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ON THE ASYMPTOTIC BEHAVIORS ASSOCIATED WITH THE DAVISON FUNCTIONAL EQUATION

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Abstract. We prove the Hyers-Ulam stability of the Davison functional equation

$$f(x+xy)+f(y)=f(x+y)+f(xy),$$

for a class of mappings from a normed algebra $\mathcal{A}$ (with a unit element 1) into a Banach space $\mathcal{B}$, on the restricted domain $\{(x,y)\in\mathcal{A}\times\mathcal{A}:\ \min\{\|x\|,\|y\|\}\geqslant d\}$, where $d>0$ is a constant. As a result, we obtain some asymptotic behaviors of Davison mappings. In addition, we obtain the corollary that for every mapping g from a normed algebra $\mathcal{A}$ into a normed space $\mathcal{B}$, and for all positive real numbers r,s, one of the following two conditions must be valid:

$$\sup_{x,y\in\mathcal{A}}\|g(x+y)+g(xy)-g(x+xy)-g(y)\|\cdot\|x\|^r\cdot\|y\|^s=+\infty$$

or

$$g(x+y)+g(xy)=g(x+xy)+g(y).$$

1. Introduction and preliminaries

The functional equation

$$f(x+xy)+f(y)=f(x+y)+f(xy), \tag{1.1}$$

was proposed by Davison [2] at the 17th International Symposium on Functional Equations. He inquired about its general solution for mappings from a commutative field $\mathbb{F}$ to another commutative field $\mathbb{K}$. At the same symposium, Benz [1] provided the general continuous solution to the functional equation (1.1) when f is an unknown mapping from the real numbers to the real numbers. In 2000, Girgensohn and Lajkó [4] characterized the general solution of (1.1) without requiring any regular condition.

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They showed that if $f : \mathbb{R} \to \mathbb{R}$ is a solution of (1.1), then f can be expressed as $f(x) = A(x) + b$, where $A : \mathbb{R} \to \mathbb{R}$ is an additive mapping and $b \in \mathbb{R}$ is any constant. Furthermore, they derived the general solution of the pexiderized version of (1.1). In a separate work, Davison [3] determined the solution of (1.1) when the domain of the unknown mapping f is the ring of integers $\mathbb{Z}$ or the set of natural numbers $\mathbb{N}$. Najati and Sahoo [9] introduced two pexiderized functional equations of Davison type and obtained their general solutions.

Jung and Sahoo [6] were the first to study the Hyers-Ulam stability of the Davison functional equation (1.1). The pexiderized functional equation

$$f(xy) + f(x + y) = g(xy + x) + g(y)$$

was investigated for the Hyers-Ulam stability in [5]. Studying the Hyers-Ulam stability of Davison functional equation (1.1) and its pexiderized version on restricted domains would be interesting topics. Let $\mathcal{A}$ be a normed algebra and consider

$$\begin{aligned}
D_1 &:= \{(x, y) \in \mathcal{A} \times \mathcal{A} : \ \min\{\|x\|, \|y\|\} \geqslant d\}, \\
D_2 &:= \{(x, y) \in \mathcal{A} \times \mathcal{A} : \ \|x\| \geqslant d\}, \\
D_3 &:= \{(x, y) \in \mathcal{A} \times \mathcal{A} : \ \|y\| \geqslant d\}, \\
D_4 &:= \{(x, y) \in \mathcal{A} \times \mathcal{A} : \ \|x\| + \|y\| \geqslant d\}, \\
D_5 &:= \{(x, y) \in \mathcal{A} \times \mathcal{A} : \ \max\{\|x\|, \|y\|\} \geqslant d\},
\end{aligned}$$

where $d > 0$ is a real constant. It is clear that $D_1 \subseteq D_j$ for $2 \leqslant j \leqslant 5$. The primary objective of this current paper is to investigate the Hyers-Ulam stability of (1.1) on the unbounded restricted domain D_1. As a consequence, we obtain a hyperstability result for the Davison functional equation (1.1). This leads us to deduce the slightly surprising result that for any mapping f, from a normed algebra $\mathcal{A}$ into a normed space $\mathcal{B}$, and for all positive real numbers $r, s > 0$ one of the following two conditions must hold true:

(i) $\sup_{x,y\in\mathcal{A}} \|f(x + xy) + f(y) - f(x + y) - f(xy)\| \cdot \|x\|^r \cdot \|y\|^s = +\infty$,
(ii) $f(x + xy) + f(y) = f(x + y) + f(xy), \ x, y \in \mathcal{A}$.

