Kragujevac Journal of Mathematics
Volume 50(4) (2026), Pages 579–596.

FIXED POINT THEOREMS AND CONTINUITY CHARACTERIZATION FOR LINEAR MAPS IN COLOMBEAU ALGEBRAS

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Abstract. In this article, we present a novel characterization of the continuity of linear maps within Colombeau algebras. Additionally, we introduce an alternative representation for the contraction of these maps. Moreover, we put forth a new concept of fixed-point theorems in Colombeau algebra, extending classical fixed-point theorems, including those of Banach, Chatterjea, and Kannan. To underscore the practical relevance of our findings, we offer various examples and applications.

1. Introduction

The fixed point theorems are regarded as an effective tool for solving differential equations, for example, see [21–26]. In the literature, the embedding of fixed point theory in the framework of Colombeau algebra is based on the famous theorem of J. A. Marti see [16]. The idea of J. A. Marti is based on the Banach fixed point theorem in classical metric spaces, with several assumptions to make sense of the contraction of mappings. Our new idea is based on this result of J. A. Marti, but we will lighten the assumptions. Indeed: We have defined a new contraction in which we don't need all these assumptions. Our contraction is intended for mappings defined between the Colombeau algebras $(\mathcal{G}_E, (\mathcal{P}_i)_{i\in I}$, where $(\mathcal{P}_i)_{i\in I}$, $\mathcal{P}_i(u) = e^{-v_{p_i}(u_\epsilon)}$ and v_{p_i} is the valuation function associated with the ultra pseudo-seminorms family which makes E a locally convex space. We have also extended the two theorems of Chattergia and Kannan in the framework of Colombeau algebras. On the other hand,

Key words and phrases. Fixed point, generalized function, Colombeau algebra, conctraction mapping, ultra pseudo-seminorm.
2020 *Mathematics Subject Classification.* 58C30, 47H10, 37C25.
DOI
Received: June 19, 2023.
Accepted: January 21, 2024.

our paper contains a very legitimate notion of contraction, which will allow us to cite some fundamental theorems of fixed point in the algebra of generalized functions of Colombeau. We have also introduced a characterization of the continuity notion of $\widetilde{\mathbb{C}}$-linear maps between Colombeau algebras, based on the sharp topology defined on Colombeau algebras seen as modules on $\widetilde{\mathbb{C}}$. In this article, we have proved three well-known fixed point theorems in the classical framework, namely Banach's fixed point theorem, Kannan's fixed point theorem, and that of Chatterjea (see [2, 14]). Fixed point theorems are very useful to know if an equation has a solution, finding a solution to a differential equation can then be interpreted as finding a map and proving that it has a fixed point, which will be the solution to the problem.

The present work is formulated in the framework of Colombeau generalized functions. This will allow us to use the powerful tools from this theory to combine generalized function data with nonlinearities and measure regularity. We give some properties in the theory of topological $\widetilde{\mathbb{C}}$-modules and locally convex topological $\widetilde{\mathbb{C}}$-modules, and illustrated this with application to an evolution problem. In [8] the authors, shows that in one space dimension, an initial singularity at the origin propagates along the characteristic lines emanating from the origin, as in the linear case. The proof is based on a fixed point theorem in a suitable ultrametric topology on the subset of Colombeau solutions possessing the required regularity. J. A. Marti in [16] and several other authors have found together a lot of results on the existence and uniqueness of generalized fixed point by a method which consists of transforming the problem given in the sense of Colombeau to its equivalence in classic, after they showed that its solutions are moderate and therefore deduce that the generalized solution exists. But our point of view is to use a definition of contraction in the sense of generalized functions to mount existence and uniqueness of fixed point in this frame of Colombeau generalized functions. Starting from the notions of $\widetilde{\mathbb{C}}$-linear and locally convex $\widetilde{\mathbb{C}}$-linear topologies which are introduced and described from their neighborhoods, a characterization of the continuity of $\widetilde{\mathbb{C}}$-linear maps from a locally convex topological $\widetilde{\mathbb{C}}$-modules into another $\widetilde{\mathbb{C}}$-modules is given in this work. We are inspired by the analogous statement involving seminormes and locally convex vector spaces, making use of the concept of ultra pseudo-seminorm, for example, the continuity of $\widetilde{\mathbb{C}}$-modules is studied in this paper. Our main result is to give a necessary and sufficient condition in order to a linear map from a Colombeau algebra to another be continuous, and we focus on proving a new theorems of fixed point in this context.

The present paper is organized as follows. After this introduction, we will recall some basic properties concerning Colombeaus algebra in Section 2. The notion of $\widetilde{\mathbb{C}}$-modules topology and some properties are presents in Section 3. In Section 4, we are talking about continuity, contraction and fixed point in $\widetilde{\mathbb{C}}$-modules. Finally, Section 5 is devoted to the existence-uniqueness result of a differential equation.

2. Preliminaries

Before describing our results in more detail, a few words about Colombeau algebras are in order. The elements of Colombeau algebra $\mathcal{G}$ are equivalence classes of nets of smooth functions satisfying asymptotic conditions in the regularization parameter ϵ, for more details [3–6, 12, 13, 17]. We define the set $\mathcal{E}(\mathbb{R}^n) = (\mathcal{C}^\infty(\mathbb{R}^n))^{(0,1]}$. The set of moderate functions is given as follows

$$\mathcal{E}_M(\mathbb{R}^n) = \Big\{(\sigma_\epsilon)_\epsilon \in \mathcal{E}(\mathbb{R}^n) \mid \text{ for all } K \subset\subset \mathbb{R}^n,\ \alpha \in \mathbb{N}, \text{there exists } N \in \mathbb{N} : \sup_{x\in K} |D^\alpha \sigma_\epsilon(x)| = O_{\epsilon\to 0}(\epsilon^{-N})\Big\},$$

where D^α is the differential operator of order α. The ideal of negligible functions is defined by

$$\mathcal{N}(\mathbb{R}^n) = \Big\{(\sigma_\epsilon)_\epsilon \in \mathcal{E}(\mathbb{R}^n) \mid \text{for all } K \subset\subset \mathbb{R}^n,\ \alpha \in \mathbb{N}, \text{for all } q \in \mathbb{N} : \sup_{x\in K} |D^\alpha \sigma_\epsilon(x)| = O_{\epsilon\to 0}(\epsilon^{q})\Big\}.$$

