} \leq \mu\langle h(w), v\rangle. \tag{3.4}$$

Using hypothesis (H2) and Young's inequality, we get the following estimates

$$\begin{aligned} \|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 &\leq \int_\Omega (C_3 + C_4|w|)\,|v|dx \\ &\leq C_4\|w\|_2\|v\|_2 + C_5\|v\|_2 \\ &\leq C_6\|w\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} + C_5\|v\|_2 \\ &\leq \frac{\tau}{2}\|v\|^2_{H_0^1(\Omega)} + \frac{1}{2\tau}C_6^2\|w\|^2_{H_0^1(\Omega)} + C_5\|v\|_2. \end{aligned}$$

Then,

$$\|b(v)\|_2^2 + \frac{\tau}{2}\|\nabla v\|_2^2 \leq \frac{1}{2\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + C_5\|v\|_2 .$$

It follows that

$$2\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + 2C_5\|v\|_2 .$$

Hence,

$$\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + 2C_5\|v\|_2 .$$

Using again Poincaré's inequality, we obtain

$$\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + 2C_5C_7\|\nabla v\|_2 .$$

By using Young's inequality, we get that

$$\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + \frac{(2C_5C_7)^2}{\tau} + \frac{\tau}{2}\|\nabla v\|_2^2 .$$

Then,

$$\begin{aligned}\|b(v)\|_2^2 + \frac{\tau}{2}\|\nabla v\|_2^2 &\leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + \frac{(2C_5C_7)^2}{\tau},\\ \|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 &\leq 2\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{2}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + \frac{2(2C_5C_7)^2}{\tau}.\end{aligned}$$

Then, there exist positive constants C_8 and C_9, such that

$$\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq C_8\|w\|_{H_0^1(\Omega)}^2 + C_9. \tag{3.5}$$

Then,

$$\|b(v)\|_2^2 \leq C_8\|w\|_{H_0^1(\Omega)}^2 + C_9. \tag{3.6}$$

Since w is bounded in $H_0^1(\Omega)$, then there exists a positive constant C_{10} such that

$$\|v\|_2^2 \leq C_{10}. \tag{3.7}$$

Recall from (3.5), that

$$\tau\|\nabla v\|_2^2 \leq \|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq C_8\|w\|_{H_0^1(\Omega)}^2 + C_9. \tag{3.8}$$

Combining (3.7) and (3.8), we get

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \leq C_8\|w\|_{H_0^1(\Omega)}^2 + C_{11}. \tag{3.9}$$

Else we have $v \leq b(v)$, then from (3.2), we get

$$\begin{aligned}\|v\|_2^2 + \tau\|\nabla v\|_2^2 &\leq \int_\Omega (C_3 + C_4|w|)\,|v|dx\\ &\leq C_4\|w\|_2\|v\|_2 + C_5\|v\|_2\\ &\leq C_6\|w\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} + C_5\|v\|_2\\ &\leq \frac{\tau}{2}\|v\|_{H_0^1(\Omega)}^2 + \frac{1}{2\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + C_5\|v\|_2 .\end{aligned} \tag{3.10}$$

Then,

$$\|v\|_2^2 + \frac{\tau}{2}\|\nabla v\|_2^2 \le \frac{1}{2\tau}C_6^2\|w\|^2_{H_0^1(\Omega)} + C_5\|v\|_2.$$

Hence,

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \le 2\|v\|_2^2 + \tau\|\nabla v\|_2^2 \le \frac{1}{\tau}C_6^2\|w\|^2_{H_0^1(\Omega)} + 2C_5\|v\|_2.$$

Using together Poincaré's inequality with Young's inequality, we obtain

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \le \frac{1}{\tau}C_6^2\|w\|^2_{H_0^1(\Omega)} + \frac{2(C_5C_7)^2}{\tau} + \frac{\tau}{2}\|\nabla v\|_2^2.$$

It follows that

$$\|v\|_2^2 + \frac{\tau}{2}\|\nabla v\|_2^2 \le \frac{1}{\tau}C_6^2\|w\|^2_{H_0^1(\Omega)} + \frac{2(C_5C_7)^2}{\tau}.$$

Then,

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \le 2\|v\|_2^2 + \tau\|\nabla v\|_2^2 \le \frac{2}{\tau}C_6^2\|w\|^2_{H_0^1(\Omega)} + 4\frac{(C_5C_7)^2}{\tau}.$$

Then, there exist positive constants C_{12} and C_{13}, such that

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \le C_{12}\|w\|^2_{H_0^1(\Omega)} + C_{13}. \tag{3.11}$$

From (3.9) and (3.11), there exist positive constants C_{14} and C_{15} such that

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \le C_{14}\|w\|^2_{H_0^1(\Omega)} + C_{15}. \tag{3.12}$$

We can see that $\mu \mapsto A(\mu, v)$ is a continuous function and we have that $A(0, v) = V$ for any v if and only if $V = 0$. By using the theory of degree topology, we get that $v \in H_0^1(\Omega)$ and satisfies Problem (3.1). □

The following lemmas play a key role in the proof of Theorem 4.1.

Lemma 3.1. *If $v_0 \in L^\infty(\Omega)$, then $v_n \in L^\infty(\Omega)$ for all $n = 1, 2, 3, \ldots, N$.*

The proof of Lemma 3.1, is similar to the one used by de Thelin in [6] in a different problem; we shall give here only a sketch. Suppose $m \ge 2$ and define

$$\gamma = \begin{cases} \frac{2m}{m-2}, & \text{if } m > 2, \\ 4, & \text{if } m = 2. \end{cases}$$

Let $p_1 = \gamma$ and let

$$p_k = \left\{\left(\frac{\gamma}{2}\right)^{k-1}(\gamma - 2)\right\}\frac{\gamma}{\gamma - 2}, \quad k \ge 2.$$

Then, we have

$$p_{k+1} = p_k\frac{\gamma}{2}, \quad \text{for all } k \in \mathbb{N}^*.$$

Lemma 3.2. *For k in $\mathbb{N}^*$, $v_n \in L^{p_k}(\Omega)$ and*

$$\|v_n\|_\infty = \lim_{k\to+\infty}\sup\|v_n\|_{p_k} < +\infty. \tag{3.13}$$

Proof. For simplicity, we write $V = v_n$, $K(x) = b(v_{n-1})$. From (3.1), we have

$$b(V) - \tau\Delta V = K(x) + \frac{\tau\lambda g(V)}{(\int_\Omega g(V)dx)^2} =: H(V).$$