Also (ii) is equivalent to

$$\sup_{x,y\in\mathcal{A}} \|f(x + xy) + f(y) - f(x + y) - f(xy)\|(\|x\|^r + \|y\|^s) = +\infty.$$

2. Stability and Hyperstability

The following lemma plays a key role in proving the main theorem.

Lemma 2.1. *Let $\varepsilon \geqslant 0$ and $d > 0$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a mapping from a normed algebra $\mathcal{A}$ (with unit element 1) to a normed space $\mathcal{B}$ satisfying*

$$\|f(x + y) + f(xy) - f(x + xy) - f(y)\| \leqslant \varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d. \tag{2.1}$$

Then,

$$\|f(x + 4y) + f(x + 4y + 1) - f(4y) - f(4y + 1) - f(2x + 2y) + f(2y)\| \leqslant 3\varepsilon, \tag{2.2}$$

for all $x, y \in \mathcal{A}$*, with* $\min\{\|x\|, \|y\|\} \geqslant d+1$*. Moreover,*

$$\Big\|f(-2x)+f(2x)-f(x)-f(-x)\Big\| \leqslant 39\varepsilon, \quad \|x\| \geqslant 4d+4, \tag{2.3}$$

$$\|-f(-4x+1)-f(2x)+f(-2x)+f(1)\| \leqslant 12\varepsilon, \quad \|x\| \geqslant 2d+2, \tag{2.4}$$

$$\|f(2x)-2f(x)+f(0)\| \leqslant 213\varepsilon, \quad \|x\| \geqslant 12d+12. \tag{2.5}$$

Proof. Replace y by $y+1$ in (2.1) to obtain

$$\|f(x+y+1)+f(xy+x)-f(2x+xy)-f(y+1)\| \leqslant \varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d+1. \tag{2.6}$$

Adding (2.6) and (2.1), one obtains

$$\|f(x+y+1)+f(x+y)+f(xy)-f(2x+xy)-f(y)-f(y+1)\| \leqslant 2\varepsilon, \tag{2.7}$$

for all $x, y \in \mathcal{A}$, with $\min\{\|x\|, \|y\|\} \geqslant d+1$. By substituting $4y$ for y in (2.7), we obtain

$$\|f(x+4y+1)+f(x+4y)+f(4xy)-f(2x+4xy)-f(4y)-f(4y+1)\| \leqslant 2\varepsilon, \tag{2.8}$$

for all $x, y \in \mathcal{A}$, with $\min\{\|x\|, \|y\|\} \geqslant d+1$. Replacing x by $2x$ and y by $2y$ in (2.1), one obtains

$$\|f(2x+2y)+f(4xy)-f(2x+4xy)-f(2y)\| \leqslant \varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d. \tag{2.9}$$

Using (2.8) and (2.9), we get (2.2).

By substituting $-2x$ for x and x for y in (2.2), we obtain

$$\|2f(2x)+f(2x+1)-f(4x)-f(4x+1)-f(-2x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.10}$$

Also, replacing x by $2x$ and y by $\frac{x}{2}$ in (2.2), we get

$$\|f(4x)+f(4x+1)-f(2x)-f(2x+1)-f(5x)+f(x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.11}$$

Adding (2.10) and (2.11), we obtain

$$\|f(2x)-f(-2x)-f(5x)+f(x)\| \leqslant 6\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.12}$$

By substituting $-3x$ for x and $\frac{x}{2}$ for y in (2.2), we have

$$\|f(-x)+f(-x+1)-f(2x)-f(2x+1)-f(-5x)+f(x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.13}$$

Add (2.10) and (2.13), to get

$$\begin{aligned} &\|f(2x)-f(4x)-f(4x+1)-f(-2x)+f(-x) \\ &\quad + f(-x+1)-f(-5x)+f(x)\| \leqslant 6\varepsilon, \quad \|x\| \geqslant 2d+2. \end{aligned} \tag{2.14}$$

Replacing x by $3x$ and y by $-x$ in (2.2), we have

$$\|f(-x)+f(-x+1)-f(-4x)-f(-4x+1)-f(4x)+f(-2x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.15}$$