The Colombeau algebra is defined as a factor algebra

$$\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n). \tag{2.1}$$

We denote $\mathbb{C}$ the field of complex numbers. Let $\widetilde{\mathbb{C}}$ be the ring of complex generalized numbers obtained by factorizing,

$$\mathcal{E}_{\mathbb{C}} = \Big\{(r_\epsilon)_\epsilon \in \mathbb{C}^{(0,1]} \mid \text{there exists } N \in \mathbb{N},\ |r_\epsilon| = O_{\epsilon\to 0}(\epsilon^{-N})\Big\}, \tag{2.2}$$

with respect to the ideal

$$\mathcal{N}_{\mathbb{C}} = \Big\{(r_\epsilon)_\epsilon \in \mathbb{C}^{(0,1]} \mid \text{for all } q \in \mathbb{N},\ |r_\epsilon| = O_{\epsilon\to 0}(\epsilon^{q})\Big\}. \tag{2.3}$$

Also, the algebra of generalized complex numbers is defined as follows

$$\widetilde{\mathbb{C}} = \mathcal{E}_{\mathbb{C}}/\mathcal{N}_{\mathbb{C}}. \tag{2.4}$$

3. Definition and Basic Properties of $\widetilde{\mathbb{C}}$-Module $\mathcal{G}$

It is clear that $\widetilde{\mathbb{C}}$ is trivially a module over itself and it can be endowed with a structure of a topological ring (see [10, 11]). In the sequel, we need the following function, which is inspired by non-standard analysis [17, 20] and the previous work in this field [1, 18, 19] we define the following function

$$v : \begin{cases} \mathcal{E}_M(\mathbb{R}^n) \to (-\infty, +\infty], \\ (\sigma_\epsilon)_\epsilon \to \sup\Big\{ l \in \mathbb{R} \mid |\sigma_\epsilon| = O_{\epsilon\to 0}(\epsilon^{l})\Big\}. \end{cases} \tag{3.1}$$

It satisfies the following conditions:

(i) $v((\sigma_\epsilon)_\epsilon) = +\infty$ if and only if $(\sigma_\epsilon)_\epsilon \in \mathcal{N}(\mathbb{R}^n)$;

(ii) $v((\sigma_\epsilon)_\epsilon(\varrho_\epsilon)_\epsilon) \geqslant v((\sigma_\epsilon)_\epsilon) + v((\varrho_\epsilon)_\epsilon)$, for all $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{E}_M(\mathbb{R}^n)$;

(ii') $v\left((\sigma_\epsilon)_\epsilon(\varrho_\epsilon)_\epsilon\right) = v\left((\sigma_\epsilon)_\epsilon\right) + v\left((\varrho_\epsilon)_\epsilon\right)$, for all $\varrho_\epsilon = c\epsilon^b$, $c \in \mathbb{C}$, $b \in \mathbb{R}$;
(iii) $v\left((\sigma_\epsilon)_\epsilon + (\varrho_\epsilon)_\epsilon\right) \geqslant \min\left\{v\left((\sigma_\epsilon)_\epsilon\right), v\left((\varrho_\epsilon)_\epsilon\right)\right\}$, for all $(\sigma_\epsilon)_\epsilon$, $(\varrho_\epsilon)_\epsilon \in \mathcal{E}_M(\mathbb{R}^n)$.

Note that if $\left(\sigma_\epsilon - \sigma'_\epsilon\right)_\epsilon \in \mathcal{N}(\mathbb{R}^n)$, (i) combined with (iii) yields $v\left((\sigma_\epsilon)_\epsilon\right) = v\left((\sigma'_\epsilon)_\epsilon\right)$. This means that we can use the function v to define the valuation

$$v_{\widetilde{\mathbb{C}}}(\sigma) = v\left((\sigma_\epsilon)_\epsilon\right), \tag{3.2}$$

of a complex generalized number $\sigma = [(\sigma_\epsilon)_\epsilon]$, and that all the previous properties hold for elements of $\widetilde{\mathbb{C}}$. Now let us consider the following map

$$|\cdot|_e : \begin{cases} \widetilde{\mathbb{C}} \to [0, +\infty), \\ \sigma \to |\sigma|_e = e^{-v_{\widetilde{\mathbb{C}}}(\sigma)}. \end{cases} \tag{3.3}$$

Definition 3.1. Let E be a locally convex topological vector space equipped through the family of seminorms $\{p_i\}_{i\in I}$. The elements of

$$\mathcal{M}_E = \left\{(\sigma_\epsilon)_\epsilon \in E^{(0,1]} \mid \text{for all } i \in I,\ \text{there exists } N \in \mathbb{N},\ p_i(\sigma_\epsilon) = O_{\epsilon\to 0}(\epsilon^{-N})\right\} \tag{3.4}$$

and

$$\mathcal{N}_E = \left\{(\sigma_\epsilon)_\epsilon \in E^{(0,1]} \mid \text{for all } i \in I,\ q \in \mathbb{N},\ p_i(\sigma_\epsilon) = O_{\epsilon\to 0}(\epsilon^q)\right\}, \tag{3.5}$$

are called respectively $\mathcal{M}_E$-moderate space and $\mathcal{N}_E$-negligible space, respectively.