Multiplying the above identity by $|V|^{p_k-2}V$, to obtain

$$\int_\Omega b(V)|V|^{p_k-2}V\ dx - \tau\int_\Omega \Delta V|V|^{p_k-2}V dx = \int_\Omega H(V)\cdot|V|^{p_k-2}V dx. \tag{3.14}$$

In view of the fact that

$$\begin{aligned} -\tau\int_\Omega \Delta V|V|^{p_k-2}V\ dx &= \tau\int_\Omega \nabla V\cdot\nabla(|V|^{p_k-2}V)\ dx \\ =&\tau\int_\Omega |\nabla V|^2|V|^{p_k-2}\ dx + \tau\int_\Omega (p_k-2)|\nabla V|^2|V|^{p_k-2}\ dx \\ =&\tau\int_\Omega (p_k-1)|\nabla V|^2|V|^{p_k-2}\ dx, \end{aligned}$$

and from (3.14), we get that

$$\tau\int_\Omega (p_k-1)|\nabla V|^2|V|^{p_k-2}\ dx \le -\int_\Omega b(V)|V|^{p_k-2}V + \int_\Omega H(V)|V|^{p_k-2}V\ dx. \tag{3.15}$$

Keeping (3.15) in mind and using the fact that $H(V) \le C_0|V| + C_1$, we obtain that

$$\begin{aligned} &\tau\int_\Omega (p_k-1)|\nabla V|^2|V|^{p_k-2}\ dx \\ \le& -\int_\Omega b(V)|V|^{p_k-2}V\ dx + \int_\Omega (C_0|V|+C_1)|V|^{p_k-1}\ dx. \end{aligned} \tag{3.16}$$

Let us deal with the first term on the right side of the inequality (3.16).
If $b(V) \le V$ then we get

$$-\int_\Omega b(V)|V|^{p_k-2}V\ dx \le -\int_\Omega (b(V))^2|V|^{p_k-2}\ dx \le 0,$$

else we have $b(V) \ge V$,

$$-\int_\Omega b(V)|V|^{p_k-2}V\ dx \le -\int_\Omega V^2|V|^{p_k-2}\ dx \le 0.$$

In both cases, we have

$$-\int_\Omega b(V)|V|^{p_k-2}V\ dx \le 0.$$

This implies that

$$\begin{aligned} \tau\int_\Omega (p_k-1)|\nabla V|^2|V|^{p_k-2}\ dx &\le \int_\Omega (C_0|V|+C_1)|V|^{p_k-1}\ dx \\ &\le C_0\int_\Omega |V|^{p_k}\ dx + C_1\int_\Omega |V|^{p_k-1}\ dx \\ &\le C_0\int_\Omega |V|^{p_k}\ dx + C_1\int_\Omega |V|^{p_k-1}\ dx. \end{aligned} \tag{3.17}$$

Using Young's inequality, we get that

$$C_1\int_\Omega |V|^{p_k-1}\ dx \le \int_\Omega C_1^{p_k}\ dx + \frac{p_k-1}{p_k}||V||_{p_k}^{p_k} \le C_{16} + \frac{p_k-1}{p_k}||V||_{p_k}^{p_k}. \tag{3.18}$$

Combining (3.17) with (3.18), we get

$$\tau \int_\Omega (p_k - 1)|\nabla V|^2 \cdot |V|^{p_k-2}\, dx \le C_{16} + \frac{p_k-1}{p_k}||V||_{p_k}^{p_k} + C_0 \int_\Omega |V|^{p_k}\, dx \le C_{16} + C_{17}||V||_{p_k}^{p_k}. \tag{3.19}$$

On the other hand, by using Poincaré's inequality, we have for all $V \in H_0^1(\Omega)$ the following inequality

$$||V||_\gamma \le C_{18}||\nabla V||_{L^2(\Omega)}.$$

Hence,

$$\begin{aligned} \|V\|_{p_{k+1}}^{p_k} &= \left(\int_\Omega \|V\|^{p_{k+1}}\, dx\right)^{\frac{p_k}{p_{k+1}}} = \left(\int_\Omega \|V\|^{\frac{p_k\gamma}{2}}\, dx\right)^{\frac{2}{\gamma}} \\ &= \left(\int \left(|V|^{1+\frac{p_k-2}{2}}\right)^\gamma dx\right)^{\frac{2}{\gamma}} = \left\||V|^{\frac{p_k}{2}}\right\|_\gamma^2 \\ &\le C_{18}^2 \int_\Omega \left|\nabla\left(|V|^{\frac{p_k}{2}}\right)\right|^2 \\ &\le \left(\frac{p_k}{2}\right)^2 C_{18}^2 \int_\Omega |\nabla V|^2 |V|^{p_k-2} dx. \end{aligned} \tag{3.20}$$

From (3.19) and (3.20), it follows that

$$\|V\|_{p_{k+1}}^{p_k} \le p_k^2 \left(C_{19} + C_{20}||V||_{p_k}^{p_k}\right).$$

By induction on k, we get that $V \in L^{p_k}(\Omega)$, $1 \le k \le m+1$. Then, we have the following

$$\left(\|V\|_{p_{m+1}}^{p_{m+1}}\right)^{\frac{2}{\gamma}} \le \left(C_{19} + C_{20}\,\|V\|_{p_m}^{p_m}\right) p_m^2. \tag{3.21}$$

The rest of the proof follows the same lines as in [29]. As follows: for $\delta =: p_1$, we remark that

$$p_m \le \delta \left(\frac{\delta}{2}\right)^{m-1}.$$

Setting $a =: \frac{\delta}{2}$, $b =: \frac{\delta}{2} \log\max\{1, C_{19} + C_{20}\}$, $E_m = p_m \log\max\{1, ||V||_{p_m}\}$ and $r_m = b + \delta(m-1)\log a$. We obtain that

$$\begin{aligned} E_{m+1} \le & r_m + aE_m \le r_m + ar_{m-1} + \cdots + a^{m-1}r_1 + a^m E_1, \\ E_{m+1} \le & a^m \left\{E_1 + \frac{b}{a-1} + \frac{\delta \log a}{(a-1)^2}\right\} =: da^m. \end{aligned}$$

From which we conclude that

$$\|V\|_\infty \le \limsup_{m\to+\infty} \exp\left(\frac{E_m}{p_m}\right) \le \limsup_{m\to+\infty} \exp\left(\frac{da^{m-1}}{p_m}\right) \le \exp\left(\frac{d}{\delta}\right).$$

This concludes the proof of (3.13). □

4. Stability Results

In this section, we focus on the stability analysis for the semi-discretized problem. For this purpose, we consider V_0 in $L^\infty(\Omega)$.