By (2.14) and (2.15), we conclude

$$\begin{aligned} &\|f(-4x)+f(-4x+1)-2f(-2x)+f(2x) \\ &\quad - f(4x+1)-f(-5x)+f(x)\| \leqslant 9\varepsilon, \quad \|x\| \geqslant 2d+2. \end{aligned} \tag{2.16}$$

By substituting $-x$ for x in equation (2.10) and then combining the result with inequalities (2.10) and (2.16), we arrive at

$$\begin{aligned} \|f(-2x) - 2f(2x) - f(-5x) + f(x) + f(-2x+1) \\ - f(2x+1) + f(4x)\| \leqslant 15\varepsilon, \quad \|x\| \geqslant 2d+2. \end{aligned} \tag{2.17}$$

If we substitute $-4x$ for x and $\frac{x}{2}$ for y in (2.2), we can obtain

$$\|f(-2x) + f(-2x+1) - f(2x) - f(2x+1) - f(-7x) + f(x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.18}$$

It can be inferred from equations (2.17) and (2.18) that

$$\| - f(2x) - f(-5x) + f(4x) + f(-7x)\| \leqslant 18\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.19}$$

Replacing x by $-x$ in (2.19) and adding the resultant to (2.12), we obtain

$$\|f(7x) + f(-4x) - f(2x) - f(x)\| \leqslant 24\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.20}$$

Replacing x by $2x$ and y by $\frac{3x}{2}$ in (2.2), we have

$$\|f(8x)+f(8x+1)-f(6x)-f(6x+1)-f(7x)+f(3x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.21}$$

Also, replacing x by $-2x$ and y by $2x$ in (2.2), we get

$$\|f(6x)+f(6x+1)-f(8x)-f(8x+1)-f(0)+f(4x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.22}$$

Add (2.21) and (2.22), to obtain

$$\| - f(7x) + f(3x) + f(4x) - f(0)\| \leqslant 6\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.23}$$

Also, adding (2.20) and (2.23), we arrive at

$$\|f(-4x) + f(4x) + f(3x) - f(2x) - f(x) - f(0)\| \leqslant 30\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.24}$$

Putting $y = \frac{x}{2}$ in (2.2), we have

$$\|f(3x+1) - f(2x) - f(2x+1) + f(x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.25}$$

Now, replacing x by $\frac{x}{2}$ in (2.22) and combining the resultant to (2.25), we conclude

$$\|f(3x)-f(4x)-f(4x+1)+2f(2x)+f(2x+1)-f(x)-f(0)\| \leqslant 6\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.26}$$

It follows from (2.10) and (2.26) that

$$\|f(3x) + f(-2x) - f(x) - f(0)\| \leqslant 9\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.27}$$

So, by combining (2.24) and (2.27), we get (2.3).

By substituting $3x - y$ for x in (2.2), we get the following inequality:

$$\|f(3x+3y) + f(3x+3y+1) - f(6x) - f(4y) - f(4y+1) + f(2y)\| \leqslant 3\varepsilon,$$

for all $x, y \in \mathcal{A}$, with $\min\{\|3x-y\|, \|y\|\} \geqslant d+1$. If we put $y = -x$ in the above inequality, we can rewrite it as:

$$\|f(0)+f(1)-f(6x)-f(-4x)-f(-4x+1)+f(-2x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d+1. \tag{2.28}$$

Finally, by substituting $2x$ for x in (2.27) and adding the result to (2.28), we arrive at inequality (2.4).

Replacing x by $-x$ in (2.10) and then combining the resultant inequality with (2.4), one obtains

$$\|f(-4x) - f(-2x+1) - f(-2x) + f(1)\| \leqslant 15\varepsilon, \quad \|x\| \geqslant 2d+2. \tag{2.29}$$

Also, replacing x by $\frac{x}{2}$ in (2.4) and then combining the resultant inequality with (2.29), we conclude

$$\|f(-4x) - f(-2x) + f(x) - f(-x)\| \leqslant 27\varepsilon, \quad \|x\| \geqslant 4d+4. \tag{2.30}$$

Substitute $2x$ for x in (2.27) and then combining the obtained inequality with (2.30), we obtain the following inequality:

$$\|f(6x) + f(-2x) - f(x) + f(-x) - f(2x) - f(0)\| \leqslant 36\varepsilon, \quad \|x\| \geqslant 4d+4. \tag{2.31}$$

