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DIFFERENTIAL INVARIANTS OF COUPLED HIROTA-SATSUMA KDV EQUATIONS

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Abstract. In this paper, we consider a generalized coupled Hirota-Satsuma KdV (CHSK) system of equations. We apply the moving frames method to find a finite generating set of differential invariants for the Lie symmetry group of CHSK equations. Once the generating set of differential invariants is located, we obtain recurrence relations and syzygies among the generating differential invariants. Our approach provides a complete characterization of the structure of algebras of differential invariants of CHSK equations.

1. Introduction

The equivalence moving frames method was introduced by E. Cartan to solve the equivalence problems on submanifolds under the action of a transformation group. In 1974, P. A. Griffiths has paid to the uniqueness and existence problem on geometric differential equations by using the Cartan method of Lie groups and moving frames [25]. Later on, in the 1990s, Fels and Olver have presented the moving co-frame method as a new formulation of the classical Cartan method for finite-dimensional Lie group actions on manifolds [10, 11]. In the last two decades, the moving frames method has been developed in the general algorithmic and equivariant framework which gives several new powerful tools for finding and classifying the equivalence and symmetry properties of submanifolds, differential invariants, and their syzygies (for instance, see [20–22]).

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The equivalence and symmetries of submanifolds are subject to their differential invariants, which have the same symmetry properties and allow us to determine the exact solutions of differential equations [4, 16]. In general, differential invariants are used to solve a broad range of problems appearing in nonlinear theory, mathematical physics, computer science and so on. A significant step for developing these applications is to study the structure of the algebra of differential invariants.

The KdV equations are well-known nonlinear evolution equations (NLEEs) which are a model for many physical phenomena. The simplest form of the KdV equation is $u_t + uu_x + u_{xxx} = 0$, where the uu_x term models nonlinear effects and the u_{xxx} term models dispersive effects of a wave propagation. The KdV equations can give a clear interpretation of both nonlinear effects and dispersive effects of propagation of long waves [3]. In the present paper, we consider a generalized coupled Hirota-Satsuma KdV (CHSK) system as [31]:

$$\begin{cases} u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3(vw)_x = 0, \\ v_t + v_{xxx} - 3uv_x = 0, \\ w_t + w_{xxx} - 3uw_x = 0, \end{cases} \tag{1.1}$$

which is the mathematical model of interactions of two long waves with different dispersion relations [6]. Especially, when $v = w$, the system (1.1) gives the well-known coupled Hirota-Satsuma KdV system.

System (1.1) has been studied via several methods such as the classical Lie group method [1], non-local symmetries [5], collocation method with quintic b-spline method [27]. These works mainly focused on obtaining solutions. Recently, a Lie group analysis on the time-fractional (CHSK) system has been done to obtain exact solutions and conservation laws [28]. In analytical works, differential invariants appeared. Indeed, these methods reduce the system with the aid of differential invariants (e.g. [1, 5, 28]). Knowledge of the algebraic structure of the differential invariants enables us to obtain not only the reduced equations but also to construct a class of differential equations which has the same symmetry properties, and it is significant from the physical point of view.

As far as we know, a comprehensive structure of algebras of differential invariants of equations (1.1) is not obtained so far. In this paper, using the moving frames method, we consider the structure of algebras of differential invariants of System (1.1). The advantage of our approach is that we yield the structure of the differential invariants algebra of System (1.1) only by using the infinitesimal determining equations and choosing a proper cross-section. Further, we do not need additional efforts for integration. Moreover, our approach contains straightforward calculations, yet it is more powerful since it yields the relations among the invariants.

The paper has the following organization. In Section 2, first in subsection 2.1, we remember the concept of differential invariants and some results on them. Then, in subsection 2.2, we explain the moving frames method and how one can apply the method to analyze the algebraic structure of groups. In Section 3, we apply the method to the CHSK system and we analyze the algebraic structure of its Lie symmetry. In fact, using the moving frames method, we locate a finite set of generating differential invariants for the CHSK system and we obtain recurrence relations and the syzygies among the generation of differential invariants.

2. Preliminaries

In this section, we present the preliminary concepts of differential invariants and moving frames method. We assume the reader to be familiar with the concepts of Lie symmetry method which is described in [15] and is used in many papers (e.g. [2, 12, 13, 28, 29]).

