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SOME k-FRACTIONAL INTEGRAL INEQUALITIES FOR p-CONVEX FUNCTIONS

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Abstract. In this paper, we use Riemann-Liouville k-fractional and k-fractional confomable integrals to prove Hermite-Hadamard inequality, an identity and Hermite-Hadamard type inequality for p-convex functions. Some special cases are also discussed. Our work is extensions of other related previous results.

1. Introduction

Convex functions have been used to investigate various scientific problems. Many refinements have been built for convex functions in order to study problems of pure and applied sciences (see [3, 4, 8, 14–16].)

The Hermite-Hadamard inequality [6, 7] for a convex function $\mathcal{F} : \mathcal{H} \to \mathbb{R}$ on an interval $\mathcal{H}$ is defined by

$$\mathcal{F}\left(\frac{h_1+h_2}{2}\right) \leq \frac{1}{h_2-h_1}\int_{h_1}^{h_2} \mathcal{F}(g)dg \leq \frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}, \tag{1.1}$$

for all $h_1, h_2 \in \mathcal{H}$ with $h_1 < h_2$. Due to extensive applicability of Hermite-Hadamard type inequalities and fractional integrals, number of researchers expand their research involving generalized fractional integrals for diverse classes of convex functions. For instance see [12, 13, 17–19, 23, 25, 26] etc.

Fractional integral inequalities are helpful in estimating the uniqueness of solutions for specific fractional partial differential equations. These inequalities also ensure upper and lower bounds for solutions of the fractional boundary value problems. Our

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aim is to prove several Hermite-Hadamard type inequalities for p-convex functions via Riemann-Liouville k-fractional and k-fractional confomable integrals.

2. Preliminaries

Here we give some basic definitions from the literature. For $k \in (0, \infty)$ and $h \in \mathbb{C}$, the k-gamma function is given by (see [1, 21])

$$\Gamma_k(h) = \lim_{n \to \infty} \frac{n! k^n n k^{\frac{h}{k}-1}}{h_{n,k}}$$

in terms of

$$\tau_{n,k} = \begin{cases} 1, & n = 0, \\ \tau(\tau + k) \cdots (\tau + (n-1)k), & n \in \mathbb{N}, \end{cases}$$

where the integral representaion of $\Gamma_k(\cdot)$ is given as:

$$\Gamma_k(\beta) = \int_0^\infty t^{\beta-1} e^{-\frac{t^k}{k}} dt.$$

Definition 2.1 ([11])**.** Let $\mathcal{F} \in L_1[h_1, h_2]$. The left and right sided Riemann-Liouville fractional integrals $J^\alpha_{h_1+}\mathcal{F}$ and $J^\alpha_{h_2-}\mathcal{F}$ of order $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,(\alpha) > 0$ and $h_2 > h_1 \geq 0$ are defined by

$$J^\alpha_{h_1+}\mathcal{F}(g) = \frac{1}{\Gamma(\alpha)} \int_{h_1}^{g} (g-t)^{\alpha-1} \mathcal{F}(t) dt, \quad g > h_1, \tag{2.1}$$

and

$$J^\alpha_{h_2-}\mathcal{F}(g) = \frac{1}{\Gamma(\alpha)} \int_{g}^{h_2} (t-g)^{\alpha-1} \mathcal{F}(t) dt, \quad g < h_2, \tag{2.2}$$

respectively, where $\Gamma(\cdot)$ is the Gamma function.

Mubeen and Habibullah [20] defined the following generalized fractional integrals.

Definition 2.2 ([20])**.** Let $\mathcal{F} \in L_1[h_1, h_2]$. The left and right sided Riemann-Liouville k-fractional integrals $J^\alpha_{k,h_1+}\mathcal{F}$ and $J^\alpha_{k,h_2-}\mathcal{F}$ of order $\alpha \in \mathbb{C}$ and $h_2 > h_1 \geq 0$ are defined by

$$J^\alpha_{k,h_1+}\mathcal{F}(g) = \frac{1}{k\Gamma_k(\alpha)} \int_{h_1}^{g} (g-t)^{\alpha/k-1} \mathcal{F}(t) dt, \quad g > h_1, \tag{2.3}$$

and

$$J^\alpha_{k,h_2-}\mathcal{F}(g) = \frac{1}{k\Gamma_k(\alpha)} \int_{g}^{h_2} (t-g)^{\alpha/k-1} \mathcal{F}(t) dt, \quad g < h_2, \tag{2.4}$$

respectively, with $\mathrm{Re}\,(\alpha)$, $k > 0$.

Definition 2.3 ([10])**.** Let $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) > 0$, then the left and right sided fractional conformable integral operators are characterised as:

$$ {}^{\beta}_{h_1}\mathcal{J}^{\alpha}\mathcal{F}(g) = \frac{1}{\Gamma(\beta)} \int_{h_1}^{g} \left(\frac{(g-h_1)^{\alpha} - (t-h_1)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\mathcal{F}(t)}{(t-h_1)^{1-\alpha}} dt, \tag{2.5} $$

$$ {}^{\beta}\mathcal{J}^{\alpha}_{h_2}\mathcal{F}(g) = \frac{1}{\Gamma(\beta)} \int_{g}^{h_2} \left(\frac{(h_2-g)^{\alpha} - (h_2-t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\mathcal{F}(t)}{(h_2-t)^{1-\alpha}} dt, \tag{2.6} $$

respectively, with $\alpha > 0$.