Theorem 4.1. *Under hypotheses* (H1)-(H3) *there exists positive constants* $C\left(M,V_0\right)$ *and* $C_i\left(M,V_0\right)$, $i=1,2$, *such that*

$$\left\|V_n\right\|_\infty \leq C\left(M,V_0\right), \tag{4.1}$$

$$\int_\Omega \psi^*\left(b\left(V_n\right)\right)\,dx+\tau\sum_{k=1}^{n}\left\|V_k\right\|_{1,p}^p \leq C_1\left(M,V_0\right), \tag{4.2}$$

$$\sum_{k=1}^{n}\left\|b(V_k)-b\left(V_{k-1}\right)\right\|_2^2 \leq C_2\left(M,V_0\right). \tag{4.3}$$

Proof. From Lemma 3.2, we have that $V_n\in L^\infty(\Omega)$. Multiplying the first equation of (3.1) by $|b(V_n)|^k b(V_n)$, we get

$$\begin{aligned}&\int_\Omega \left|b\left(V_n\right)\right|^k\left|b\left(V_n\right)\right|^2\,dx-\tau\int_\Omega \Delta V_n\left|b\left(V_n\right)\right|^k b\left(V_n\right)\,dx\\ =&\int_\Omega b\left(V_{n-1}\right)\left|b\left(V_n\right)\right|^k b\left(V_n\right)dx+\int_\Omega h(V_n)\left|b\left(V_n\right)\right|^k b\left(V_n\right)dx.\end{aligned} \tag{4.4}$$

It can be shown that

$$-\int_\Omega \Delta V_n\left|b\left(V_n\right)\right|^k b\left(V_n\right)\leq 0. \tag{4.5}$$

Combining (4.4) with (4.5), we get

$$\left\|b\left(V_n\right)\right\|_{k+2}^{k+2}\leq\int_\Omega b\left(V_{n-1}\right)\left|b\left(V_n\right)\right|^k b\left(V_n\right)\,dx+\frac{\tau\lambda}{(\sigma\ \text{meas}\ (\Omega))^2}\int_\Omega g\left(V_n\right)\left|b\left(V_n\right)\right|^{k+1}\,dx.$$

It follows that

$$\left\|b\left(V_n\right)\right\|_{k+2}^{k+2}\leq\left\|b\left(V_n\right)\right\|_{k+2}^{k+1}\left\|b\left(V_{n-1}\right)\right\|_{k+2}+\frac{\tau\lambda}{(\sigma\,\text{meas}(\Omega))^2}\int_\Omega g\left(V_n\right)\left|b\left(V_n\right)\right|^{k+1}\,dx. \tag{4.6}$$

To deal with the second term on the above inequality, using hypotheses (H3)-(H4), we obtain

$$\begin{aligned}\int_\Omega g(V_n)\left|b\left(V_n\right)\right|^{k+1}\,dx &\leq \int_\Omega(|g(V_n))-g(0)|+g(0))|b(V_n)|^{k+1}\,dx\\ &\leq \int_\Omega \left|g\left(V_n\right)-g(0)\right|\cdot|b(V_n)|^{k+1}\,dx+g(0)\int_\Omega |b(V_n)|^{k+1}\,dx\\ &\leq C_\sigma\int_\Omega \left|b\left(V_n\right)-b(0)\right|\cdot\left|b\left(V_n\right)\right|^{k+1}\,dx+g(0)\int_\Omega \left|b\left(V_n\right)\right|^{k+1}dx\\ &\leq C_\sigma\int_\Omega \left|b\left(V_n\right)\right|^{k+2}\,dx+g(0)\int_\Omega \left|b\left(V_n\right)\right|^{k+1}dx\\ &\leq C_\sigma\left\|b\left(V_n\right)\right\|_{k+2}^{k+2}+g(0)\int_\Omega \left|b\left(V_n\right)\right|^{k+1}dx.\end{aligned}$$

Then, inequality (4.6) becomes

$$\|b(V_n)\|_{k+2}^{k+2} \le \|b(V_n)\|_{k+2}^{k+1} \|b(V_{n-1})\|_{k+2} + C_\sigma \|b(V_n)\|_{k+2}^{k+2} + g(0)\int_\Omega |b(V_n)|^{k+1}\, dx.$$

Under the hypothesis (H4), we get that

$$(1-C_\sigma)\|b(V_n)\|_{k+2}^{k+2} \le \|b(V_n)\|_{k+2}^{k+1} \|b(V_{n-1})\|_{k+2} + C_{21}\|b(V_n)\|_{k+1}^{k+1}.$$

Then, there exists a positive constant C such that

$$C\|b(V_n)\|_{k+2} \le \|b(V_{n-1})\|_{k+2} + C_{22}.$$

By induction, we get that

$$\|b(V_n)\|_{k+2} \le \left(\frac{1}{C}\right)^N \|b(V_0)\|_{k+2} + C_{23}N.$$

We tend k to infinity, we get

$$\|b(V_n)\|_\infty \le \left(\frac{1}{C}\right)^N \|b(V_0)\|_\infty + C_{23}N.$$

This implies that

$$||V_n||_\infty \le C(M, V_0).$$

Now, we prove the second inequality (4.2) in Theorem 4.1. Toward this end, multiplying (3.1) by V_k and integrating over Ω, we get

$$\int_\Omega (b(V_k) - b(V_{k-1}))\cdot V_k\, dx - \tau\int_\Omega \Delta V_k V_k\, dx = \lambda\int_\Omega \frac{g(V_k)V_k}{\left(\int_\Omega g(V_k)dx\right)^2}\, dx. \tag{4.7}$$

Since

$$\int_\Omega \Psi^*(b(V_k))\, dx - \int_\Omega \Psi^*(b(V_{k-1}))\, dx \le (b(V_k) - b(V_{k-1}), V_k)_{L^2(\Omega)}. \tag{4.8}$$