First, we remember the concept of jet space. By definition, the *jet space of order* n, $J^n = J^n(M, p)$, is the equivalence classes of p-dimensional submanifolds of a manifold M (of dimension m) under the equivalence relation of nth order contact. For instance, let we consider the local coordinates $z = (x, u)$ on manifold M, such that, the components of $x = (x^1, \dots, x^p)$ are assumed as independent variables and the components of $u = (u^1, \dots, u^q)$ are regarded as dependent ones. So, in these coordinates, a p-submanifold is realized as the graph of a function $u = f(x)$ [17]. Two such submanifolds are equivalent at a point $(x_0, u_0) = (x_0, f(x_0))$ if and only if they have the same nth order Taylor polynomials at x_0 [17]. The induced coordinates on the jet space J^n are denoted by $z^{(n)} = (x, u^{(n)})$, consisting of independent variables x^i, dependent variables u^α, and their derivatives u^α_J, of order $\#J \leq n$.

Now, let G be a local group of transformations acting on M. The induced local action of G on the nth jet space $J^n(M)$ is called the nth prolongation of G denoted by G^n. This prolongation transforms $u = f(x)$ and its derivatives. Studying the infinitesimal generators of prolonged group transformations is much easier than working with the explicit formula for the prolonged group transformations. Therefore, we work with the infinitesimal generators of prolonged group transformations.

If G is assumed to be a connected transformation group, then its infinitesimal generators form the Lie algebra of vector fields as

$$X = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

The nth prolongation of X is

$$X^{(n)} = \sum_{i=1}^{p} \xi^i(x, u^{(n)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} \eta^\alpha_J(x, u^{(n)}) \frac{\partial}{\partial u^\alpha_J},$$

where

$$\eta^\alpha_J = D_J \left(\eta^\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i \right) + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i},$$

and $D_J = D_{j_1} \cdots D_{j_n}$. Here D_j denotes the total derivative with respect to x^j as

$$D_j = \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^{q} \sum_J u^\alpha_{J,j} \frac{\partial}{\partial u^\alpha_J}. \tag{2.1}$$

2.1. **Differential Invariants.** A real-valued function $F : J^n \to \mathbb{R}$ is a *differential invariant* of a group transformation G, if it is remained unchanged by the prolongation of G, i.e. $F(g^{(n)}.z^{(n)}) = F(z^{(n)})$, for all $z^{(n)} \in J^n$ and all $g \in G$ [23].

A well-known theorem ([9, Theorem 42]) of S. Lie and S. Scheffers says that under appropriate assumptions, all the differential invariants can be generated by a finite number of low order invariants and their derivatives. Lie and Scheffers presented the finite-dimensional Lie group case. Then, in 1894, A. Tress extends the theorem to infinite-dimensional pseudo-groups [30]. Indeed, there exists a finite set of generating differential invariants, and p invariant differential operators that preserve the differential invariant algebra, such that any other differential invariant can be locally written as a function of the generating invariants and their invariant derivatives. The order of differentiation is important, since the invariant differential operators need not commute. Furthermore, the differentiated invariants are generally not functionally independent, but are govern by certain functional relations or *syzygies* [24].

To have a complete investigation of the algebra of differential invariants, we need to find a finite set of generating differential invariant, their functional relationships or their syzygies, and the commutation relationships between invariant operators.

2.2. **Moving Frames Method.** In order to describe the moving frames method, first we recall that a moving frame is an equivariant map $\rho : J^n \to G$ from the jet space J^n to the group G satisfying $\rho(g^{(n)} \cdot z^{(n)}) = g \cdot \rho(z^{(n)})$ for every $z^{(n)} \in J^n$ and $g \in G$. However, only free actions have moving frames. To prove the necessity of freeness, let g be an arbitrary element in the isotropy subgroup and ρ be a moving frame, then $\rho(z^{(n)}) = \rho(g^{(n)} \cdot z^{(n)}) = g \cdot \rho(z^{(n)})$. Therefore, the isotropy subgroup must be $G_{z^{(n)}} = \{e\}$ for each $z^{(n)} \in J^{(n)}$, meaning that the action must be free.