Inequality (2.3) gives us

$$\Big\|2f(-4x) + 2f(4x) - 2f(2x) - 2f(-2x)\Big\| \leqslant 78\varepsilon, \quad \|x\| \geqslant 4d+4.$$

By (2.3), (2.31) and the above inequality, we conclude

$$\|f(6x) - 2f(2x) - 2f(x) + 2f(4x) + 2f(-4x) - f(0)\| \leqslant 153\varepsilon, \quad \|x\| \geqslant 4d+4. \tag{2.32}$$

By multiplying (2.24) by 2 and adding the result to (2.32), we get

$$\|f(6x) - 2f(3x) + f(0)\| \leqslant 213\varepsilon, \quad \|x\| \geqslant 4d+4.$$

This can be rewritten as inequality (2.5), which is the desired result. □

Now we are ready to prove the main theorem.

Theorem 2.1. *Take $\varepsilon \geqslant 0$, $d > 0$. Let $\mathcal{A}$ be a normed algebra* (*with unit element* 1) *and $\mathcal{B}$ a Banach space. If a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies* (2.1)*, then there is a unique additive mapping $\varphi : \mathcal{A} \to \mathcal{B}$ such that*

$$\|f(x) - \varphi(x) - f(0)\| \leqslant 640\varepsilon, \quad x \in \mathcal{A}. \tag{2.33}$$

Proof. By Lemma 2.1, f fulfills (2.5). Then, for all integers n, m with $n \geqslant m \geqslant 0$, we have

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m} + \sum_{i=m}^{n} \frac{f(0)}{2^{i+1}}\right\| \leqslant \sum_{i=m}^{n} \frac{213\varepsilon}{2^{i+1}}, \quad \|x\| \geqslant 12d+12. \tag{2.34}$$

Therefore, $\{\frac{f(2^n x)}{2^n}\}_n$ is a Cauchy sequence for all $x \in \mathcal{A}$. Define $\varphi : \mathcal{A} \to \mathcal{B}$ by $\varphi(x) = \lim_{n\to+\infty} \frac{f(2^n x)}{2^n}$ for all $x \in \mathcal{A}$. Obviously, $\varphi(2x) = 2\varphi(x)$ for all $x \in \mathcal{A}$. Therefore, by (2.3) we infer that φ is odd. So, by (2.4), we conclude

$$\varphi(x) = \lim_{n\to+\infty} \frac{f(2^n x + 1)}{2^n}, \quad x \in \mathcal{A}.$$

Hence, it follows from (2.2) that

$$2\varphi(x+4y) + \varphi(2y) = 2\varphi(4y) + \varphi(2x+2y), \quad x, y \in \mathcal{A}. \tag{2.35}$$

Since $\varphi(2x) = 2\varphi(x)$, (2.35) can be written as

$$\varphi(2x+8y) = 3\varphi(2y) + \varphi(2x+2y), \quad x, y \in \mathcal{A}. \tag{2.36}$$

Putting $x = -y$ in (2.36) and using $\varphi(0) = 0$, we conclude

$$\varphi(3y) = 3\varphi(y), \quad y \in \mathcal{A}. \tag{2.37}$$

Hence, (2.36) and (2.37) yield

$$\varphi(2x+8y) = \varphi(6y) + \varphi(2x+2y), \quad x, y \in \mathcal{A}. \tag{2.38}$$

Replacing y by $\frac{y}{6}$ and x by $\frac{x}{2} - \frac{y}{6}$ in (2.38), we deduce that φ is an additive mapping.
By setting $m = 0$ and letting n approach infinity in (2.34), we arrive at

$$\|f(x) - \varphi(x) - f(0)\| \leqslant 213\varepsilon, \quad \|x\| \geqslant 12d + 12. \tag{2.39}$$

For $y \in \mathcal{A} \setminus \{0\}$ we can choose $x \in \mathcal{A}$ such that

$$\min\{\|x\|, \|xy\|, \|x+y\|, \|x+xy\|\} \geqslant 12d + 12.$$

By (2.39), we have the following inequalities

$$\begin{aligned}
\| -f(x+y) + \varphi(x+y) + f(0)\| &\leqslant 213\varepsilon,\\
\| -f(xy) + \varphi(xy) + f(0)\| &\leqslant 213\varepsilon,\\
\|f(x+xy) - \varphi(x+xy) - f(0)\| &\leqslant 213\varepsilon.
\end{aligned}$$

Combining the previous inequalities and (2.1), we get

$$\|f(y) - \varphi(y) - f(0)\| \leqslant 640\varepsilon.$$

Since this inequality holds for $y = 0$, we deduce (2.33) which is what we wanted to prove. □

As a result, we conclude that if a mapping f satisfies (1.1) on a certain subset $D \subseteq \mathcal{A}$, then f fulfills (1.1) on the entire $\mathcal{A}$.