Keeping (4.7)–(4.8) and hypotheses on g in mind, we obtain

$$\int_\Omega \Psi^*(b(V_k))\, dx - \int_\Omega \psi^*(b(V_{k-1}))\, dx + \tau\|\nabla V_k\|_{L^2(\Omega)}^2 \le \frac{\tau\lambda}{(\sigma\,\mathrm{meas}(\Omega))^2}\int_\Omega |g(V_k)|\cdot|V_k|\, dx. \tag{4.9}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega g(V_k)|V_k|\, dx &\le \int_\Omega \frac{\tau\lambda}{(\sigma\,\mathrm{meas}(\Omega))^2}(|g(V_k)-g(0)|\cdot|V_k| + |g(0)|\cdot|V_k|)\, dx\\ &\le \frac{\tau\lambda L_1}{(\sigma\,\mathrm{meas}(\Omega))^2}\left[\int_\Omega |V_k|^2\, dx + g(0)|\int_\Omega |V_k|\, dx\right]\\ &\le C_{24}\|V_k\|_{L^2(\Omega)} + C_{25}\|V_k\|_{L^1(\Omega)}.\end{aligned}$$

From the above inequality and Höder's inequality combining with (4.9), it yields that

$$\int_\Omega \Psi^*(b(V_k))\, dx - \int_\Omega \Psi^*(b(V_{k-1}))\, dx + \tau\|\nabla V_k\|_{L^2(\Omega)}^2 \le C_{26}\|V_k\|_{L^2(\Omega)}. \tag{4.10}$$

Summing (4.10) from $k = 1$ to n, we obtain

$$\sum_{k=0}^{n}\left[\int_\Omega \Psi^*\left(b\left(V_k\right)\right)dx - \int_\Omega \Psi^*(b\left(V_{k-1}\right))dx\right] + \tau\sum_{k=0}^{n}\|\nabla V_k\|_2^2 \le C_{26}\sum_{k=0}^{n}\|V_k\|_2 .$$

Then,

$$\int_\Omega \Psi^*\left(b\left(V_n\right)\right)dx - \int_\Omega \Psi^*(b\left(V_0\right))dx + \tau\sum_{k=0}^{n}\|\nabla V_k\|_2^2 \le C_{26}\sum_{k=0}^{n}\|V_k\|_2.$$

Hence,

$$\int_\Omega \Psi^*\left(b\left(V_n\right)\right)dx + \tau\sum_{k=0}^{n}\|\nabla V_k\|_2^2 \le C_{26}\sum_{k=0}^{n}\|V_k\|_2 + \int_\Omega \Psi^*(b\left(V_0\right))dx.$$

Keeping this in mind and using (4.1), there exists a positive constant $C_1(M, V_0)$ such that

$$\int_\Omega \Psi^*\left(b\left(V_n\right)\right)dx + \tau\sum_{k=0}^{n}\|\nabla V_k\|_2^2 \le C_1(M, V_0). \tag{4.11}$$

Now, let us prove that

$$\sum_{k=1}^{n}\left\|b(V^k) - b\left(V^{k-1}\right)\right\|_2^2 \le C_2\left(M, V_0\right).$$

To this end, multiplying the first equation of (3.1) by $b(V_k)$ and integrating over Ω, we get that

$$\int_\Omega \left(b\left(V_k\right) - b\left(V_{k-1}\right)\right)b\left(V_k\right)\ dx + \tau\langle -\Delta V_k, b\left(V_k\right)\rangle = \tau\lambda\int_\Omega \frac{g\left(V_k\right)b\left(V_k\right)}{(\int_\Omega g(V_k)dx)^2}\ dx.$$

Applying the following identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ in the first term on the above identity, we get that

$$\begin{aligned}&\|b\left(V_k\right)\|_2^2 - \|b\left(V_{k-1}\right)\|_2^2 + \|b\left(V_k\right) - b\left(V_{k-1}\right)\|_{L^2(\Omega)}^2 - \tau\int_\Omega \Delta V_k b\left(V_k\right)\\ =&\tau\lambda\int_\Omega \frac{g\left(V_k\right)b\left(V_k\right)}{(\int_\Omega g(V_k)dx)^2}\ dx.\end{aligned}$$

Knowing that

$$-\tau\int_\Omega \Delta V_k b\left(V_k\right)\ dx = \tau\int_\Omega \nabla V_k\nabla\left(b\left(V_k\right)\right)\ dx = \tau\int_\Omega |\nabla V_k|^2 b'(V_k)\ dx \ge 0.$$

It yields that

$$\|b\left(V_k\right)\|_2^2 - \|b\left(V_{k-1}\right)\|_2^2 + \|b\left(V_k\right) - b\left(V_{k-1}\right)\|_{L^2(\Omega)}^2 \le \lambda M\int_\Omega \frac{g\left(V_k\right)b\left(V_k\right)}{(\int_\Omega g(V_k)dx)^2}\ dx. \tag{4.12}$$

On the other hand, by using hypotheses on g, we get that

$$\begin{aligned}\int_\Omega g\left(V_k\right)\left|b\left(V_k\right)\right| dx &\leq \int_\Omega \left[\left|g\left(V_k\right)-g(0)\right|+|g(0)|\right] \cdot\left|b\left(V_k\right)\right| \, dx \\ &\leq C_\sigma \int_\Omega \left|b\left(V_k\right)\right|^2 dx+|g(0)| \int_\Omega \left|b\left(V_k\right)\right| dx \\ &\leq C_\sigma \left\|b\left(V_k\right)\right\|_2^2+|g(0)| \cdot\left\|b\left(V_k\right)\right\|_{L^1(\Omega)} .\end{aligned}$$

Then,

$$\int_\Omega g\left(V_k\right)\left|b\left(V_k\right)\right| dx \leq C_\sigma \left\|b\left(V_k\right)\right\|_2^2+|g(0)| \cdot\left\|b\left(V_k\right)\right\|_{L^1(\Omega)} . \tag{4.13}$$