We can make actions free by prolonging the group. A theorem, which was presented by Ovsiannikov and improved by Olver, states that if a group acts (locally) effectively on subsets, then there exists an integer k such that the prolongation of the group action is locally free on an open and dense subset of the k-th order jet space [14, 19]. In cases where G does not act effectively, without loss of any generality, we can replace G with the effectively acting quotient group G/G^*_M, where G^*_M is the global isotropy subgroup [17]. Therefore, in order to make a action free, we prolong the group action to a sufficiently high order jet space. The prolongation makes it possible to apply the moving frames method to any group.

Once the freeness is achieved, we choose a specific local cross-section to the prolonged group orbits. Based on the chosen cross-section, we construct a moving frame. After constructing a moving frame, we use the invariantization process to produce complete systems of differential invariants and invariant differential operators. So, we start with definition of cross-section.

A *cross-section* is a submanifold $\mathcal{K}^n \subset J^n$, that intersect the prolonged group orbits transversally. The cross-section is called *regular* if $\mathcal{K}^n$ intersects each orbit at most once. The corresponding moving frame associates to each $z^{(n)} \in J^n$ is the unique group element $g = \rho^{(n)}(z^{(n)}) \in G$ that maps $z^{(n)}$ to the cross-section $g \cdot z^n = \rho^{(n)}(z^{(n)}) \cdot z^{(n)} \in \mathcal{K}$ [22].

For simplicity, we can choose $\mathcal{K} = \{z_1 = c_1, \ldots, z_r = c_r\}$ as *coordinate cross-section*, which prescribed by setting the $r = \dim G$ coordinates to proper constants.

Given local coordinates $z^{(n)} = (z, u^{(n)})$ on J^n, let $w(g, z^{(n)}) = g \cdot z^{(n)}$ be the explicit formulae for the group action. The right moving frame $g = \rho^{(n)}(z^{(n)})$ associated with the coordinate cross-section

$$\mathcal{K} = \{z_1 = c_1, \ldots, z_r = c_r\},$$

is obtained by solving the *normalization equations*

$$w_1(g, z^{(n)}) = c_1, \ldots, w_r(g, z^{(n)}) = c_r. \tag{2.2}$$

Substituting the moving frame formulae for the group parameters into the remaining action rules provides a complete system of functionally independent differential invariants [26].

$$I(z^{(n)}) = w(\rho^{(n)}(z^{(n)}), z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)}.$$

In particular, the normalization components (2.2) of w will be constant, which are called the *phantom differential invariants.* Other components (2.2) are called *fundamental invariants.*

In particular,

$$H^i(x, u^{(n)}) = \iota(x^i), \quad I^\alpha_J(x, u^{(n)}) = \iota(u^\alpha_J),$$

will denote the normalized differential invariants.

To carry on the moving frames method, we use the concept of invariantization and begin the invariantization process. The invariantization

$$\iota : F(x, u^{(n)}) \to I(x, u^{(n)}) = F(\rho^{(n)}(x, u^{(n)}) \cdot (x, u^{(n)})),$$

maps the differential function F to the differential invariant $I = \iota(F)$ [24].

Separating the local coordinates (x, u) on M into independent and dependent variables splits the one-forms on J^∞ into *horizontal forms*, which are spanned by $dx^1, \ldots, dx^p$, and *vertical forms*, which are spanned by the basic contact one-forms [17]

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i} dx^i, \quad \alpha = 1, \ldots, q, \#J \geq 0.$$

The pull back of the dual Maurer-Cartan forms $\mu^1, \ldots, \mu^r$ on G via the moving frame map: $v^k = \rho^* \mu^k$ produces the *invariantized Maurer-Cartan forms* [8]. We split the invariantized Maurer-Cartan forms into horizontal forms and vertical forms:

$$v^k = \sum_{i=1}^{p} R_i^{\kappa} (\bar{\omega})^i + \sum_{\alpha,J} S_{\alpha}^{\kappa,J} \vartheta_j^{\alpha},$$

where R_i^{κ} and $S_{\alpha}^{\kappa,J}$ are certain differential invariants, $\bar{\omega}^i = \iota(dx^i)$ denote the invariantized horizontal one-forms and the invariantized basis contact forms are denoted by

$$\vartheta_J^{\alpha} = \iota(\theta_J^{\alpha}), \quad \alpha = 1, \ldots, q, \# J \geq 0.$$

The R_i^{κ} are called the *Maurer-Cartan invariants* [7]. The Maurer-Cartan invariants will appear in the recurrence formula which will introduce later.