We introduce the space of generalized functions based on E as the factor space $\mathcal{G}_E = \mathcal{E}_E/\mathcal{N}_E$

It is clear that the definition of $\mathcal{G}_E$ does not depend on the family of seminorms which determines the locally convex topology of E. We adopte the notation $\sigma = \left[(\sigma_\epsilon)_\epsilon\right]$ for the class σ of $(\sigma_\epsilon)_\epsilon$ in $\mathcal{G}_E$, and $\mathcal{C}^\infty(\mathbb{R})$ embedded into this algebra via the constant embedding $f \mapsto \left[(f)_\epsilon\right]$. By the properties of seminorms on E we may define the product between complex generalized numbers and elements of $\mathcal{G}_E$ via the map $\widetilde{\mathbb{C}} \times \mathcal{G}_E \to \mathcal{G}_E$. It is natural to introduce the p_i-valuation of $(\sigma_\epsilon)_\epsilon \in \mathcal{M}_E$ as

$$v_{p_i}\left((\sigma_\epsilon)_\epsilon\right) = \sup\left\{b \in \mathbb{R} \mid \text{for all } i \in I,\ p_i(\sigma_\epsilon) = O_{\epsilon\to 0}(\epsilon^b)\right\}. \tag{3.6}$$

Note that $v_{p_i}\left((\sigma_\epsilon)_\epsilon\right) = v\left((p_i(\sigma_\epsilon))_\epsilon\right)$, (where the function v) gives the valuation on $\mathcal{M}_E$. Clearly v_{p_i} maps $\mathcal{M}_E$ into $(-\infty, +\infty)$ and the following properties hold:

(i) $v_{p_i}\left((\sigma_\epsilon)_\epsilon\right) = +\infty$ for all $i \in I$ if and only if $((\sigma_\epsilon)_\epsilon) \in \mathcal{N}_E$;
(ii) $v_{p_i}\left((\varrho_\epsilon\sigma_\epsilon)_\epsilon\right) \geqslant v_{p_i}\left((\varrho_\epsilon)_\epsilon\right) + v_{p_i}\left((\sigma_\epsilon)_\epsilon\right)$, for all $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{M}_E$;
(iii) $v_{p_i}\left((\lambda_\epsilon\sigma_\epsilon)_\epsilon\right) = v_{p_i}\left((\lambda_\epsilon)_\epsilon\right) + v_{p_i}\left((\sigma_\epsilon)_\epsilon\right)$ for all $\lambda_\epsilon = c\epsilon^b, c \in \mathbb{C}, b \in \mathbb{R}$;
(iv) $v_{p_i}\left((\sigma_\epsilon)_\epsilon + (\varrho_\epsilon)_\epsilon\right) \geqslant \min\left\{v_{p_i}\left((\sigma_\epsilon)_\epsilon\right), v_{p_i}\left((\varrho_\epsilon)_\epsilon\right)\right\}$, for all $(\sigma_\epsilon)_\epsilon$, $(\varrho_\epsilon)_\epsilon \in \mathcal{M}_E$.

Last assertion (i) combined with (iv) shows that

$$v_{p_i}\left((\sigma_\epsilon)_\epsilon\right) = v_{p_i}\left((\sigma'_\epsilon)_\epsilon\right), \quad \text{if } \left(\sigma_\epsilon - \sigma'_\epsilon\right)_\epsilon \text{ is } \mathcal{N}_E\text{-negligible}.$$

This means that we can use (3.6) for defining the p_i-valuation of a generalized function $\sigma = \left[(\sigma_\epsilon)_\epsilon\right] \in \mathcal{G}_E$ by

$$v_{p_i}(\sigma) = v_{p_i}\left((\sigma_\epsilon)_\epsilon\right).$$

And thus $\mathcal{P}_i(\sigma) = e^{-v_{p_i}(\sigma)}$ is an ultra pseudo-seminorm on the $\widetilde{\mathbb{C}}$-module $\mathcal{G}_E$. By [10, Theorem 1.10] $\mathcal{G}_E$ endowed with the topology of ultra pseudo-seminorms $(\mathcal{P}_i)_{i\in I}$ is a locally convex topological $\widetilde{\mathbb{C}}$-module. The topology induced by the ultra pseudo-seminorms $(\mathcal{P}_i)_{i\in I}$ called the sharp topology on $\mathcal{G}_E$. A basis of 0-neighbourhood is the set of all balls

$$\mathcal{B}(i,\gamma) = \Big\{\sigma \in \mathcal{G}_E \mid \mathcal{P}_i(\sigma) < \gamma\Big\}, \quad i \in I \text{ and } \gamma > 0.$$

4. Main Results

This section is devoted to the important results of this paper. Let's start with the subsection of continuity.

4.1. Continuity and contraction in $\widetilde{\mathbb{C}}$-modules.

First, we are looking if it is possible to define a $\widetilde{\mathbb{C}}$-linear map $\mathcal{S} : \mathcal{G}_E \to \mathcal{G}_F$ by means of a given family $(\mathcal{S}_\epsilon)_{\epsilon\in(0,1]}$ of $\mathbb{C}$-linear maps $\mathcal{S}_\epsilon : E \to F$, where E and F are locally convex topological vector spaces. The general requirement is given in the following.

Lemma 4.1 ([16]). *Let $(\mathcal{S}_\epsilon)_{\epsilon\in(0,1]}$ be a given family of $\mathbb{C}$-linear maps $\mathcal{S}_\epsilon : E \to F$. Suppose that*

1. $(\sigma_\epsilon)_\epsilon \in \mathcal{M}_E$ *implies* $(\mathcal{S}_\epsilon\sigma_\epsilon)_\epsilon \in \mathcal{M}_F$*;*
2. $(\sigma_\epsilon)_\epsilon \in \mathcal{N}_E$ *implies* $(\mathcal{S}_\epsilon\sigma_\epsilon)_\epsilon \in \mathcal{N}_F$*.*

Then, $\widetilde{\mathbb{C}}$-linear map $\mathcal{S} : (\mathcal{G}_E, (\mathcal{P}_i)_{i\in I}) \to (\mathcal{G}_F, (\mathcal{Q}_j)_{j\in I})$ is well defined by

$$\mathcal{S}\sigma = [(\mathcal{S}_\epsilon\sigma_\epsilon)_\epsilon], \quad \textit{for all } \sigma \in \mathcal{G}_E.$$

Now we turn up to the continuity of mappings in Colombeau algebras. A function $f : \mathcal{G}_E \to \mathcal{G}_F$ between Colombeau algebras is said to be continuous in σ_0 in $\mathcal{G}_E$, if for all $\gamma > 0$ there exists $\delta_\gamma > 0$ such that $\sigma - \sigma_0 \in \mathcal{B}(i, \delta_\gamma)$ implies that $f(\sigma) - f(\sigma_0) \in \mathcal{B}(j,\gamma)$.