From (4.12) and (4.13), we get

$$\begin{aligned}&\left\|b\left(V_k\right)\right\|_2^2-\left\|b\left(V_{k-1}\right)\right\|_2^2+\left\|b\left(V_k\right)-b\left(V_{k-1}\right)\right\|_{L^2(\Omega)}^2 \\ &\leq \frac{M C_\sigma \lambda}{(\sigma \operatorname{meas}(\Omega))^2}\left\|b\left(V_k\right)\right\|_2^2+\frac{M|g(0)| \lambda}{(\sigma \operatorname{meas}(\Omega))^2}\left\|b\left(V_k\right)\right\|_{L^1(\Omega)} .\end{aligned}$$

We know that $(V_k)_k$ is a bounded sequence in $L^\infty(\Omega)$. Furthermore, we have that b is a Liptshitz function, then there exists a positive constant C_{27} such that

$$\left\|b\left(V_k\right)\right\|_2^2-\left\|b\left(V_{k-1}\right)\right\|_2^2+\left\|b\left(V_k\right)-b\left(V_{k-1}\right)\right\|_{L^2(\Omega)}^2 \leq C_{27}.$$

This implies that

$$\sum_{k=1}^n\left(\left\|b\left(V_k\right)\right\|_2^2-\left\|b\left(V_{k-1}\right)\right\|_2^2\right)+\sum_{k=1}^n\left\|b(V_k)-b\left(V_{k-1}\right)\right\|_2^2 \leq n C_{27}.$$

It follows that

$$\left\|b\left(V_n\right)\right\|_2^2-\left\|b\left(V_0\right)\right\|_2^2+\sum_{k=1}^n\left\|b(V_k)-b\left(V_{k-1}\right)\right\|_2^2 \leq n C_{27}.$$

Hence,

$$\left\|b\left(V_n\right)\right\|_2^2+\sum_{k=1}^n\left\|b(V_k)-b\left(V_{k-1}\right)\right\|_2^2 \leq n C_{27}+\left\|b\left(V_0\right)\right\|_2^2 .$$

Then, there exists a positive constant $C_2\left(M, V_0\right)$ such that

$$\sum_{k=1}^n\left\|b(V^k)-b\left(V^{k-1}\right)\right\|_2^2 \leq C_2\left(M, V_0\right).$$

This concludes the proof. □

5. Error Estimates for Solutions

In this section, we will study the error estimate. If y is a continuous function (resp., summable), defined in $]0, M[$, with values in $H^{-1}(\Omega)$ or $L^2(\Omega)$ or $H_0^1(\Omega)$, we define $y^n=y\left(t^n, \cdot\right)$, $\bar{y}^n=(1/\tau) \int_{J_n} y(t, \cdot) dt$, $\bar{y}^0=y^0=y(0, \cdot)$ and the error $e_v=v(t)-V_n$, $e_b=b(v(t))-b\left(V_n\right)$ for all $t \in I^n, n=1, \ldots, N$. For notational simplicity, we denote $u_n=b\left(v_n\right)$ and $U_n=b\left(V_n\right)$ and introduce the local errors e_v^n and e_b^n, defined by

$$e_v^n=\bar{v}_n-V_n, \quad e_b^n=b\left(v_n\right)-b\left(V_n\right)=u_n-U_n .$$

Let's $(-\Delta)^{-1}$ be the Green operator satisfying

$$\left\langle \nabla(-\Delta)^{-1}v, \nabla w\right\rangle = \langle v, w\rangle, \quad \text{for all } v \in H_0^1(\Omega), w \in H^{-1}(\Omega). \tag{5.1}$$

By arguments inspired from [4, 13, 28], we get the following result hold.

Theorem 5.1. *Assume that the hypotheses* (H1)-(H5) *holds. Then,*

$$\|e_b^m\|_{H^{-1}(\Omega)} \leq C_{28}\tau^{1/2}, \tag{5.2}$$

$$\|e_b^n\|^2_{L^\infty(0,M,H^{-1}(\Omega))} + \int_0^M \|e_b^n\|_2^2\, dt \leq C_{29}\tau, \tag{5.3}$$

$$\left\|\nabla \int_0^{t^m} e_v dt\right\|^2_{L^2(\Omega)} \leq C_{30}\tau^{1/4}.$$

Proof. The discrete problem (3.1) has the following variational formulation

$$\begin{aligned}&\langle b(v_n) - b\left(v_{n-1}\right), \varphi\rangle_{L^2(\Omega)} + \tau\left\langle \nabla v_n, \nabla\varphi\right\rangle_{L^2(\Omega)}\\ =&\frac{\tau\lambda}{(\int_\Omega(g(v_n)dx)^2}\langle g\left(v_n\right), \varphi\rangle, \quad \text{for all } \varphi \in H_0^1(\Omega).\end{aligned} \tag{5.4}$$

Integrating the continuous problem (1.2) over J_n, we obtain that its solution verifies the following variational identity with respect to the notation introduced in the introduction

$$b\left(V_n\right) - b\left(V_{n-1}\right) - \Delta\left(\int_{J_n} V_n dt\right) = \lambda\int_{J_n}\frac{g\left(V_n\right)}{\left(\int_\Omega g(V_n)dx\right)^2}dt.$$

Then,

$$b\left(V_n\right) - b\left(V_{n-1}\right) - \tau\Delta\bar{V}_n = \lambda\int_{J_n}\frac{g\left(V_n\right)}{\left(\int_\Omega g\left(V_n\right)dx\right)^2}dt. \tag{5.5}$$

Multiplying (5.5) by φ and integrating over Ω, we get

$$\langle b\left(V_n\right) - b\left(V_{n-1}\right), \varphi\rangle - \tau\left\langle \Delta\bar{V}^n, \varphi\right\rangle = \left\langle \lambda\int_{J_n}\frac{g\left(V_n\right)}{\left(\int_\Omega g\left(V_n\right)dx\right)^2}dt, \varphi\right\rangle,$$

for all $\varphi \in H_0^1(\Omega)$. From Green's formula, we obtain

$$\langle b\left(V_n\right) - b\left(V_{n-1}\right), \varphi\rangle + \tau\left\langle \nabla\bar{V}_n, \nabla\varphi\right\rangle = \lambda\left\langle \int_{J_n}\frac{g\left(V_n\right)}{(\int_\Omega(g(V_n)dx)^2}dt, \varphi\right\rangle,$$

for all $\varphi \in H_0^1(\Omega)$. Then,

$$\langle b\left(V_n\right) - b\left(V_{n-1}\right), \varphi\rangle + \tau\left\langle \nabla\bar{V}_n, \nabla\varphi\right\rangle = \frac{\tau\lambda}{(\int_\Omega g(V_n)dx)^2}\langle\overline{g(V_n)}, \varphi\rangle, \tag{5.6}$$

for all $\varphi \in H_0^1(\Omega)$. Substracting (5.6) from (5.4), we get

$$\begin{aligned}&\langle b\left(v_n\right) - b\left(v_{n-1}\right) - \left(b\left(V_n\right) - b\left(V_{n-1}\right)\right), \varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \tau\left\langle \nabla\bar{V}_n - \nabla v_n, \nabla\varphi\right\rangle_{L^2(\Omega)}\\ =&\tau\lambda\left\langle \frac{g\left(v_n\right)}{(\int_\Omega(g(v_n)dx)^2} - \frac{\overline{g(V_n)}}{(\int_\Omega g(V_n)dx)^2}, \varphi\right\rangle.\end{aligned}$$