Though invariantization respect all algebraic operators, it does not respect differentiation, i.e., $D[\iota(F)] \neq \iota[D(F)]$. However, there is an explicit formula known as the *recurrence formula* which determines the effect of invariantization on derivatives [11]. Given a differential function $F(x, u^{(n)})$ and $\iota(F)$ its moving frame invariantization. Then the recurrence formula will be

$$D_i[\iota(F)] = \iota[D_i(F)] + \sum_{\kappa=1}^{r} R_i^{\kappa} \iota[X_{\kappa}^{(n)}(F)], \tag{2.3}$$

where R_i^{κ} are the Maurer-Cartan invariants and $X_{\kappa}^{(n)}$ are the nth prolongations of the infinitesimal generators X_{κ} [26]. In our approach, the recurrence formula (2.3) is the key to study the algebra of differential invariants.

The invariant differential operators $\mathcal{D}_i$ map differential invariants to differential invariants. In most cases, they do not commute, but they satisfy in linear commutation relations of the form

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} Y_{ij}^k \mathcal{D}_k, \quad i, j = 1, \ldots, p, \tag{2.4}$$

where the coefficients Y_{ij}^k are certain differential invariants called the *commutator invariants* [24]. One can obtain the commutator invariants Y_{ij}^k by differentiating the recurrence formula (2.3).

In general, if $K = (k_1, \ldots, k_l)$ is an ordered multi-index, then, as a result of (2.4),

$$\mathcal{D}_{\pi(K)} = \mathcal{D}_K + \sum_{\# J < \# K} Y_{\pi,K}^J \mathcal{D}_J, \tag{2.5}$$

for any permutation π of the entries of K. For example,

$$\begin{aligned} \mathcal{D}_i \mathcal{D}_j \mathcal{D}_k &= \mathcal{D}_i \left(\mathcal{D}_k \mathcal{D}_j + \sum_{l=1}^{p} Y_{jk}^l \mathcal{D}_l \right) \\ &= \mathcal{D}_k \mathcal{D}_i \mathcal{D}_j + \sum_{l=1}^{p} \left[Y_{ik}^l \mathcal{D}_l \mathcal{D}_j + Y_{jk}^l \mathcal{D}_i \mathcal{D}_l + (\mathcal{D}_i Y_{jk}^l) \mathcal{D}_l \right]. \end{aligned}$$

Using the commutator formulae (2.5), we can construct an infinite number of *commutator syzygies* by applying (2.5) on any one of our generating differential invariants.

3. Invariants of Generalized Coupled Hirota-Satsuma KdV Equations

First, we consider the Lie point symmetries of System (1.1). The infinitesimal Lie transformations for equations (1.1) are of the form:

$$\begin{aligned} x &\mapsto x + \lambda\xi^x(x,t,u,v,w),\\ t &\mapsto t + \lambda\xi^t(x,t,u,v,w),\\ u &\mapsto u + \lambda\eta^u(x,t,u,v,w),\\ v &\mapsto v + \lambda\eta^v(x,t,u,v,w),\\ w &\mapsto w + \lambda\eta^w(x,t,u,v,w),\end{aligned}$$

with the symmetry generator

$$X = \xi^x\frac{\partial}{\partial x} + \xi^t\frac{\partial}{\partial t} + \eta^u\frac{\partial}{\partial u} + \eta^v\frac{\partial}{\partial v} + \eta^w\frac{\partial}{\partial w}.$$