Definition 4.1. Let $\mathcal{S} : \mathcal{G}_E \to \mathcal{G}_F$ be a map with $(\mathcal{G}_E, (\mathcal{P}_i)_{i\in I})$ and $(\mathcal{G}_F, (\mathcal{Q}_j)_{j\in J})$ are two locally convex topological $\widetilde{\mathbb{C}}$-modules. We say that $\mathcal{S}$ is continuous if and only if, for every $j \in J$, there exist $i_0 \in I$, $c > 0$ such that

$$\mathcal{Q}_j(\mathcal{S}\sigma - \mathcal{S}\varrho) \leqslant c\mathcal{P}_{i_0}(\sigma - \varrho), \quad \text{for all } \sigma, \varrho \in \mathcal{G}_E.$$

Example 4.1. Let $E = \mathcal{C}^\infty(\mathbb{R})$, $\mathcal{P}_i(\cdot) = e^{-v_{p_i}(\cdot)}$ and $p_i(\cdot) = \|\cdot\|_\infty$ for all $i \in I$. The mapping

$$\mathcal{S} : \begin{cases} \mathcal{G}_E \to \mathcal{G}_E, \\ \sigma \to \mathcal{S}\sigma = [(\mathcal{S}_\epsilon\sigma_\epsilon)_\epsilon] = [(\frac{1}{\epsilon^2}\sin(\sigma_\epsilon))_\epsilon], \end{cases}$$

is continuous in $\mathcal{G}_E$. Indeed, for all $\sigma, \varrho \in \mathcal{G}_E$, we have

$$|\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon| \leq \frac{c}{\epsilon^2}|\sigma_\epsilon - \varrho_\epsilon|, \quad c > 0.$$

And thus, $v_{p_i}(\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon) \geq -2 + v_{p_{i_0}}(\sigma_\epsilon - \varrho_\epsilon)$. It follows that,

$$\mathcal{P}_i(\mathcal{S}\sigma - \mathcal{S}\varrho) \leq c\mathcal{P}_{i_0}(\sigma - \varrho), \quad c = e^2 > 0.$$

In particular for the continuity of a linear map on Colombeau's algebra we have the following.

Definition 4.2. Let $\mathcal{S} : \mathcal{G}_E \to \mathcal{G}_F$ be a $\widetilde{\mathbb{C}}$-linear map with $(\mathcal{G}_E, (\mathcal{P}_i)_{i\in I})$ and $(\mathcal{G}_F, (\mathcal{Q}_j)_{j\in J})$ are two locally convex topological $\widetilde{\mathbb{C}}$-modules. We say that $\mathcal{S}$ is continuous if and only if, for every $j \in J$, *there exist* $i_0 \in I$ and $c > 0$ such that

$$\mathcal{Q}_j(\mathcal{S}\sigma) \leqslant c\mathcal{P}_{i_0}(\sigma), \quad \text{for all } \sigma. \tag{4.1}$$

Theorem 4.1. *Under the some notations above, let $(\mathcal{S}_\epsilon)_\epsilon$ be a $\mathbb{C}$-linear maps family given by the constant family $(s)_\epsilon$ (i.e, $\mathcal{S}\sigma = [(s\sigma_\epsilon)_\epsilon]$). If $s : E \to F$ is continuous, then $\mathcal{S} : \mathcal{G}_E \to \mathcal{G}_F$ is a well defined $\widetilde{\mathbb{C}}$-linear and continuous.*

Proof. First, show that $\mathcal{S}$ is well defined. Let $(\sigma_\epsilon)_\epsilon \in \mathcal{M}_E$. Then, for any $i \in I$, there exists $N \in \mathbb{N}$ such that $p_i(\sigma_\epsilon) = O_{\epsilon\to 0}(\epsilon^{-N})$, and $\sigma_\epsilon \in E$, for all $\epsilon \in (0, 1]$. So, $s\sigma_\epsilon \in F$. By continuity of s, for each $j \in J$, there exist $i_0 \in I$, $c > 0$, such that

$$q_j(s\sigma_\epsilon) \leqslant cp_i(\sigma_\epsilon) \leqslant c \times c^{'}\epsilon^{-N} = c_2\epsilon^{-N}.$$

Then, for $j \in J$, there exists $N \in \mathbb{N}$, with $q_j(s\sigma_\epsilon) = O_{\epsilon\to 0}(\epsilon^{-N})$. Hence, $(s\sigma_\epsilon)_\epsilon \in \mathcal{M}_F$, which implies that $\mathcal{S}\sigma \in \mathcal{G}_F$.

Let $(\varrho_\epsilon)_\epsilon \in \mathcal{N}_E$, i.e., for each $i \in I$, $p_i(\varrho_\epsilon) = O_{\epsilon\to 0}(\epsilon^q)$, for all $q \in \mathbb{N}$ and $\varrho_\epsilon \in F$, for all $\epsilon \in (0, 1]$. Since s is linear and continuous, then for any $j \in J$ there exist $i_0 \in I$ and $c > 0$ such that

$$q_j\left(s\varrho_\epsilon\right) \leqslant cp_{i_0}(\varrho_\epsilon) \leqslant c^{'}\epsilon^q, \quad \text{where } c^{'} > 0.$$

Hence, $q_j(\mathcal{S}_\epsilon\varrho_\epsilon) = O_{\epsilon\to 0}(\epsilon^q)$, and thus $(s\sigma_\epsilon)_\epsilon \in \mathcal{N}_F$. From Lemma 4.1, $\mathcal{S}$ is well defined.