Then,

$$
\begin{aligned}
&\sum_{n=1}^{m}\left\langle e_b^n-e_b^{n-1},\varphi\right\rangle+\tau\sum_{n=1}^{m}\left(\nabla e_v^n,\nabla\varphi\right)\\
\leq &C_{31}\tau\left|\sum_{n=1}^{m}\left(\overline{g(v)}^n-g\left(V_n\right),\varphi\right)\right|+C_{32}\tau\left|\sum_{n=1}^{m}\left(g\left(V_n\right),\varphi\right)\right|.
\end{aligned}\tag{5.7}
$$

We set

$$
\begin{aligned}
I_1^n&=\left\langle b\left(v_n\right)-b\left(V_n\right)-\left(b\left(v_{n-1}\right)-b(V_{n-1})\right),\varphi\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\
&=\left\langle e_b^n-e_b^{n-1},\varphi\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)},\\
I_2^n&=\tau\langle\nabla e_v^n,\nabla\varphi\rangle_{L^2(\Omega)},\\
I_3^m&=C_{31}\tau\left|\sum_{n=1}^{m}\left(\overline{g(v)}^n-g\left(V_n\right),\varphi\right)\right|+C_{32}\tau\left|\sum_{n=1}^{m}\left(g\left(V_n\right),\varphi\right)\right|=I_{3,1}^m+I_{3,2}^m.
\end{aligned}
$$

Choosing $\varphi=\nabla(-\Delta)^{-1}e_b^n$ and adding from $n=1$ to m with $m\leq N$, we have

$$
\begin{aligned}
\sum_{n=1}^{m}I_1^n=&\sum_{n=1}^{m}\left\langle\nabla(-\Delta)^{-1}\left[e_b^n-e_b^{n-1}\right],\nabla(-\Delta)^{-1}e_b^n\right\rangle_{L^2(\Omega)}\\
=&\frac{1}{2}\left\langle\nabla(-\Delta)^{-1}e_b^m,\nabla(-\Delta)^{-1}e_b^m\right\rangle_{L^2(\Omega)}\\
&+\frac{1}{2}\sum_{n=1}^{m}\left\langle\nabla\left((-\Delta)^{-1}\left[e_b^n-e_b^{n-1}\right]\right),\nabla\left((-\Delta)^{-1}\left[e_b^n-e_b^{n-1}\right]\right)\right\rangle\\
=&\frac{1}{2}\left\|e_b^m\right\|_{H^{-1}(\Omega)}^2+\frac{1}{2}\sum_{n=1}^{m}\left\|e_b^n-e_b^{n-1}\right\|_{H^{-1}(\Omega)}^2.
\end{aligned}
$$

From (5.1), we get

$$
I_2^n=\tau\langle\nabla e_v^n,\nabla\varphi\rangle_{L^2(\Omega)}=\tau\left\langle\nabla e_v^n,\nabla(-\Delta)^{-1}e_b^n\right\rangle_{L^2(\Omega)}=\langle e_v^n,e_b^n\rangle_{L^2(\Omega)}.
$$

Again adding from $n=1$ to m with $m\leq N$, we obtain

$$
\begin{aligned}
\sum_{n=1}^{m}I_2^n=&\tau\sum_{n=1}^{m}\langle e_v^n,e_b^n\rangle_{L^2(\Omega)}\\
=&\sum_{n=1}^{m}\left\langle\int_{J_n}\left(v(t)-V_n\right)dt,u(t)-U_n\right\rangle_{L^2(\Omega)}\\
&+\sum_{n=1}^{m}\int_{J_n}\langle v(t)-V_n,u_n-u(t)\rangle_{L^2(\Omega)}\,dt\\
=&I_4^m+I_5^m.
\end{aligned}
$$

Since b is an increasing function, it follows that

$$
I_4^m=\sum_{n=1}^{m}\int_{J_n}\langle v(t)-v_n,b(v(t))-b(v_n)\rangle_{L^2(\Omega)}\geq 0.
$$

Now, we split I_5^m into two parts and we estimate each one separately

$$\begin{aligned}I_5^m =&\sum_{n=1}^{m}\int_{J_n}\langle v(t)-V_n, u_n-u(t)\rangle_{L^2(\Omega)}\,dt\\ =&\sum_{n=1}^{m}\int_{J_n}\langle v(t), u(t)-U_n\rangle_{L^2(\Omega)}\,dt\\ &-\sum_{n=1}^{m}\int_{J_n}\langle V_n, u_n-u(t)\rangle_{L^2(\Omega)}\,dt\\ =&I_6^m+I_7^m.\end{aligned}$$

We start by estimating I_6^m

$$\begin{aligned}|I_6^m| &= \left|\sum_{n=1}^{m}\int_{J_n}\left\langle v(t),\int_t^{t^n}\frac{\partial u}{\partial s}ds\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}dt\right|\\ &\le \sum_{n=1}^{m}\int_{J_n}\left(\int_t^{t^n}\left\|\frac{\partial u}{\partial s}\right\|_{H^{-1}(\Omega)}ds\right)\cdot\|v(t)\|_{H_0^1(\Omega)}\,dt\\ &\le \sum_{n=1}^{m}\int_{J_n}\left(\int_t^{t^n}\left\|\frac{\partial u}{\partial s}\right\|_{H^{-1}(\Omega)}ds\right)\cdot\|v(t)\|_{H_0^1(\Omega)}\,dt\\ &\le \tau\left\|\frac{\partial b(u)}{\partial s}\right\|_{L^2(0,t^m H^{-1}(\Omega))}\le C_{33}\tau.\end{aligned}\tag{5.8}$$