In [1], using Lie's method, the infinitesimal Lie transformations for equations (1.1) are obtained by solving the following determining system:

$$\begin{aligned} &\xi^x_t = \xi^x_u = \xi^x_v = \xi^x_w = 0,\ \ \xi^t_x = \xi^t_u = \xi^t_v = \xi^t_w = 0,\ \ \xi^x_x = \frac{1}{3}\xi^t_t,\\ &\eta^v_x = \eta^v_t = \eta^v_u = \eta^v_w = 0,\ \ \eta^u = -\frac{2}{3}\xi^t_t u,\ \ \eta^w = -\frac{1}{3}\cdot\frac{(4v\xi^t_t + 3\eta^v)w}{v}.\end{aligned} \tag{3.1}$$

Solving (3.1) yields the following coefficients of the vector field X:

$$\xi^x = a_1 + \frac{1}{3}xa_4,\quad \xi^t = a_1 + t\,a_4,\quad \eta^u = -\frac{2}{3}u\,a_4,\quad \eta^v = a_3\,v\quad ,\eta^w = -a_3\,w - a_4\frac{4}{3}w.$$

where the a_1, a_2, a_3, a_4 are constants. Thus, the Lie algebra of the symmetries is generated by the following four vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x},\\ X_2 &= \frac{\partial}{\partial t},\\ X_3 &= v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w},\\ X_4 &= \frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}u\frac{\partial}{\partial u} - \frac{4}{3}w\frac{\partial}{\partial w}.\end{aligned} \tag{3.2}$$

The action of the symmetry group of equations (1.1) can be yielded by composing the flows of the vector fields (3.2) and is given by

$$(X,T,U,V,W) = \exp(\lambda_1 X_1)\circ\exp(\lambda_2 X_2)\circ\exp(\lambda_3 X_3)\circ\exp(\lambda_4 X_4), \tag{3.3}$$

where $\lambda_1,\ldots,\lambda_4$ are the group parameters. Calculating (3.3) leads to

$$X = (x+\lambda_1)\,\mathrm{e}^{1/3\,\lambda_4},\quad T = (t+\lambda_2)\,\mathrm{e}^{\lambda_4},\quad U = u\mathrm{e}^{-2/3\,\lambda_4},$$

$$V = v\mathrm{e}^{\lambda_3}, \quad W = w\mathrm{e}^{-\lambda_3}\mathrm{e}^{-\frac{4}{3}\lambda_4}.$$

As noted in Section 2.2, we should choose an appropriate coordinate cross-section. Since the dimension of group action is four, we can choose a coordinate cross-section by setting four arbitrary coordinates equal to constants. Here, we set the coordinate cross-section as $\mathcal{K} = \{x = 1,\, t = 0\,, v = 1,\, w = 1\}$. Our chosen coordinate cross-section defines the following four normalization equations

$$X = 1, \quad T = 0, \quad V = 1, \quad W = 1, \tag{3.4}$$

As a result of our choice of normalization (3.4) the phantom differential invariants are

$$H^1 = \iota(x) = 1, \quad H^2 = \iota(t) = 0, \quad J_{00} = \iota(v) = 1, \quad K_{00} = \iota(w) = 1.$$

Using Maple, we found that the normalization equations (3.4) can be solved by the following group parameters:

$$\lambda_1 = -\frac{x\,(vw)^{1/4} - 1}{(vw)^{1/4}}, \quad \lambda_2 = -t, \quad \lambda_3 = -\ln v, \quad \lambda_4 = \frac{3}{4}\ln(vw). \tag{3.5}$$

The dual implicit differential operators are defined as follow [17]

$$D_{X^i} = \sum_{j=1}^{p} W_j^i D_{x^j}, \quad W_i^j = \left(D_{x^j} X^i\right)^{-1}, \tag{3.6}$$

where D_{x^i} are the total derivative and are (2.1).

From (3.6), we have

$$D_X = \mathrm{e}^{-\frac{1}{3}\lambda_4} D_x, \quad D_T = \mathrm{e}^{-\lambda_4}\, D_t. \tag{3.7}$$