The continuity. Let $h = \sup\left\{b \in \mathbb{R} \mid q_j\left(s\sigma_\epsilon\right) = O_{\epsilon\to 0}(\epsilon^b)\right\}$. For every $j \in J$, we have

$$v_{q_j}\left((s\sigma_\epsilon)_\epsilon\right) = v\Big(\left(q_j(s\sigma_\epsilon)\right)_\epsilon\Big) = \sup\left\{b \in \mathbb{R} \mid q_j\left(s\sigma_\epsilon\right) = O_{\epsilon\to 0}(\epsilon^b)\right\} = h.$$

And s is a linear continuous mapping, then for all $j \in J$, there exist $i_0 \in I, c > 0$ *such that* $q_j(s\sigma_\epsilon) \leqslant cp_{i_0}(\sigma_\epsilon)$. Now we have to consider

$$d = v_{p_i}((\sigma_\epsilon)_\epsilon) = v\left((p_i(\sigma_\epsilon))_\epsilon\right) = \sup\left\{b \in \mathbb{R} \mid p_i(\sigma_\epsilon) = O_{\epsilon\to 0}(\epsilon^b)\right\}.$$

Then, we have

$$q_j\left(s\sigma_\epsilon\right) \leqslant c \times c^{''}\epsilon^d = c_0\epsilon^d, \quad \text{where } c_0, c'' > 0.$$

So, $d \leqslant h$, which implies $v_{p_{i_0}}((\sigma_\epsilon)_\epsilon) \leqslant v_{q_j}((s\sigma_\epsilon)_\epsilon)$. Hence, $e^{-v_{q_j}((s\sigma_\epsilon)_\epsilon)} \leqslant e^{-v_{p_{i_0}}((\sigma_\epsilon)_\epsilon)}$, and we get $\mathcal{Q}_j(\mathcal{S}\sigma) \leqslant \mathcal{P}_{i_0}(\sigma)$. Conlusion $\mathcal{S}$ is $\widetilde{\mathbb{C}}$-linear continuous. □

Theorem 4.2. *With the previous notations, if the following map*

$$\mathcal{S}_\epsilon : \begin{cases} E \to F, \\ \sigma_\epsilon \to \mathcal{S}_\epsilon\sigma_\epsilon, \end{cases}$$

is linear and contraction with the constant of contraction $k_\epsilon = M\epsilon^{k_0}$ and $k_0, M > 0$, then $\mathcal{S} : \mathcal{G}_E \to \mathcal{G}_F$ is contraction.

Proof. For the proof of this result, we will follow the same procedure as in [7]. Let $\epsilon \in (0,1]$, and let the linear mapping $\mathcal{S}_\epsilon : (E,(p_i)_{i\in I}) \to (F,(q_j)_{j\in J})$ be contraction. Then for all $j \in J$ there exist $i_0 \in I$ and $k_\epsilon \in (0,1)$ such that for all $\sigma_\epsilon, \varrho_\epsilon \in E$, we have

$$q_j(\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon) \leqslant k_\epsilon p_{i_0}(\sigma_\epsilon - \varrho_\epsilon). \tag{4.2}$$

Setting

$$\begin{aligned} h &= v_{q_j}\Big((\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon)_\epsilon\Big) = v\Big((q_j(\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon))_\epsilon\Big) \\ &= \sup\Big\{b \in \mathbb{R} \mid \ q_j(\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon) = O_{\epsilon\to 0}(\epsilon^b)\Big\} \end{aligned}$$

and

$$\begin{aligned} d &= v_{p_{i_0}}\Big((\sigma_\epsilon - \varrho_\epsilon)_\epsilon\Big) = v\Big((p_{i_0}(\sigma_\epsilon - \varrho_\epsilon))_\epsilon\Big) \\ &= \sup\Big\{b \in \mathbb{R} \mid p_{i_0}(\sigma_\epsilon - \varrho_\epsilon) = O_{\epsilon\to 0}(\epsilon^b)\Big\}. \end{aligned}$$

Thank's to (4.2), we get

$$q_j(\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon) \leqslant k_\epsilon p_{i_0}(\sigma_\epsilon - \varrho_\epsilon) \leqslant \epsilon^{k_0} M\, c\, \epsilon^d = c_1 \epsilon^{d+k_0},$$

where $c_1 > 0$. Hence, $d + k_0 \leqslant h$, which implies that

$$\begin{gathered} v_{q_j}\Big((\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon)_\epsilon\Big) \geqslant v_{p_{i_0}}\Big((\sigma_\epsilon - \varrho_\epsilon)_\epsilon\Big) + k_0, \\ e^{-v_{q_j}((\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon)_\epsilon)} \leqslant e^{-k_0}\, e^{-v_{p_{i_0}}((\sigma_\epsilon - \varrho_\epsilon)_\epsilon)}, \\ \mathcal{Q}_j(\mathcal{S}\sigma - \mathcal{S}\varrho) \leqslant e^{-k_0}\, \mathcal{P}_{i_0}(\sigma - \varrho). \end{gathered}$$

We conclude that $\mathcal{S}$ is contraction. ☐

Example 4.2. Let $\mathcal{G}_E = \widetilde{\mathbb{R}}$, $\mathcal{P}_i(\cdot) = e^{-v_{p_i}(\cdot)}$ and $p_i(\cdot) = |\cdot|$ for all $i \in I$. The mapping

$$\mathcal{S} : \begin{cases} \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}, \\ \sigma \to \mathcal{S}\sigma = [(\mathcal{S}_\epsilon\sigma_\epsilon)_\epsilon] = [(\epsilon^r e^{-\sigma_\epsilon})_\epsilon], \end{cases}$$

where $r > 0$, is continuous in $\widetilde{\mathbb{R}}$. Indeed, for all $\sigma, \varrho \in \widetilde{\mathbb{R}}$, we have

$$|\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon| \leq k_\epsilon|\sigma_\epsilon - \varrho_\epsilon|, \quad k_\epsilon = \epsilon^r.$$