In the subsequent results, $\mathcal{A}$ denotes a normed algebra with unit element and $\mathcal{B}$ is a normed space.

Corollary 2.1. *Suppose that a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies one of the following assertions:*

(i) $f(x+y) + f(xy) - f(x+xy) - f(y) = 0,\ \min\{\|x\|, \|y\|\} \geqslant d,$
(ii) $f(x+y) + f(xy) - f(x+xy) - f(y) = 0,\ \max\{\|x\|, \|y\|\} \geqslant d,$
(iii) $f(x+y) + f(xy) - f(x+xy) - f(y) = 0,\ \|x\| + \|y\| \geqslant d,$
(iv) $f(x+y) + f(xy) - f(x+xy) - f(y) = 0,\ \|x\| \geqslant d,$
(v) $f(x+y) + f(xy) - f(x+xy) - f(y) = 0,\ \|y\| \geqslant d,$

for some constant $d > 0$. Then, $f - f(0)$ is additive on $\mathcal{A}$.

Proof. Since $(ii) - (v)$ imply (i), we only need to deal with (i). Applying Lemma 2.1 for $\varepsilon = 0$ we deduce

$$f(2x) = 2f(x) - f(0), \quad \|x\| \geqslant 12d + 12.$$

By induction on n, one obtains

$$f(2^n x) = 2^n f(x) - (2^n - 1)f(0), \quad \|x\| \geqslant 12d + 12.$$

This yields the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ is convergent for all $x \in \mathcal{A}$. We define

$$\varphi(x) := \lim_{n \to +\infty} \frac{f(2^n x)}{2^n}, \quad x \in \mathcal{A}.$$

By applying some parts of the proof of Theorem 2.1, we deduce that φ is additive and $\varphi(x) = f(x) - f(0)$ for all $x \in \mathcal{A}$. This ends the proof. $\square$

In the following, we investigate a result that concerns some asymptotic properties related to Davison mappings.

Corollary 2.2. *Suppose that a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies one of the following conditions:*

(i) $\lim_{\min\{\|x\|,\|y\|\} \to +\infty} [f(x+y) + f(xy) - f(x+xy) - f(y)] = 0,$

(ii) $\lim_{\max\{\|x\|,\|y\|\} \to +\infty} [f(x+y) + f(xy) - f(x+xy) - f(y)] = 0,$

(iii) $\lim_{\|x\|+\|y\| \to +\infty} [f(x+y) + f(xy) - f(x+xy) - f(y)] = 0,$

(iv) $\lim_{\|x\| \to +\infty} \sup_{y \in \mathcal{A}} [f(x+y) + f(xy) - f(x+xy) - f(y)] = 0,$

(v) $\lim_{\|y\| \to +\infty} \sup_{x \in \mathcal{A}} [f(x+y) + f(xy) - f(x+xy) - f(y)] = 0.$

Then, $f - f(0)$ is additive on $\mathcal{A}$.

Proof. It is clear that (i) is a consequence of $(ii) - (v)$. Therefore, we only consider (i). Let $\varepsilon > 0$ be any given real number and $\widetilde{\mathcal{B}}$ be the completion of $\mathcal{B}$. From (i), we can find $d_\varepsilon > 0$ such that

$$\|f(x+y) + f(xy) - f(x+xy) - f(y)\| < \varepsilon, \quad \min\{\|x\|, \|y\|\} \geqslant d_\varepsilon.$$

By applying Theorem 2.1 we obtain a constant $K > 0$ and an additive mapping $\varphi_\varepsilon : \mathcal{A} \to \widetilde{\mathcal{B}}$ that satisfy

$$\|\varphi_\varepsilon(x) - f(x) + f(0)\| \leqslant K\varepsilon, \quad x \in \mathcal{A}.$$