In similar manner, to derive an estimate of I_7^m, we use inequality (4.2) to get

$$\begin{aligned}|I_7^m| &= \left|\sum_{n=1}^{m}\int_{J_n}\left\langle V_n,\int_t^{t^n}\frac{\partial u}{\partial s}ds\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}dt\right|\\ &\le \tau\left\|\frac{\partial b(u)}{\partial s}\right\|_{L^2(0,t^m;H^{-1}(\Omega))}\left(\tau\sum_{n=1}^{m}\|V_n\|^2_{H_0^1(\Omega)}\right)^{1/2}\\ &\le C_{34}\tau.\end{aligned}\tag{5.9}$$

From (5.8)–(5.9), we get that

$$|I_5^m|\le C_{35}\tau.\tag{5.10}$$

Next, we estimate $I_{3,1}^m$ by using Hölder's, Young's inequalities and hypotheses on g

$$\begin{aligned}\left|I_{3,1}^m\right| &\le \left|\sum_{n=1}^{m}\left\langle\int_{J_n}[g(v)-g(V_n)]\,dt,(-\Delta)^{-1}(e_b^n)\right\rangle\right|\\ &\le C_{36}\tau^{1/2}\sum_{n=1}^{m}\left(\int_{J_n}\|g(v)-g(V_n)\|_2^2\,dt\right)^{1/2}\|e_b^n\|_{H^{-1}(\Omega)}\\ &\le \eta\sum_{n=1}^{m}\left(\int_{J_n}\|g(v)-g(V_n)\|_2^2\,dt\right)+\frac{C_{37}}{\eta}\tau\sum_{n=1}^{m}\|e_b^n\|^2_{H^{-1}(\Omega)}\end{aligned}$$

$$\leq C_{38}\eta \sum_{n=1}^{m}\left(\int_{J_n}\|b(v)-b(V_n)\|_2^2\,dt\right)+\frac{C_{37}}{\eta}\tau\sum_{n=1}^{m}\|e_b^n\|_{H^{-1}(\Omega)}^2 .$$

Knowing that b is an increasing Lipschitz function, we get

$$|I_{3,1}^m| \leq C_{39}\eta\sum_{n=1}^{m}\int_{J_n}\int_{\Omega}(b(v)-b(V_n))(v-V_n)dxdt + c(\eta)\tau\sum_{n=1}^{m}\|e_b^n\|_{H^{-1}(\Omega)}^2 .$$

Moreover, we have

$$|I_{3,2}^m| \leq C\tau + \tau\sum_{n=1}^{m}\|e_b^n\|_{H^{-1}(\Omega)}^2 .$$

From (5.7), we get

$$\sum_{n=1}^{m} I_1^n + \sum_{n=1}^{m} I_2^n \leq I_{3,1}^m + I_{3,2}^m. \tag{5.11}$$

It follows that

$$\begin{aligned}&\sum_{n=1}^{m} I_1^n - C\tau + \sum_{n=1}^{m}\int_{J_n}\int_{\Omega}(b(v)-b(V_n))(v-V_n)dxdt\\ \leq& C_{39}\eta\sum_{n=1}^{m}\int_{J_n}\int_{\Omega}(b(v)-b(V_n))(v-V_n)dxdt + c(\eta)\tau\sum_{n=1}^{m}\|e_b^n\|_{H^{-1}(\Omega)}^2 .\end{aligned}$$

Then choosing η small enough to absorb the first term of the right hand side of the above estimate, we get that

$$\begin{aligned}&\frac{1}{2}\|e_b^m\|_{H^{-1}(\Omega)}^2 + \frac{1}{2}\sum_{n=1}^{m}\left\|e_b^n - e_b^{n-1}\right\|_{H^{-1}(\Omega)}^2 + \sum_{n=1}^{m}\int_{J_n}\int_{\Omega}(b(v)-b(V_n))(v-V_n)dxdt\\ \leq& C\tau + C_{40}\tau\sum_{n=1}^{m}\|e_b^n\|_{H^{-1}(\Omega)}^2 .\end{aligned} \tag{5.12}$$

Putting

$$y^m = \sum_{n=1}^{m}\|e_b^n\|_{H^{-1}(\Omega)}^2 , \quad y^m - y^{m-1} \leq C\cdot r + C_{40}\cdot\tau\cdot y^m$$

and applying discrete Gronwall's inequality, we get that $y^m \leq C(T)$

$$\|e_b^m\|_{H^{-1}(\Omega)} \leq C_{41}\tau^{1/2},$$

which proves the inequality (5.2). On the other hand, we have

$$\sup_{t\in(0,t_m)}\|e_b^n(t)\|_{H^{-1}(\Omega)} - c_{41}\tau^{1/2} \leq \max_{1\leq n\leq m}\|e_b^n(t^n)\|_{H^{-1}(\Omega)} = \max_{1\leq n\leq m}\|e_b^n\|_{H^{-1}(\Omega)} .$$

Thus,

$$\|e_b^n\|_{L^\infty(0,T;H^{-1}(\Omega))} - C_{41}\tau^{1/2} \leq \max_{1\leq n\leq m}\|e_b^n\|_{H^{-1}(\Omega)} .$$

From the above inequality, we obtain

$$\|e_b\|_{L^\infty(0,T;H^{-1}(\Omega))}^2 + \sum_{n=1}^{N}\int_{J_n}\left(b(v(t)) - b\left(V_n\right), v(t)-V_n\right)_{L^2(\Omega)}dt \leq C_{42}\tau.$$

That achieve the proof of the second point in Theorem 5.3.