Qi et al. [22] defined k-fractional conformable fractional integrals as follows.

Definition 2.4 ([22])**.** Let $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) > 0$, then the left and right sided k-fractional conformable integrals are characterised as:

$$ {}^{\beta}_{k,h_1}\mathcal{J}^{\alpha}\mathcal{F}(g) = \frac{1}{k\Gamma_k(\beta)} \int_{h_1}^{g} \left(\frac{(g-h_1)^{\alpha} - (t-h_1)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}-1} \frac{\mathcal{F}(t)}{(t-h_1)^{1-\alpha}} dt, \tag{2.7} $$

$$ {}^{\beta}\mathcal{J}^{\alpha}_{k,h_2}\mathcal{F}(g) = \frac{1}{k\Gamma_k(\beta)} \int_{g}^{h_2} \left(\frac{(h_2-g)^{\alpha} - (h_2-t)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}-1} \frac{\mathcal{F}(t)}{(h_2-t)^{1-\alpha}} dt, \tag{2.8} $$

respectively, with $\alpha, k > 0$.

Definition 2.5 ([8])**.** Consider an interval $\mathcal{H} \subset (0, \infty)$ and $p \in \mathbb{R}\backslash\{0\}$. A function $\mathcal{F} : \mathcal{H} \to \mathbb{R}$ is called p-convex if

$$ \mathcal{F}\left([rh_1^p + (1-r)h_2^p]^{\frac{1}{p}} \right) \leq r\mathcal{F}(h_1) + (1-r)\mathcal{F}(h_2), \tag{2.9} $$

for all $h_1, h_2 \in \mathcal{H}$ and $r \in [0, 1]$. If (2.9) is reversed then $\mathcal{F}$ is called p-concave.

3. Inequalities for k-Fractional Integrals

First we prove the k-fractional Hadamard's inequality for p-convex function.

Theorem 3.1. *Let $\mathcal{F} : [h_1, h_2] \subset (0, \infty) \to \mathbb{R}$ be a p-convex function such that $\mathcal{F} \in L_1[h_1, h_2]$. Then*

(i) for $p > 0$ we have

$$ \begin{aligned} \mathcal{F}\left(\left[\frac{h_1^p + h_2^p}{2} \right]^{1/p} \right) &\leq \frac{\Gamma_k(\alpha+k)}{2(h_2^p - h_1^p)^{\frac{\alpha}{k}}} \left[J^{\alpha}_{k,h_1^p+}(\mathcal{F} \circ \mu)(h_2^p) + J^{\alpha}_{k,h_2^p-}(\mathcal{F} \circ \mu)(h_1^p) \right] \\ &\leq \frac{\mathcal{F}(h_1) + \mathcal{F}(h_2)}{2}, \end{aligned} \tag{3.1} $$

where $\mu(g) = g^{\frac{1}{p}}$ for all $g \in [h_1^p, h_2^p]$;

(ii) for $p<0$ *we have*

$$\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(h_1^p-h_2^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p-}(\mathcal{F}\circ\mu)(h_2^p)+J^{\alpha}_{k,h_2^p+}(\mathcal{F}\circ\mu)(h_1^p)\right] \tag{3.2}$$
$$\leq \frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2},$$

where $\mu(g)=g^{\frac{1}{p}}$, $g\in[h_2^p,h_1^p]$.

Proof. Since $\mathcal{F}$ is p-convex on $[h_1,h_2]$, we get

$$\mathcal{F}\left(\left[\frac{u^p+w^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{\mathcal{F}(u)+\mathcal{F}(w)}{2}.$$

Taking $u^p=rh_1^p+(1-r)h_2^p$ and $w^p=(1-r)h_1^p+rh_2^p$ with $r\in[0,1]$, we get

$$\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)+\mathcal{F}\left([(1-r)h_1^p+rh_2^p]^{\frac{1}{p}}\right)}{2}. \tag{3.3}$$

Multiplying (3.3) by $r^{\frac{\alpha}{k}-1}$ on both sides with $r\in(0,1)$, $\alpha>0$, and then integrating along r over $r\in[0,1]$ and using changes of variable, we obtain

$$\frac{2k}{\alpha}\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{\frac{1}{p}}\right)$$
$$\leq\int_0^1 r^{\frac{\alpha}{k}-1}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr+\int_0^1 r^{\frac{\alpha}{k}-1}\mathcal{F}\left([rh_2^p+(1-r)h_1^p]^{\frac{1}{p}}\right)dr$$
$$=\int_{h_2^p}^{h_1^p}\left(\frac{h_2^p-w}{h_2^p-h_1^p}\right)^{\frac{\alpha}{k}-1}(\mathcal{F}\circ\mu)(w)\frac{dw}{h_1^p-h_2^p}+\int_{h_1^p}^{h_2^p}\left(\frac{z-h_1^p}{h_2^p-h_1^p}\right)^{\frac{\alpha}{k}-1}(\mathcal{F}\circ\mu)(z)\frac{dz}{h_2^p-h_1^p}$$
$$=\frac{k\Gamma_k(\alpha)}{(h_2^p-h_1^p)^{\frac{\alpha}{k}}}[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p)+J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p)],$$

that is,

$$\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}\right)\leq\frac{\Gamma_k(\alpha+k)}{2(h_2^p-h_1^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p)+J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p)\right]. \tag{3.4}$$