And thus, $v_{p_i}(\mathcal{S}_\epsilon\sigma_\epsilon - \mathcal{S}_\epsilon\varrho_\epsilon) \geq r + v_{p_{i_0}}(\sigma_\epsilon - \varrho_\epsilon)$. Consequently, we have

$$\mathcal{P}_i(\mathcal{S}\sigma - \mathcal{S}\varrho) \leq c\mathcal{P}_{i_0}(\sigma - \varrho), \quad \text{with } c = e^{-r} \in (0,1).$$

Proposition 4.1. *The space $(\mathcal{G}_E, (\mathcal{P}_i)_{i\in I})$ is a separated locally convex topological $\widetilde{\mathbb{C}}$-module.*

Proof. By the definition of the negligible space $\mathcal{N}_E$, if $u \neq 0$ in $\mathcal{G}_E$, then $v_{p_i}((\sigma_\epsilon)_\epsilon) \neq \pm\infty$ for some $i \in I$, hence $\mathcal{P}_i(u) > 0$, so, $(\mathcal{G}_E, (\mathcal{P}_i)_{i\in I})$ is separate. ☐

A sequence in Colombeau algebra is a map

$$\sigma : \mathbb{N} \to \mathcal{G}_E, \quad n \mapsto \sigma_n = [(\sigma_{n,\epsilon})_\epsilon],$$

and is denoted by $(\sigma_n)_{n\in\mathbb{N}}$. We say that $(\sigma_n)_{n\in\mathbb{N}}$ converges to $\sigma \in \mathcal{G}_E$ if for all $\gamma > 0$, there is $n_0 \in \mathbb{N}$, such that if $n > n_0$, then $\sigma_n - \sigma \in \mathcal{B}(i, \gamma)$. Such a sequence $(\sigma_n)_{n\in\mathbb{N}}$ is a Cauchy if for all $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that if $m, n > n_0$, then $\sigma_m - \sigma_n \in \mathcal{B}(i, \gamma)$. Since the sharp topology is a Hausdorff, then limits are unique whenever they exist. Which is equivalent in terms of families of seminorms to the following definition.

Definition 4.3. A sequence $(\sigma_n)_{n\in\mathbb{N}}$ in locally convex topological $\widetilde{\mathbb{C}}$-module $(\mathcal{G}_E, \mathcal{P}_{i\in I})$ is called convergent and converges to $\sigma \in \mathcal{G}_E$ if,

$$\lim_{n\to+\infty} \mathcal{P}_i(\sigma_n) = \sigma, \quad \text{for all } \ i \in I.$$

Example 4.3. If we take $\mathcal{G}_E = \widetilde{\mathbb{R}}$, $\mathcal{P}_i(\cdot) = e^{-v_{p_i}(\cdot)}$ and $p_i(\cdot) = |\cdot|$ for all $i \in I$. The sequence $\sigma_n = [(\sigma_{n,\epsilon})_\epsilon]$ in $\widetilde{\mathbb{R}}$, where $\sigma_{n,\epsilon} = \frac{1}{n}$, $n \in \mathbb{N}$ does not converge to 0 in $\widetilde{\mathbb{R}}$ with respect to the sharp topology, because if $\mathcal{P}_i(\frac{1}{n}) < \gamma$, for all $\gamma > 0$, we obtain $n < \frac{-1}{\ln(\gamma)}$, which is absurd when n is large enough. However, the sequence $\sigma_n = [(\sigma_{n,\epsilon})_\epsilon]$ in $\widetilde{\mathbb{R}}$, where $\sigma_{n,\epsilon} = \frac{r_\epsilon}{n}$ and $r_\epsilon = \epsilon^{\frac{r^2}{r^4+\epsilon^4}}$, $r > 0$ converges to 0. Indeed, for all $\gamma > 0$, take $n_0 = \lceil |r_\epsilon| \epsilon^{\ln(\gamma)} \rfloor + 1$ and $\lceil \cdot \rfloor$ symbolizes the integer part. If $n > n_0$, we have that $|\frac{\epsilon^\gamma}{n}| < \epsilon^{-\ln(\gamma)}$, then $v_{p_i}(\frac{r_\epsilon}{n}) > -\ln(\gamma)$. And thus $\mathcal{P}_i(\frac{\epsilon^\gamma}{n}) < \gamma$ which implies that $\sigma_n \in \mathcal{B}(i, \gamma)$.

Definition 4.4. A sequence $(\sigma_n)_{n\in\mathbb{N}}$ in locally convex topological $\widetilde{\mathbb{C}}$-module $(\mathcal{G}_E, \mathcal{P}_{i\in I})$ is called Cauchy if, for each $m, n \in \mathbb{N}$, where $m > n$ such that

$$\lim_{n\to+\infty} \mathcal{P}_i(\sigma_n - \sigma_m) = 0, \quad \text{for all } i \in I.$$

The following proposition has been proved in [10].

Proposition 4.2. ([10, Proposition 3.4, p. 25]) *The space $(\mathcal{G}_E, (\mathcal{P}_i)_{i\in I})$ is complete.*

Remark 4.1. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}_E$. Then $(\sigma_n)_{n\in\mathbb{N}}$ is convergent if and only if, it is a Cauchy sequence if and only if, for all $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that $n > m > n_0$ implies that $\sigma_m - \sigma_n \in \mathcal{B}(i, \gamma)$.

4.2. **Some fixed point theorems in Colombeau algebra.** The second part of this section is dealing with some new theorems of fixed point in the framework of Colombeau algebra based on the locally convex space. In this subsection to simplify the formula we take $v((\sigma_\epsilon)_\epsilon) = v(\sigma_\epsilon)$. The idea of this theorem inspired by that of the Banach fixed point theorem in a classical metric space.

Theorem 4.3. *Let $A : (\mathcal{G}_E, (\mathcal{P}_i)_{i\in I}) \to (\mathcal{G}_E, (\mathcal{P}_i)_{i\in I})$ be a mapping from an algebra of generalized functions into itself such that*

1. *there exists $\sigma_0 \in \mathcal{G}_E$, $p_i(A_\epsilon \sigma_{0,\epsilon} - \sigma_{0,\epsilon}) = O_{\epsilon\to 0}(\epsilon^c)$, where $\sigma_0 = [(\sigma_{0,\epsilon})_\epsilon]$;*

2. *for every* $i \in I$, $p_i(A_\epsilon\sigma_\epsilon - A_\epsilon\varrho_\epsilon) \le M\epsilon^{k_0}\, p_i(\sigma_\epsilon - \varrho_\epsilon)$ *for all* $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{M}_E$, *and* $M, k_0 > 0$.