Now, return to the inequality (5.7) and taking $\varphi = \sum_{n=1}^{m} e_v^n = \sum_{n=1}^{m}(\bar{v}_n - V_n)$ as test function, to get

$$\tau \int_\Omega (b(v) - b(V_n)) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dxdt + \tau^2 \left\| \sum_{n=1}^{m} \nabla (\bar{v}_n - V_n) \right\|^2_{L^2(\Omega)}$$
$$\leq C_{43}\tau^2 \left| \int_\Omega \sum_{n=1}^{m} (\overline{g(v)})^n - g(V_n) \Big) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dx \right|$$
$$+ C_{44}\tau^2 \left| \sum_{n=1}^{m} \left(g(V_n), \sum_{n=1}^{m} (\bar{v}_n - V_n) \right) \right|.$$

It follows that

$$\tau \int_\Omega (b(v) - b(V_n)) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dxdt + \left\| \nabla \int_0^{t^m} e_v dt \right\|^2_{L^2(\Omega)}$$
$$\leq C_{43}\tau^2 \left| \int_\Omega \sum_{n=1}^{m} (\overline{g(v)})^n - g(V_n) \Big) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dx \right|$$
$$+ C_{44}\tau^2 \left| \sum_{n=1}^{m} \left(g(V_n), \sum_{n=1}^{m} (\bar{v}_n - V_n) \right) \right|.$$

Then,

$$\begin{aligned}
\tau^2 \left\| \sum_{n=1}^{m} \nabla (\bar{v}_n - V_n) \right\|^2_{L^2(\Omega)} &= \left\| \nabla \int_0^{t^m} e_v dt \right\|^2_{L^2(\Omega)} \\
&\leq \tau \left| \int_\Omega (b(v) - b(V_n)) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dx \right| \\
&\quad + C_{43}\tau^2 \left| \int_\Omega \sum_{n=1}^{m} \left((\overline{g(v)})^n - g(V_n) \right) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dx \right| \\
&\quad + C_{44}\tau^2 \left| \sum_{n=1}^{m} \left(g(V_n), \sum_{n=1}^{m} (\bar{v}_n - V_n) \right) \right| \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

We begin by estimating J_2

$$\begin{aligned}
J_2 \leq & \left(\int_\Omega \left(\sum_{n=1}^{m} \int_{J_n} (g(v) - g(V_n))\, dt \right)^2 dx \right)^{1/2} \\
& \times \left(\int_\Omega \left(\sum_{n=1}^{m} \int_{J_n} (v(t) - V_n)\, dt \right)^2 dx \right)^{1/2} \\
\leq & T^2 \left(\sum_{n=1}^{m} \int_{J_n} (||g(v) - g(V_n)||_2)^2 dt \right)^{1/2} \left(\sum_{n=1}^{m} \int_{J_n} ||v(t) - V_n||_2^2 dt \right)^{1/2}
\end{aligned}$$

$$\leq T^2 \left(\sum_{n=1}^{m} \int_{J_n} ||g(v) - g(V_n)||_2^2 dt \right)^{1/2}$$
$$\times \left(2 \|v\|^2_{L^2\left(0,T;H_0^1(\Omega)\right)} + 2\tau \sum_{n=1}^{m} ||V_n||_2^2 \right)^{1/2}$$
$$\leq C_{45} \tau^{1/2}.$$

The above inequality follows by using respectively the L^∞-estimate of $v(t), V^n$ and the error bound given in Theorem 4.1. Arguing as in the previous estimate, we get

$$J_3 \leq T^2 \left(\sum_{n=1}^{m} \int_{I_n} ||g(V_n)||_2^2 dt \right)^{1/2} \left(2 \|v\|^2_{L^2\left(0,T;H_0^1(\Omega)\right)} + 2\tau \sum_{n=1}^{m} \|V_n\|_2^2 \right)^{1/2}.$$

Combining (4.2) with the estimates above and using hypothesis (H1), to obtain $J_3 \leq C_{46} \tau^{1/2}$ and

$$J_1 \leq \|e_b^m\|_{H^{-1}(\Omega)} \left(\sum_{n=1}^{m} \int_{I^n} \|v(t)\|_{H_0^1(\Omega)} dt + \tau \sum_{n=1}^{m} \|V_n\|_{H_0^1(\Omega)} \right). \tag{5.13}$$

Finally, collecting these results, it follows that

$$\left\| \nabla \int_0^T e_v^n dt \right\|_2^2 \leq C_{30} \tau^{1/2}.$$

By accumulating all of the previous results, the proof of Theorem 5.1 is completed. □

Remark 5.1. As an example of the application, we can take $b(v) = v$. If we then choose a field that satisfies the hypotheses (H1)-(H5), we will apply Theorem 3.1 to obtain the existence of a solution to the semi-discrete problem associated with the continuous problem (1.2). This result is confirmed by the result obtained in [14] with the appropriate value of "b“. We can also consider the following example:

$$g_1(x) = \sqrt{C_\sigma^2 x^2 + 5},$$

and the function $b(x) = x$. It can be shown that g_1 and b satisfy assumptions (H1) to (H5).

We now exhibit an example showing that many functions satisfy (H1)-(H4).

Example 5.1. Consider b and g two functions such that:

- $b(x) = kx$ (increasing, Lipschitz, $b(0) = 0$);
- $g(x) = \sigma + \frac{L_1}{2} \tanh(x)$ (bounded below by σ, Lipschitz).

Verification.

(a) (H1) g is Lipschitz because $\tanh(x)$ has derivative $\operatorname{sech}^2(x) \leq 1$, so $L_1 = \frac{L_1}{2} \cdot 1$.
(b) (H2) $\tanh(x) \in]-1, 1[$, so $g(x) \geq \sigma - \frac{L_1}{2}$. To ensure $g(x) \geq \sigma$, we must pick L_1 small enough (e.g., $L_1 \leq 2\sigma$).
(c) (H3) b is increasing, Lipschitz, and $b(0) = 0$.

(d) (H4)

i) Since $b(x) - b(y) = k(x - y)$, we have:

$$|g(x) - g(y)| = \frac{L_1}{2} |\tanh(x) - \tanh(y)| \leq \frac{L_1}{2} |x - y|.$$

ii) Meanwhile, $|b(x) - b(y)| = k|x - y|$.

iii) Thus, $|g(x) - g(y)| \leq \left(\frac{L_1}{2k}\right) |b(x) - b(y)|$.

Choose k small enough to ensure $\frac{L_1}{2k} < \frac{(\sigma \cdot \text{meas}(\Omega))^2}{\tau\lambda}$.

Observe that all hypotheses of the preceding theorem are satisfied, from which the error estimates immediately follow.

6. Conclusion and Perspective

In this work, we have shown the existence and uniqueness of a solution to the steadystate problem, which is the time discretization for the continuous problem (1.2). We have also proved some stability results and error estimates for a family of time discretization schemes. We would like to point out that this study is accompanied by effective numerical computations.

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