This completes the left inequality of (3.1). For the right inequality, we consider

$$\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)+\mathcal{F}\left([rh_2^p+(1-r)h_1^p]^{\frac{1}{p}}\right)\leq[\mathcal{F}(h_1)+\mathcal{F}(h_2)]. \tag{3.5}$$

Multiplying (3.5) by $r^{\frac{\alpha}{k}-1}$ on both sides with $r\in(0,1)$, $\alpha>0$, and then integrating along r over $\in[0,1]$ and using changes of variable, we obtain

$$\frac{\Gamma_k(\alpha+k)}{2(h_2^p-h_1^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p)+J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p)\right]\leq\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}. \tag{3.6}$$

This completes the second inequality of (3.1). Hence, from (3.4) and (3.6), we get (3.1).

(ii) The proof is analogous to (i). □

Remark 3.1. In Theorem 3.1

(i) if $p = 1$, then the inequality (3.1) becomes the inequality (2.1) of Theorem 2.1 in [5];

(ii) if one takes $\alpha = k = 1$, then the inequality (3.1) becomes the inequality (1.11) of Theorem 6 in [8];

(iii) if one takes $k = p = 1$, then the inequality (3.1) becomes the inequality (2.1) of Theorem 2 in [23];

(iv) if one takes $\alpha = k = p = 1$, then the inequality (3.1) becomes the inequality (1.1).

Lemma 3.1. *Consider a differentiable mapping $\mathcal{F} : [h_1, h_2] \to \mathbb{R}$ on (h_1, h_2) with $h_1 < h_2$. If $\mathcal{F}' \in L_1[h_1, h_2]$, then the following equality holds.*

(i) *For $p > 0$*

$$\begin{aligned}\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2} - \frac{\Gamma_k(\alpha+k)}{2(h_2^p-h_1^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p) + J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p)\right] \\ =\frac{h_2^p-h_1^p}{2p}\int_0^1\left((1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}}\right)A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr,\end{aligned} \tag{3.7}$$

where $A_r^{\frac{1}{p}-1} = [rh_1^p + (1-r)h_2^p]^{\frac{1}{p}}$ and $\mu(g) = g^{\frac{1}{p}}$ for all $g \in [h_1^p, h_2^p]$;

(ii) *For $p < 0$*

$$\begin{aligned}\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2} - \frac{\Gamma_k(\alpha+k)}{2(h_2^p-h_1^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p-}(\mathcal{F}\circ\mu)(h_2^p) + J^{\alpha}_{k,h_2^p+}(\mathcal{F}\circ\mu)(h_1^p)\right] \\ =\frac{h_1^p-h_2^p}{2p}\int_0^1\left((1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}}\right)B_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_2^p+(1-r)h_1^p]^{\frac{1}{p}}\right)dr,\end{aligned} \tag{3.8}$$

where $B_r^{\frac{1}{p}-1} = [rh_2^p + (1-r)k_1^p]^{\frac{1}{p}}$, $\mu(g) = g^{\frac{1}{p}}$ for all $g \in [h_2^p, h_1^p]$.

Proof. First consider

$$\begin{aligned}I =&\int_0^1\left((1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}}\right)A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr \\ =&\left[\int_0^1(1-r)^{\frac{\alpha}{k}}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\right] \\ &+\left[-\int_0^1 r^{\frac{\alpha}{k}}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\right] \\ =&I_1+I_2.\end{aligned} \tag{3.9}$$

Integrating by parts, we obtain

$$I_1 = \int_0^1(1-r)^{\frac{\alpha}{k}}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr \tag{3.10}$$

$$=\frac{p(1-r)^{\frac{\alpha}{k}}}{h_1^p-h_2^p}\mathcal{F}\left(\left[rh_1^p+(1-r)h_2^p\right]^{\frac{1}{p}}\right)\Big|_0^1$$
$$+\frac{p}{h_1^p-h_2^p}\int_0^1\frac{\alpha(1-r)^{\frac{\alpha}{k}-1}}{k}\mathcal{F}\left(\left[rh_1^p+(1-r)h_2^p\right]^{\frac{1}{p}}\right)dr$$
$$=\frac{p}{h_2^p-h_1^p}\mathcal{F}(h_2)-\frac{\alpha p}{k(h_1^p-h_2^p)}\int_{h_2^p}^{h_1^p}\left(\frac{h_1^p-w}{h_1^p-h_2^p}\right)^{\frac{\alpha}{k}-1}\frac{(\mathcal{F}\circ\mu)(w)}{h_1^p-h_2^p}dw$$
$$=\frac{p}{h_2^p-h_1^p}\mathcal{F}(h_2)-\frac{p\Gamma_k(\alpha+k)}{(h_2^p-h_1^p)^{\frac{\alpha}{k}+1}}J^{\alpha}_{h_2^p-}(\mathcal{F}\circ\mu)(h_1^p).$$