Substituting (3.5) into (3.7), the corresponding invariant differential operators are

$$\mathcal{D}_x = (vw)^{-1/4}\, D_x, \quad \mathcal{D}_t = (vw)^{-3/4}\, D_t.$$

A complete system of functionally independent normalized differential invariants is yielded by invariantizing the remaining non-phantom differential invariants:

$$I_{00} = \frac{u}{(vw)^{1/2}}, \quad I_{10} = \frac{(vw)^{1/4}\; u_x}{vw}, \quad I_{01} = \frac{(vw)^{1/4}\; u_t}{vw},$$

$$J_{10} = \frac{(vw)^{3/4}\; v_x}{vw}, \quad J_{01} = \frac{(vw)^{1/4}\; v_t}{vw},$$

$$K_{10} = \frac{(vw)^{3/4}\; w_x}{vw}, \quad K_{01} = \frac{(vw)^{1/4}\; w_t}{vw},$$

$$I_{20} = -\frac{1}{4} \cdot \frac{\sqrt{vw}\,(u_x v_x w + u_x w_x v - 4u_{xx} vw)}{v^2 w^2},$$

$$\begin{aligned} I_{30} = &-\frac{1}{8} \cdot \frac{(vw)^{1/4}}{v^3 w^3} \Big(2v_{xx} u_x v w^2 + 2w_{xx} u_x v^2 w - 8u_{xxx} v^2 w^2 - 3u_x v_x^2 w^2 - 2u_x v_x w_x vw\Big) \\ &+ \frac{1}{8} \cdot \frac{(vw)^{1/4}}{v^3 w^3} \left(-3u_x w_x^2 v^2 + 6u_{xx} v_x v w^2 + 6u_{xx} w_x v^2 w\right), \end{aligned}$$

$$
\begin{aligned}
J_{30} =& -\frac{1}{8}\cdot\frac{(vw)^{1/4}}{v^3w^3}\left(8v_{xx}v_xvw^2+6v_{xx}w_xv^2w+2w_{xx}v_xv^2w-3v_x^3w^2\right)\\
&+\frac{1}{8}\cdot\frac{(vw)^{1/4}}{v^3w^3}\left(2v_x^2w_xvw+3v_xw_x^2v^2+8v_{xxx}v^2w^2\right),\\
K_{30} =&\frac{1}{8}\cdot\frac{(vw)^{1/4}}{v^3w^3}\left(8v^2w^2w_{xxx}-8v^2ww_xw_{xx}+3v^2w_x^3-6vw^2v_xw_{xx}\right)\\
&-\frac{1}{8}\cdot\frac{(vw)^{1/4}}{v^3w^3}\left(2vw^2w_xv_{xx}-2vwv_xw_x^2-3w^2v_x^2w_x\right)\\
&\vdots
\end{aligned}
$$

where

$$I_{ij}=\iota(u_{i,j}),\quad J_{ij}=\iota(v_{i,j}),\quad K_{ij}=\iota(w_{i,j}).$$

By applying the invariantization process, System (1.1) can be rewritten in terms of the differential invariants as

$$
\begin{cases}
I_{01}-\frac{1}{2}I_{30}+3I_{00}I_{10}-3(J_{00}K_{10}+J_{10}K_{00})=0,\\
J_{01}+J_{30}-3I_{00}J_{10}=0,\\
K_{01}+K_{30}-3I_{00}K_{10}=0.
\end{cases}
$$

Next, we locate the a finite generating set of differential invariants for Equation (1.1). One can obtain higher order differential invariants by repeatedly applying the invariant differential operators to the lower order differential invariants.

According to (2.3), the recurrence formula for the differential invariants are

$$
\begin{aligned}
\mathcal{D}_1H^j &= \delta_1^j+\sum_{\kappa=1}^{r}\iota(\xi_\kappa^j)R_1^\kappa, & \mathcal{D}_2H^j &= \delta_2^j+\sum_{\kappa=1}^{r}\iota(\xi_\kappa^j)R_2^\kappa,\\
\mathcal{D}_1I_{jk} &= I_{j+1,k}+\sum_{\kappa=1}^{r}\iota(\eta_\kappa^{u,jk})R_1^\kappa, & \mathcal{D}_2I_{jk} &= I_{j,k+1}+\sum_{\kappa=1}^{r}\iota(\eta_\kappa^{u,jk})R_2^\kappa,\\
\mathcal{D}_1J_{jk} &= J_{j+1,k}+\sum_{\kappa=1}^{r}\iota(\eta_\kappa^{v,jk})R_1^\kappa, & \mathcal{D}_2J_{jk} &= J_{j,k+1}+\sum_{\kappa=1}^{r}\iota(\eta_\kappa^{v,jk})R_2^\kappa,\\
\mathcal{D}_1K_{jk} &= K_{j+1,k}+\sum_{\kappa=1}^{r}\iota(\eta_\kappa^{w,jk})R_1^\kappa, & \mathcal{D}_2K_{jk} &= K_{j,k+1}+\sum_{\kappa=1}^{r}\iota(\eta_\kappa^{w,jk})R_2^\kappa,
\end{aligned}
\tag{3.8}
$$