Then, A has a unique fixed point.

Proof. We introduce the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$\begin{cases} x_{n+1} = Ax_n, \\ x_0 = \sigma_0, \end{cases} \quad \text{is equivalent to} \quad \begin{cases} x_{n+1,\epsilon} = Ax_{n,\epsilon} + n_\epsilon \\ x_{0,\epsilon} = \sigma_{0,\epsilon} + m_\epsilon, \end{cases}$$

where $(n_\epsilon)_\epsilon, (m_\epsilon)_\epsilon \in \mathcal{N}_E$. We have,

$$e^{-v_{p_i}(x_{n+1,\epsilon} - x_{n,\epsilon})} = e^{-v_{p_i}(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon})}.$$

We set

$$h = v_{p_i}\left(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon}\right) = \sup\left\{b \in \mathbb{R} \mid p_i(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon}) = O_{\epsilon\to 0}(\epsilon^b)\right\}$$

and

$$d = v_{p_i}\left(x_{n,\epsilon} - x_{n-1,\epsilon}\right) = \sup\left\{b \in \mathbb{R} \mid p_i(x_{n,\epsilon} - x_{n-1,\epsilon}) = O_{\epsilon\to 0}(\epsilon^b)\right\}.$$

By taking $\sigma_\epsilon = x_{n,\epsilon}$ and $\varrho_\epsilon = x_{n-1,\epsilon}$ in the second condition, we get

$$p_i(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon}) \leqslant M\epsilon^{k_0} p_i(x_{n,\epsilon} - x_{n-1,\epsilon}) \leqslant M\epsilon^{k_0}\, c\,\epsilon^d = c_1\epsilon^{d+k_0},$$

where $c_1 > 0$. Hence, $d + k_0 \leqslant h$ then, it follows that

$$v_{p_i}(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon}) \geqslant v_{p_i}(x_{n,\epsilon} - x_{n-1,\epsilon}) + k_0,$$
$$e^{-v_{p_i}(A_\epsilon x_{n,\epsilon} - A_\epsilon x_{n-1,\epsilon})} \leqslant e^{-k_0}\, e^{-v_{p_i}(x_{n,\epsilon} - x_{n-1,\epsilon})}.$$

Therefore, $e^{-v_{p_i}(x_{n+1,\epsilon} - x_{n,\epsilon})} \le e^{-k_0} e^{-v_{p_i}(x_{n,\epsilon} - x_{n-1,\epsilon})}$. By induction, we obtain

$$e^{-v_{p_i}(x_{n+1,\epsilon} - x_{n,\epsilon})} \le (e^{-k_0})^n e^{-v_{p_i}(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon})}.$$

Now let us prove that (x_n) is a Cauchy sequence. Let $m, n \in \mathbb{N}$, $m > n$. We have

$$\begin{aligned}
e^{-v_{p_i}(x_{m,\epsilon} - x_{n,\epsilon})} &= e^{-v_{p_i}(x_{m,\epsilon} - x_{n,\epsilon})} \\
&\le e^{\sup\left\{-v_{p_i}(x_{m,\epsilon} - x_{m-1,\epsilon}), -v_{p_i}(x_{m-1,\epsilon} - x_{n,\epsilon}\}\right)} \\
&\le \sup\left\{e^{-v_{p_i}(x_{m,\epsilon} - x_{m-1,\epsilon})}, e^{-v_{p_i}(x_{m-1,\epsilon} - x_{n,\epsilon})}\right\} \\
&\le e^{-v_{p_i}(x_{m,\epsilon} - x_{m-1,\epsilon})} + e^{-v_{p_i}(x_{m-1,\epsilon} - x_{n,\epsilon})} \\
&\le \left(e^{-k_0}\right)^{m-1} p_i\left(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}\right) + \cdots + \left(e^{-k_0}\right)^n p_i\left(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}\right) \\
&\le (e^{-k_0})^n \Big(1 + e^{-k_0} + \cdots + e^{-k_0(m-n-1)}\Big) p_i\left(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}\right) \\
&= (e^{-k_0})^n \left(\frac{1 - e^{-k_0(m-n)}}{1 - e^{k_0}}\right) e^{-v_{p_i}(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon})} \\
&\le \frac{(e^{-k_0})^n}{1 - e^{k_0}} e^{-v_{p_i}(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon})},
\end{aligned}$$

by the first condition, we get $p_i\left(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}\right) = O_{\epsilon\to 0}(\epsilon^c)$. Then,

$$\begin{aligned} v_{p_i}\left(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}\right) &\geq c, \\ -v_{p_i}\left(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon}\right) &\leq -c, \\ e^{-v_{p_i}(A_\epsilon x_{0,\epsilon} - x_{0,\epsilon})} &\leq e^{-c} = \text{ const.} \end{aligned}$$

And thus,

$$e^{-v_{p_i}(x_{m,\epsilon} - x_{n,\epsilon})} \leq \frac{\left(e^{-k_0}\right)^n}{1 - e^{k_0}} \text{const.}$$

Since the right hand side of the last inequality tends to zero as $n \to +\infty$, it follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{G}_,(\mathcal{P}_i)_{i\in I})$ which is a complete space, by Proposition 4.2, then $(x_n)_{n\in\mathbb{N}}$ is convergent. Then there exists σ in $\mathcal{G}_E$ such that $\sigma = \lim_{n\to+\infty} x_n$. On the other hand, we have for any $\sigma, \varrho \in \mathcal{G}_E$, we can prove that A is a continuous mapping. Indeed for any $\sigma, \varrho \in \mathcal{G}_E$, we have

$$\mathcal{P}_i\left(A\sigma - A\varrho\right) = e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - A_\epsilon\varrho_\epsilon)} \leq e^{-k_0}\ e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} \leq e^{-k_0}\ \mathcal{P}_i\left(\sigma_- \varrho\right).$$