Similarly, we have

$$I_2=-\int_0^1 r^{\frac{\alpha}{k}}A_r^{\frac{1}{p}-1}\mathcal{F}'\left(\left[rh_1^p+(1-r)h_2^p\right]^{\frac{1}{p}}\right)dr \tag{3.11}$$
$$=-\frac{pr^{\frac{\alpha}{k}}}{h_1^p-h_2^p}\mathcal{F}\left(\left[rh_1^p+(1-r)h_2^p\right]^{\frac{1}{p}}\right)\Big|_0^1$$
$$+\frac{p}{h_1^p-h_2^p}\int_0^1\frac{\alpha r^{\frac{\alpha}{k}-1}}{k}\mathcal{F}\left(\left[rh_1^p+(1-r)h_2^p\right]^{\frac{1}{p}}\right)dr$$
$$=\frac{p}{h_2^p-h_1^p}\mathcal{F}(h_1)-\frac{\alpha p}{k(h_2^p-h_1^p)}\int_{h_2^p}^{h_1^p}\left(\frac{h_2^p-w}{h_2^p-h_1^p}\right)^{\frac{\alpha}{k}-1}\frac{(\mathcal{F}\circ\mu)(w)}{h_1^p-h_2^p}dw$$
$$=\frac{p}{h_2^p-h_1^p}\mathcal{F}(h_1)-\frac{p\Gamma_k(\alpha+k)}{(h_2^p-h_1^p)^{\frac{\alpha}{k}+1}}J^{\alpha}_{h_1^p+}(\mathcal{F}\circ\mu)(h_2^p).$$

Using (3.10) and (3.11) in (3.9) and then multiplying $\frac{h_2^p-h_1^p}{2p}$ on both sides, we get (3.7).

(ii) Proof is analogous to part (i). □

Remark 3.2. In Lemma 3.1

(i) if $p=1$, then the identity (3.7) becomes the identity (2.6) of Lemma 2.3 in [5];

(ii) if one takes $\alpha=k=1$, then the identity (3.7) becomes the identity (1.12) of Lemma 3 in [8];

(iii) if one takes $k=p=1$, then the identity (3.7) becomes the identity (3.1) of Lemma 2 in [23];

(iv) if one takes $\alpha=k=p=1$, then the identity (3.7) becomes the identity (2.1) of Lemma 2.1 in [2].

Theorem 3.2. *Consider a differentiable mapping $\mathcal{F}:[h_1,h_2]\to\mathbb{R}$ on (h_1,h_2) with $h_1<h_2$ such that $\mathcal{F}'\in L_1[h_1,h_2]$. If $|\mathcal{F}'|^q$ is p-convex on $[h_1,h_2]$ with $q\geq 1$, then the following inequality holds:*

(i) *for $p>1$*

$$\left|\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}-\frac{\Gamma_k(\alpha+k)}{2(h_2^p-h_1^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p)+J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p)\right]\right| \tag{3.12}$$

$$\leq \frac{k^{\frac{1}{q}}(h_2^p - h_1^p)}{2p} Q_1^{1-\frac{1}{q}} \left(\frac{|\mathcal{F}'(h_1)|^q + |\mathcal{F}'(h_2)|^q}{\alpha + k} \right)^{\frac{1}{q}},$$

where $Q_1 = \frac{h_2^{1-p}}{2}\ {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_1^p}{h_2^p}\right)$;

(*ii*) *for* $p < 1$

$$\left| \frac{\mathcal{F}(h_1) + \mathcal{F}(h_2)}{2} - \frac{\Gamma_k(\alpha + k)}{2(h_1^p - h_2^p)^{\frac{\alpha}{k}}} \left[J^{\alpha}_{k,h_1^p-}(\mathcal{F} \circ \mu)(h_2^p) + J^{\alpha}_{k,h_2^p+}(\mathcal{F} \circ \mu)(h_1^p) \right] \right| \tag{3.13}$$

$$\leq \frac{k^{\frac{1}{q}}(h_1^p - h_2^p)}{2p} Q_2^{1-\frac{1}{q}} \left(\frac{|\mathcal{F}'(h_1)|^q + |\mathcal{F}'(h_2)|^q}{\alpha + k} \right)^{\frac{1}{q}},$$

where $Q_2 = \frac{h_2^{p-1}}{2}\ {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_2^p}{h_1^p}\right)$.

Proof. Using Lemma 3.1 and p-convexity of $|\mathcal{F}'|$, we get

(3.14)

$$\left| \frac{\mathcal{F}(h_1) + \mathcal{F}(h_2)}{2} - \frac{\Gamma_k(\alpha + k)}{2(h_2^p - h_1^p)^{\frac{\alpha}{k}}} \left[J^{\alpha}_{k,h_1^p+}(\mathcal{F} \circ \mu)(h_2^p) + J^{\alpha}_{k,h_2^p-}(\mathcal{F} \circ \mu)(h_1^p) \right] \right|$$

$$= \left| \frac{h_2^p - h_1^p}{2} \int_0^1 \left((1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}} \right) A_r^{\frac{1}{p}-1} \mathcal{F}'\left([rh_1^p + (1-r)h_2^p]^{\frac{1}{p}} \right) dr \right|$$