where R_1^κ and R_2^κ are the Maurer-Cartan invariants and ξ^j, $\eta_\kappa^{u,jk},\eta_\kappa^{v,jk}$ and $\eta_\kappa^{w,jk}$ are the coefficients of $\frac{\partial}{\partial x_j}$, $\frac{\partial}{\partial u_{jk}}$, $\frac{\partial}{\partial v_{jk}}$ and $\frac{\partial}{\partial w_{jk}}$ in the prolongation of the infinitesimal generator X_κ respectively [26]. Solving the resulting phantom recurrence formula produces the Maurrer-Cartan invariants

$$
\begin{aligned}
&R_1^2=0,\quad R_2^2=-1,\quad R_1^1=-1-\frac{1}{3}R_1^4,\quad R_1^3=-J_{10},\\
&R_2^3=-J_{01},\quad R_1^4=\frac{3}{4}K_{10}+\frac{3}{4}J_{10},\quad R_2^4=\frac{3}{4}K_{01}+\frac{3}{4}J_{01}.
\end{aligned}
\tag{3.9}
$$

Substituting the Maurrer-Cartan invariants (3.9) back into (3.8) obtain all the non-phantom recurrence formula.

$$
\begin{aligned}
&\mathcal{D}_1 I_{00} = I_{10} - \frac{2}{3} I_{00}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 I_{00} = I_{01} - \frac{2}{3} I_{00}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right),\\
&\mathcal{D}_1 I_{10} = -I_{10}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right) + I_{20}, && \mathcal{D}_2 I_{10} = -I_{10}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right) + I_{11},\\
&\mathcal{D}_1 I_{01} = I_{11} - \frac{5}{3} I_{01}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 I_{01} = I_{02} - \frac{5}{3} I_{01}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right),\\
&\mathcal{D}_1 I_{20} = I_{30} - \frac{4}{3} I_{20}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 I_{20} = I_{21} - \frac{4}{3} I_{20}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right),\\
&\mathcal{D}_1 I_{02} = I_{12} - \frac{8}{3} I_{02}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 I_{02} = I_{03} - \frac{8}{3} I_{02}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right),\\
&\mathcal{D}_1 I_{11} = -2I_{02}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right) + I_{21}, && \mathcal{D}_2 I_{11} = -2I_{02}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right) + I_{12},\\
&\mathcal{D}_1 J_{10} = J_{20} - J_{10}^2 - \frac{1}{3} J_{10}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 J_{10} = J_{11} - J_{10}J_{01} - \frac{1}{3} J_{10}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right),\\
&\mathcal{D}_1 J_{01} = -J_{10}J_{01} - J_{01}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right) + J_{11}, && \mathcal{D}_2 J_{01} = -J_{01}^2 - J_{01}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right) + J_{02},\\
&\mathcal{D}_1 J_{20} = J_{30} - J_{10} - \frac{2}{3} J_{20}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 J_{20} = J_{21} - J_{01} - \frac{2}{3} J_{20}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right),\\
&\mathcal{D}_1 J_{02} = -2J_{02}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right) + J_{12} - J_{10}, && \mathcal{D}_2 J_{02} = -2J_{02}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right) + J_{03} - J_{01},\\
&\mathcal{D}_1 J_{11} = J_{21} - J_{10} - \frac{4}{3} J_{20}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 J_{11} = J_{21} - J_{01} - \frac{4}{3} J_{11}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right),\\
&\mathcal{D}_1 K_{10} = K_{20} + K_{10}J_{10} - \frac{4}{3} K_{10}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 K_{10} = K_{11} + K_{10}J_{01} - \frac{4}{3} K_{10}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right),\\
&\mathcal{D}_1 K_{01} = K_{11} + K_{01}J_{10} - \frac{7}{3} K_{01}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 K_{01} = K_{02} + K_{01}J_{01} - \frac{7}{3} K_{01}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right),\\
&\mathcal{D}_1 K_{20} = K_{20}J_{10} - 2K_{20}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right) + K_{30}, && \mathcal{D}_2 K_{20} = K_{20}J_{01} - 2K_{20}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right) + K_{21},\\
&\mathcal{D}_1 K_{11} = K_{21} + K_{11}J_{10} - \frac{8}{3} K_{11}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 K_{11} = K_{21} + K_{11}J_{01} - \frac{8}{3} K_{11}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right),\\
&\mathcal{D}_1 K_{02} = K_{12} + K_{20}J_{10} - \frac{10}{3} K_{11}\left(\frac{3}{4}K_{10} + \frac{3}{4}J_{10}\right), && \mathcal{D}_2 K_{02} = K_{03} + K_{02}J_{01} - \frac{10}{3} K_{02}\left(\frac{3}{4}K_{01} + \frac{3}{4}J_{01}\right).
\end{aligned}
\tag{3.10}
$$