So,

$$\begin{aligned}\|f(x+y) - f(x) - f(y) + f(0)\| &\leqslant \|f(x+y) - \varphi_\varepsilon(x+y) - f(0)\| \\ &\quad + \|\varphi_\varepsilon(x) - f(x) + f(0)\| \\ &\quad + \|\varphi_\varepsilon(y) - f(y) + f(0)\| \leqslant 3K\varepsilon, \quad x, y \in \mathcal{A}.\end{aligned}$$

Because ε was chosen arbitrarily, we conclude that $f(x+y) = f(x) + f(y) - f(0)$ for every $x, y \in \mathcal{A}$. This yields that $f - f(0)$ is additive on $\mathcal{A}$. $\square$

Corollary 2.3. *Take $\delta, \varepsilon \geqslant 0$ and suppose that $p, q < 0$ are real numbers and a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies*

$$\|f(x+y) + f(xy) - f(x+xy) - f(y)\| \leqslant \varepsilon\|x\|^p\|y\|^q + \delta(\|x\|^p + \|y\|^q),$$

for all $x, y \in \mathcal{A}$ with $\min\{\|x\|, \|y\|\} \geqslant d$, where $d > 0$ is a constant. Then, $f - f(0)$ is additive on $\mathcal{A}$.

As a result, we can deduce the slightly surprising result that for any mapping f, from a normed algebra $\mathcal{A}$ into a normed space $\mathcal{B}$, and for all positive real numbers $r, s > 0$ one of the following two conditions must hold true:

(i) $\sup_{x,y\in\mathcal{A}} \|f(x+xy) + f(y) - f(x+y) - f(xy)\| \cdot \|x\|^r \cdot \|y\|^s = +\infty$,
(ii) $f(x+xy) + f(y) = f(x+y) + f(xy),\ x, y \in \mathcal{A}$.

Also (ii) is equivalent to

$$\sup_{x,y\in\mathcal{A}} \|f(x+xy) + f(y) - f(x+y) - f(xy)\|(\|x\|^r + \|y\|^s) = +\infty.$$

Corollary 2.4. *Take $\delta, \varepsilon > 0$ and $d > 0$. Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a mapping such that $F(x_0, y_0) \neq 0$ for some $x_0, y_0 \in \mathcal{A}$ with $\min\{\|x_0\|, \|y_0\|\} \geqslant d$ and there are real numbers $p, q < 0$ such that*

$$\|F(x,y)\| \leqslant \varepsilon\|x\|^p\|y\|^q + \delta(\|x\|^p + \|y\|^q), \quad \min\{\|x\|, \|y\|\} \geqslant d.$$

Then, there does not exist any mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies

$$f(x+y) + f(xy) = f(x+xy) + f(y) + F(x,y). \tag{2.40}$$

Proof. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a solution of (2.40). So,

$$\|f(x+y) + f(xy) - f(x+xy) - f(y)\| \leqslant \varepsilon\|x\|^p\|y\|^q + \delta(\|x\|^p + \|y\|^q),$$

where $\min\{\|x\|, \|y\|\} \geqslant d$. Consequently, based on the previous lemma, it can be concluded that $f - f(0)$ is additive on $\mathcal{A}$, which implies that $F(x_0, y_0) = 0$. This contradicts our initial assumption. ☐

3. Conclusions

The Hyers-Ulam stability of the Davison functional equation has been investigated in previous studies [5–8]. In all of them, a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies the inequality

$$\|f(x+y) + f(xy) - f(x+xy) - f(y)\| \leqslant \varepsilon,$$

on the whole space $\mathcal{A}$. Studying the stability problems of the Davison functional equation on a restricted domain will also be an intriguing area of research. In more specific terms, we investigated whether a true additive mapping exists close to a mapping $f : \mathcal{A} \to \mathcal{B}$ that fulfills the aforementioned inequality only in the restricted domain $D_1 = \{(x,y) \in \mathcal{A} \times \mathcal{A} :\ \min\{\|x\|, \|y\|\} \geqslant d\}$. Consequently, we will be able to derive certain asymptotic behaviors of Davison mappings. Of course, it should be noted that this issue has been investigated on the domain $D_2 = \{(x,y) \in \mathcal{A} \times \mathcal{A} :\ \|x\| \geqslant d\}$, which contains D_1. The value derived from the estimate (2.33) is relatively large. It is anticipated that smaller values may be attainable through an alternative proof method. Therefore, an unresolved question arises: does the constant in inequality (2.33) represent the optimal estimate?

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