$$\leq \frac{h_2^p - h_1^p}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}} \left(\int_0^1 \left((1-r)^{\frac{\alpha}{k}} + r^{\frac{\alpha}{k}} \right) \left| \mathcal{F}'\left([rh_1^p + (1-r)h_2^p]^{\frac{1}{p}} \right) \right|^q dr \right)^{\frac{1}{q}}$$

$$\leq \frac{h_2^p - h_1^p}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}} \left(\int_0^1 \left((1-r)^{\frac{\alpha}{k}} + r^{\frac{\alpha}{k}} \right) [r|\mathcal{F}'(h_1)|^q + (1-r)|\mathcal{F}'(h_2)|^q] \, dr \right)^{\frac{1}{q}}$$

$$= \frac{h_2^p - h_1^p}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}}$$

$$\times \left(|\mathcal{F}'(h_1)|^q \int_0^1 r \left((1-r)^{\frac{\alpha}{k}} + r^{\frac{\alpha}{k}} \right) + |\mathcal{F}'(h_2)|^q \int_0^1 (1-r) \left((1-r)^{\frac{\alpha}{k}} + r^{\frac{\alpha}{k}} \right) dr \right)^{\frac{1}{q}}.$$

Since

$$\int_0^1 A_r^{\frac{1}{p}-1} dr = \frac{h_2^{1-p}}{2}\ {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_1^p}{h_2^p}\right), \tag{3.15}$$

$$\int_0^1 r(1-r)^{\frac{\alpha}{k}} dr = \int_0^1 (1-r) r^{\frac{\alpha}{k}} dr = \frac{k^2}{(\alpha + k)(\alpha + 2k)} \tag{3.16}$$

and

$$\int_0^1 r^{\frac{\alpha}{k}+1} dr = \int_0^1 (1-r)^{\frac{\alpha}{k}+1} dr = \frac{k}{\alpha + 2k}, \tag{3.17}$$

by using (3.15)−(3.17) in (3.14), we get (3.12). Hence, theorem is proved.

(*ii*) Proof is analogous to part (*i*). □

By taking $p = -1$ in Theroem 3.1, Lemma 3.1 and Theorem 3.2, one can get new results for harminically convex functions via k-fractional integrals.

4. Inequalities for k-Fractional Conformable Integrals

Here our aim is to prove Hadamard's inequalities for p-convex function via k-fractional conformable integrals.

Theorem 4.1. *Let $\mathcal{F} : [h_1, h_2] \subset (0, \infty) \to \mathbb{R}$ be a p-convex function such that $\mathcal{F} \in L_1[h_1, h_2]$.*

(i) Then for $p > 0$ we have

$$\mathcal{F}\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{1/p}\right) \leq \frac{\alpha^{\beta/k}\Gamma(\beta + k)}{2(h_2^p - h_1^p)^{\alpha\beta/k}} \left[{}^{\beta}_{k,h_1^p}\mathcal{J}^{\alpha}(\mathcal{F} \circ \mu)(h_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{k,h_2^p}(\mathcal{F} \circ \mu)(h_1^p)\right] \leq \frac{\mathcal{F}(h_1) + \mathcal{F}(h_2)}{2}, \tag{4.1}$$

where $\mu(g) = g^{\frac{1}{p}}$ for all $g \in [h_1^p, h_2^p]$.

(ii) Then for $p < 0$ we have

$$\mathcal{F}\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{1/p}\right) \leq \frac{\alpha^{\beta/k}\Gamma(\beta + k)}{2(h_1^p - h_2^p)^{\alpha\beta/k}} \left[{}^{\beta}\mathcal{J}^{\alpha}_{k,h_1^p}(\mathcal{F} \circ \mu)(h_2^p) + {}^{\beta}_{k,h_2^p}\mathcal{J}^{\alpha}(\mathcal{F} \circ \mu)(h_1^p)\right] \leq \frac{\mathcal{F}(h_1) + \mathcal{F}(h_2)}{2}, \tag{4.2}$$

where $\mu(g) = g^{\frac{1}{p}}$, $g \in [h_2^p, h_1^p]$.

Proof. Since $\mathcal{F}$ is p-convex on $[h_1, h_2]$, we can have

$$\mathcal{F}\left(\left[\frac{x^p + u^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\mathcal{F}(x) + \mathcal{F}(u)}{2}.$$

Taking $x^p = rh_1^p + (1 - r)h_2^p$ and $u^p = (1 - r)h_1^p + rh_2^p$ with $r \in [0, 1]$, we get

$$\mathcal{F}\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\mathcal{F}\left([rh_1^p + (1 - r)h_2^p]^{\frac{1}{p}}\right) + \mathcal{F}\left([(1 - r)h_1^p + rh_2^p]^{\frac{1}{p}}\right)}{2}. \tag{4.3}$$

Multiplying (4.3) by $\left(\frac{1-r^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1} r^{\alpha-1}$ on both sides with $r \in (0, 1)$, $\alpha > 0$, and then integrating along r over $r \in [0, 1]$, we obtain

$$2\mathcal{F}\left(\left[\frac{h_1^p + h_2^p}{2}\right]^{\frac{1}{p}}\right) \int_0^1 \left(\frac{1 - r^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1} r^{\alpha-1} dr \tag{4.4}$$

$$\leq \int_0^1 \left(\frac{1 - r^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1} r^{\alpha-1}\mathcal{F}\left([rh_1^p + (1 - r)h_2^p]^{\frac{1}{p}}\right) dr$$

$$+\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}\mathcal{F}\left([(1-r)h_1^p+rh_2^p]^{\frac{1}{p}}\right)dr$$
$$=I_1+I_2.$$