Theorem 3.1. *The entire differential invariant algebra of the CHSK equations* (1.1) *is generated by the following set:*

$$\{I_{00}, J_{10}, J_{01}, K_{10}, K_{01}\}. \tag{3.11}$$

Proof. From the recurrence formula (3.10), we find that any differential invariants up to third order can be generated by a function composition of $I_{00}, J_{10}, J_{01}, K_{10}, K_{01}$ and their derivatives. By differentiating the differential invariants (3.11), one can find that any higher order differential invariants are also generated by the generating set (3.11). □

Finally, we obtain the commutator invariants which satisfy the commutator relation

$$[\mathcal{D}_1, \mathcal{D}_2] = Y_1\mathcal{D}_1 + Y_2\mathcal{D}_2, \tag{3.12}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the invariant differential operators. As a result of general recurrence formulae, [11, 18], we have

$$Y_1 = \sum_{\kappa=1}^{r} \left[R_2^{\kappa} \iota(D_x \xi_{\kappa}^1) - R_1^{\kappa} \iota(D_t \xi_{\kappa}^1) \right], \quad Y_2 = \sum_{\kappa=1}^{r} \left[R_2^{\kappa} \iota(D_x \xi_{\kappa}^2) - R_1^{\kappa} \iota(D_t \xi_{\kappa}^2) \right],$$

in which ξ_{κ}^i is the coefficients of ∂_{x^i}, in the infinitesimal generator X_{κ}.

Substituting our formula (3.9) for the Maurer-Cartan invariants yields

$$Y_1 = \frac{1}{4}\left(K_{01} + J_{01}\right), \quad Y_2 = -\frac{3}{4}\left(K_{10} - J_{10}\right). \tag{3.13}$$

Thus, from (3.12) and (3.13), we have

$$[\mathcal{D}_1, \mathcal{D}_2] = \frac{1}{4}\left(K_{01} + J_{01}\right)\mathcal{D}_1 - \frac{3}{4}\left(K_{10} - J_{10}\right)\mathcal{D}_2. \tag{3.14}$$

Indeed, the generating differential invariants $\{I_{00}, J_{10}, J_{01}, K_{10}, K_{01}\}$, the recurrence formulas (3.10), along with the commutation relations (3.14), provide a complete specification of the structure of the differential invariant algebra of CHSK equations (1.1).

4. Conclusions

In this paper, using the moving frames method, we located a finite generating set of differential invariants and the invariant differential operators for the Lie symmetry group of a generalized coupled Hirota-Satsuma KdV equations (CHSK), and then we obtained the recurrence relations as well as syzygies among the generating differential invariants. In particular, we proved that the differential invariant algebra of CHSK equations can be generated by five differential invariants. The main application of the differential invariants is to construct a class of PDEs, which possess the same symmetry properties, which is important for both mathematics and physical interpretation. Since the CHSK system is the mathematical model of interactions of two long waves with different dispersion relations, our results are applicable to study the invariant properties of interactions of two long waves. In our approach, we also obtained the Maurer-Cartan invariants.

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