By setting $w=rh_1^p+(1-r)h_2^p$, we have

$$\begin{aligned}(4.5)\quad I_1&=\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\\
&=\int_{h_2^p}^{h_1^p}\left(\frac{1-\left(\frac{w-h_2^p}{h_1^p-h_2^p}\right)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\left(\frac{w-h_2^p}{h_1^p-h_2^p}\right)^{\alpha-1}(\mathcal{F}\circ\mu)(w)\frac{dw}{h_1^p-h_2^p}\\
&=\frac{1}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\int_{h_1^p}^{h_2^p}\left(\frac{(h_2^p-h_1^p)^\alpha-(h_2^p-w)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(h_2^p-w)^{\alpha-1}(\mathcal{F}\circ\mu)(w)dw\\
&=\frac{k\Gamma_k(\beta)}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\;{}^{\beta}\mathcal{J}^{\alpha}_{k,h_2^p}(\mathcal{F}\circ\mu)(h_1^p).\end{aligned}$$

Similarly, by setting $w=rh_2^p+(1-r)h_1^p$, we have

$$\begin{aligned}(4.6)\quad I_2&=\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}\mathcal{F}\left([(1-r)h_1^p+rh_2^p]^{\frac{1}{p}}\right)dr\\
&=\int_{h_1^p}^{h_2^p}\left(\frac{1-\left(\frac{w-h_1^p}{h_2^p-h_1^p}\right)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\left(\frac{w-h_1^p}{h_2^p-h_1^p}\right)^{\alpha-1}(\mathcal{F}\circ\mu)(w)\frac{dw}{h_2^p-h_1^p}\\
&=\frac{1}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\int_{h_1^p}^{h_2^p}\left(\frac{(h_2^p-h_1^p)^\alpha-(w-h_1^p)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(w-h_1^p)^{\alpha-1}(\mathcal{F}\circ\mu)(w)dw\\
&=\frac{k\Gamma_k(\beta)}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\;{}^{\beta}_{k,h_1^p}\mathcal{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p).\end{aligned}$$

Also, we have

$$\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}dr=\frac{k}{\beta\alpha^{\beta/k}}.$$

Thus, by putting values of I_1 and I_2 in (4.4), we get

$$(4.7)\quad \frac{k}{\alpha^{\beta/k}\beta}\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{\frac{1}{p}}\right)\le\frac{k\Gamma_k(\beta)}{(h_2^p-h_1^p)^{\alpha\beta/k}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{k,h_2^p}(\mathcal{F}\circ\mu)(h_1^p)+{}^{\beta}_{k,h_1^p}\mathcal{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p)\right].$$

This completes the first inequality of (4.1). For second inequality, we know that

$$(4.8)\qquad \mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)+\mathcal{F}\left([rh_2^p+(1-r)h_1^p]^{\frac{1}{p}}\right)\le[\mathcal{F}(h_1)+\mathcal{F}(h_2)].$$

Multiplying (4.8) by $\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k-1} r^{\alpha-1}$ on both sides with $r \in (0,1)$, $\alpha > 0$, and then integrating with respect to r on interval $[0,1]$, we obtain the following inequality

$$\frac{k\Gamma_k(\beta)}{(h_2^p - h_1^p)^{\alpha\beta/k}} \left[{}^\beta\mathcal{J}^\alpha_{h_2^p}(\mathcal{F}\circ\mu)(h_1^p) + {}^\beta_{h_1^p}\mathcal{J}^\alpha(\mathcal{F}\circ\mu)(h_2^p)\right] \leq \frac{k}{\alpha^{\beta/k}\beta}(\mathcal{F}(h_1)+\mathcal{F}(h_2)). \tag{4.9}$$

This completes the second inequality of (4.1). Hence, the proof is completed.

(ii) The proof is parallel to (i). □

Remark 4.1. In Theorem 4.1

(i) if we take $k = 1$, then we get Thoerem 2.1 in [18];

(ii) by letting $p = k = 1$, we find Theorem 2.1 in [24];

(iii) by letting $p = k = 1$ and $\alpha = 1$, we obtain Theorem 2 in [23];

(iv) by letting $p = -1$ and $k = \alpha = 1$, we get Theorem 4 in [9].

Corollary 4.1. *With the parallel assumption of Theorem* 4.1*, if we take* $p = -1$*, then we get the following inequality*

$$\begin{aligned}\mathcal{F}\left(\frac{2h_1h_2}{h_1+h_2}\right) &\leq \frac{(h_1h_2)^{\frac{\alpha\beta}{k}}\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(h_2-h_1)^{\frac{\alpha\beta}{k}}}\left[{}^\beta\mathcal{J}^\alpha_{k,1/h_1}(\mathcal{F}\circ\mu)\left(\frac{1}{h_2}\right) + {}^\beta_{k,1/h_2}\mathcal{J}^\alpha(\mathcal{F}\circ\mu)\left(\frac{1}{h_1}\right)\right] \\ &\leq \frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2},\end{aligned} \tag{4.10}$$

where $\mu(g) = \frac{1}{g}$, $g \in \left[\frac{1}{h_2}, \frac{1}{h_1}\right]$.

Lemma 4.1. *Let* $\mathcal{F} : [h_1,h_2] \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* (h_1,h_2) *with* $h_1 < h_2$ *such that* $\mathcal{F}' \in L_1[h_1,h_2]$*, then we have*

(i) *for* $p > 0$

$$\begin{aligned}&\left(\frac{\mathcal{F}(h_1^p)+\mathcal{F}(h_2^p)}{2}\right) - \frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{2(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\left[{}^\beta_{k,h_1^p}\mathcal{J}^\alpha(\mathcal{F}\circ\mu)(h_2^p) + {}^\beta\mathcal{J}^\alpha_{k,h_2^p}(\mathcal{F}\circ\mu)(h_1^p)\right] \\ &= \frac{(h_2^p-h_1^p)\alpha^{\beta/k}}{2p}\int_0^1\left[\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k} - \left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}\right] \\ &\quad \times A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr,\end{aligned} \tag{4.11}$$

where $A_r = [rh_1^p + (1-r)h_2^p]$;

(ii) *for* $p < 0$

$$\begin{aligned}&\left(\frac{\mathcal{F}(h_1^p)+\mathcal{F}(h_2^p)}{2}\right) - \frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{2(h_1^p-h_2^p)^{\frac{\alpha\beta}{k}}}\left[{}^\beta\mathcal{J}^\alpha_{k,h_1^p}(\mathcal{F}\circ\mu)(h_2^p) + {}^\beta_{k,h_2^p}\mathcal{J}^\alpha(\mathcal{F}\circ\mu)(h_1^p)\right] \\ &= \frac{(h_1^p-h_2^p)\alpha^{\beta/k}}{2p}\int_0^1\left[\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k} - \left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}\right]\end{aligned} \tag{4.12}$$

$$\times B_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_2^p+(1-r)h_1^p]^{\frac{1}{p}}\right)dr,$$

where $B_r = [rh_2^p + (1-r)h_1^p]$.

Proof. (i) Consider

$$\int_0^1\left[\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k}-\left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}\right]A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr \tag{4.13}$$

$$=\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr$$

$$-\int_0^1\left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr$$

$$=I_1-I_2.$$

Then applying by parts integration, we achieve

$$\begin{aligned}
I_1=&\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\\
=&\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k}\frac{p}{h_1^p-h_2^p}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)\Big|_0^1\\
&-\frac{p}{h_2^p-h_1^p}\int_0^1\frac{\beta}{k}\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k-1}r^{\alpha-1}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\\
=&\frac{p}{\alpha^{\beta/k}(h_2^p-h_1^p)}\mathcal{F}(h_2^p)-\frac{p\beta}{(h_2^p-h_1^p)}\frac{\Gamma_k(\beta)}{(h_2^p-h_1^p)^{\alpha\beta}}\,{}^{\beta}\mathcal{J}^{\alpha}_{h_2^p}(\mathcal{F}\circ\mu)(h_1^p)\\
=&\frac{p}{h_2^p-h_1^p}\left[\frac{\mathcal{F}(h_2^p)}{\alpha^{\beta/k}}-\frac{\Gamma_k(\beta+k)}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\,{}^{\beta}\mathcal{J}^{\alpha}_{k,h_2^p}(\mathcal{F}\circ\mu)(h_1^p)\right].
\end{aligned}\tag{4.14}$$

Similarly,

$$\begin{aligned}
I_2=&\int_0^1\left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\\
=&\left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}\frac{p}{h_1^p-h_2^p}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)\Big|_0^1\\
&-\frac{p}{h_1^p-h_2^p}\int_0^1\frac{\beta}{k}\left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(1-r)^{\alpha-1}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\\
=&-\frac{p}{\alpha^{\beta/k}(h_2^p-h_1^p)}\mathcal{F}(h_1^p)+\frac{p\beta}{h_2^p-h_1^p}\frac{\Gamma_k(\beta)}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\,{}^{\beta}_{h_1^p}\mathcal{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p)\\
=&-\frac{p}{h_2^p-h_1^p}\left[\frac{\mathcal{F}(h_2^p)}{\alpha^{\beta/k}}-\frac{\Gamma_k(\beta+k)}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\,{}^{\beta}_{k,h_1^p}\mathcal{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p)\right].
\end{aligned}\tag{4.15}$$

Here we apply change of variable by taking $w = 1 - r$. Hence, adding I_1, $-I_2$ and then by multiplying by $\frac{\alpha^{\beta/k}(h_2^p - h_1^p)}{2p}$, on both sides, we get (4.11).

(*ii*) The proof is similar to (*i*). □

Remark 4.2. In Lemma 4.1

(*i*) by letting $k = 1$, then one gets Lemma 2.4 in [18];

(*ii*) by letting $p = k = 1$, then one gets Lemma 3.1 in [24];

(*iii*) by letting $p = k = 1$ and $\alpha = 1$, then one gets Lemma 2 in [23].

Theorem 4.2. *Let $\mathcal{F} : [h_1, h_2] \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on (h_1, h_2), $h_1 < h_2$, such that $\mathcal{F}' \in 1[h_1, h_2]$. If $|\mathcal{F}'|^q$, where $q \geq 1$, is p-convex, then*

(*i*) *for $p > 0$*

$$\left|\left(\frac{\mathcal{F}(h_1^p) + \mathcal{F}(h_2^p)}{2}\right) - \frac{\Gamma_k(\beta + k)\alpha^{\beta/k}}{2(h_2^p - h_1^p)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k,h_1^p}\mathcal{J}^{\alpha}(\mathcal{F} \circ \mu)(h_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{k,h_2^p}(\mathcal{F} \circ \mu)(h_1^p)\right]\right| \tag{4.16}$$

$$\leq \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p}\left(\frac{h_2^{1-p}}{2}\,{}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_1^p}{h_2^p}\right)\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\alpha^{\frac{\beta}{k}+1}}B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right)\left[|\mathcal{F}'(h_1)|^q + |\mathcal{F}'(h_2)|^q\right]\right)^q;$$

(*ii*) *for $p < 0$*

$$\left|\left(\frac{\mathcal{F}(h_1^p) + \mathcal{F}(h_2^p)}{2}\right) - \frac{\Gamma(\beta + 1)\alpha^{\beta/k}}{2(h_1^p - h_2^p)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{k,h_1^p}(\mathcal{F} \circ \mu)(h_2^p) + {}^{\beta}_{k,h_2^p}\mathcal{J}^{\alpha}(\mathcal{F} \circ \mu)(h_1^p)\right]\right| \tag{4.17}$$

$$\leq \frac{(h_1^p - h_2^p)\alpha^{\beta/k}}{2p}\left(\frac{h_1^{1-p}}{2}\,{}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_2^p}{h_1^p}\right)\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\alpha^{\frac{\beta}{k}+1}}B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right)\left[|\mathcal{F}'(h_1)|^q + |\mathcal{F}'(h_2)|^q\right]\right)^q,$$

where B and ${}_2F_1$ are classical Beta and Hypergeometric functions, respectively.

Proof. Applying Lemma 4.1, modulus property, Hölder's inequality and p-convexity of $|\mathcal{F}'|^q$, we achieve

(4.18)

$$\left|\left(\frac{\mathcal{F}(h_1^p) + \mathcal{F}(h_2^p)}{2}\right) - \frac{\Gamma_k(\beta + k)\alpha^{\beta/k}}{2(h_2^p - h_1^p)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{h_1^p}\mathcal{J}^{\alpha}(\mathcal{F} \circ \mu)(h_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{h_2^p}(\mathcal{F} \circ \mu)(h_1^p)\right]\right|$$

$$= \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p}\left|\int_0^1\left[\left(\frac{1 - r^{\alpha}}{\alpha}\right)^{\beta/k} - \left(\frac{1 - (1 - r)^{\alpha}}{\alpha}\right)^{\beta/k}\right]\right.$$

$$\left.\times A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p + (1 - r)h_2^p]^{\frac{1}{p}}\right)dr\right|$$

$$\leq \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p} \left| \int_0^1 \left[\left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} + \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} \right] \right.$$
$$\left. \times A_r^{\frac{1}{p}-1} \mathcal{F}' \left([rh_1^p + (1-r)h_2^p]^{\frac{1}{p}} \right) \right| dr$$
$$\leq \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \left[\left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} + \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} \right] \left| \mathcal{F}' \left([rh_1^p + (1-r)h_2^p]^{\frac{1}{p}} \right) \right|^q dr \right)^{1/q}$$
$$\leq \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \left[\left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} + \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} \right] (r|\mathcal{F}'(h_1)|^q + (1-r)|\mathcal{F}'(h_2)|^q) \, dr \right)^{1/q}$$
$$= \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p} \nu^{1-\frac{1}{q}} \left(|\mathcal{F}'(h_1)|^q \int_0^1 \left[r \left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} + r \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} \right] dr \right.$$
$$\left. + |\mathcal{F}'(h_2)|^q \int_0^1 \left[(1-r) \left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} + (1-r) \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} \right] dr \right)^{1/q},$$

where

$$\nu = \int_0^1 A_r^{\frac{1}{p}-1} dr = \frac{h_2^{1-p}}{2} \, {}_2F_1 \left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_1^p}{h_2^p} \right),$$

and from changes of variables, $x = r^\alpha$ and $y = (1-r)^\alpha$, we find

$$\int_0^1 r \left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} dr = \frac{1}{\alpha^{\frac{\beta}{k}+1}} B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right),$$
$$\int_0^1 r \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} dr = \frac{1}{\alpha^{\frac{\beta}{k}+1}} \left[B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right) - B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right) \right],$$
$$\int_0^1 (1-r) \left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} dr = \frac{1}{\alpha^{\frac{\beta}{k}+1}} \left[B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right) - B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right) \right],$$
$$\int_0^1 (1-r) \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} dr = \frac{1}{\alpha^{\frac{\beta}{k}+1}} B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right).$$

Thus, by using above equalities in (4.18), we obtain the inequality (4.16).

(ii) Proof is similar to (i). □

Remark 4.3. In Theorem 4.2, if we take $k = 1$, then we get Thoerem 2.6 in [18].

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