We deduce that A is continuous. And we have, $x_{n+1} = Ax_n \to \sigma$ as $n \to +\infty$ which implies that $\sigma = A\sigma$. Therefore, A has a fixed point. Now, assume that A has another fixed point $\varrho \in \mathcal{G}_E$. Then $A\varrho = \varrho$, we can write

$$e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} = \mathcal{P}_i\left(\sigma - \varrho\right) = \mathcal{P}_i\left(A\sigma - A\varrho\right) = e^{-v_{p_i}(A\sigma_\epsilon - A\varrho_\epsilon)} \leq e^{-k_0}\ e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)},$$

this implies $e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} = 0$, which signifies that $v_{p_i}\left(\sigma_\epsilon - \varrho_\epsilon\right) = +\infty$, and thus, $\left(\sigma_\epsilon - \varrho_\epsilon\right)_\epsilon \in \mathcal{N}_E$. Therefore, $\sigma = \varrho$ in $\mathcal{G}_E$. ☐

The theorem below is an extended of Kannan's fixed point theorem in classical metric space.

Theorem 4.4. *Let $A : (\mathcal{G}_E, (\mathcal{P}_i)_{i\in I}) \to (\mathcal{G}_E, (\mathcal{P}_i)_{i\in I})$ be a mapping from an algebra of generalized functions equipped with a family of ultra pseudo-norms $(\mathcal{P}_i)_{i\in I}$ into itself, and satisfying the two following conditions:*

1. *there exists $\sigma_0 \in \mathcal{G}_E$, $p_i(A_\epsilon\sigma_{0,\epsilon} - \sigma_{0,\epsilon}) = O_{\epsilon\to 0}(\epsilon^d)$, with $d > 0$;*
2. *for every $i \in I$, $p_i(A_\epsilon\sigma_\epsilon - A_\epsilon\varrho_\epsilon) \leq M\epsilon^\lambda\Big[p_i(A_\epsilon\sigma_\epsilon - \sigma_\epsilon) + p_i(A_\epsilon\varrho_\epsilon - \varrho_\epsilon)\Big]$, for all $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{M}_E$, $\lambda > \ln(2)$, $M > 0$.*

Then A has a unique fixed point in $\mathcal{G}_E$.

Proof. Let us consider the following sequence

$$\begin{cases} y_{n+1} = Ay_n, \\ y_0 = \sigma_0, \end{cases} \qquad \text{is equivalent to} \qquad \begin{cases} y_{n+1,\epsilon} = Ay_{n,\epsilon} + n_\epsilon, \\ y_{0,\epsilon} = \sigma_{0,\epsilon} + m_\epsilon, \end{cases}$$

where $(n_\epsilon)_\epsilon, (m_\epsilon)_\epsilon \in \mathcal{N}_E$. We have

$$e^{-v_{p_i}(y_{n+1,\epsilon} - y_{n,\epsilon})} = e^{-v_{p_i}(y_{n+1,\epsilon} - y_{n,\epsilon})} = e^{-v_{p_i}(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon})}.$$

By the definition of the valuation function v_{p_i}, we can write

$$p_i\left(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon}\right) \leq M\epsilon^\lambda \ \epsilon^{\min\left\{v_{p_i}(A_\epsilon y_{n,\epsilon}-y_{n,\epsilon}),v_{p_i}(A_\epsilon y_{n-1,\epsilon}-y_{n-1,\epsilon})\right\}}.$$

Then,

$$v_{p_i}\left(A_\epsilon y_{n+1,\epsilon} - A_\epsilon y_{n-1,\epsilon}\right) \geq \lambda + \min\left\{v_{p_i}(A_\epsilon y_{n,\epsilon} - y_{n,\epsilon}), v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n-1,\epsilon})\right\}.$$

Thus,

$$\begin{aligned} e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-A_\epsilon y_{n-1,\epsilon})} &\leq e^{-\lambda}\ e^{-\min\left\{v_{p_i}(A_\epsilon y_{n,\epsilon}-y_{n,\epsilon}),v_{p_i}(A_\epsilon y_{n-1,\epsilon}-y_{n-1,\epsilon})\right\}} \\ &\leq e^{-\lambda}\ \max\left\{e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-y_{n,\epsilon})}, e^{-v_{p_i}(A_\epsilon y_{n-1,\epsilon}-y_{n-1,\epsilon})}\right\} \\ &\leq e^{-\lambda}\ \left(e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-y_{n,\epsilon})} + e^{-v_{p_i}(A_\epsilon y_{n-1,\epsilon}-y_{n-1,\epsilon})}\right). \end{aligned}$$

It follows,

$$e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-A_\epsilon y_{n-1,\epsilon})} \leq \frac{e^{-\lambda}}{1-e^{-\lambda}}\ e^{-v_{p_i}(A_\epsilon y_{n-1,\epsilon}-y_{n-1,\epsilon})}.$$

By induction, we can conclude that

$$e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-A_\epsilon y_{n-1,\epsilon})} \leq \left(\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)^n\ e^{-v_{p_i}(A_\epsilon y_{0,\epsilon}-y_{0,\epsilon})}.$$

Now we have to prove that $(y_n)_n$ is a Cauchy sequence. Let $n, p \in \mathbb{N}$,

$$\begin{aligned} e^{-v_{p_i}(x_{n+p,\epsilon}-y_{n,\epsilon})} =& e^{-v_{p_i}(x_{n+p,\epsilon}-y_{n,\epsilon})} \\ \leq& e^{\max\left\{-v_{p_i}(y_{n+p,\epsilon}-y_{n+p-1,\epsilon}),-v_{p_i}(y_{n+p-1,\epsilon}-y_{n,\epsilon})\right\}} \\ \leq& \max\left\{e^{-v_{p_i}(y_{n+p,\epsilon}-y_{n+p-1,\epsilon})}, e^{-v_{p_i}(y_{n+p-1,\epsilon}-y_{n,\epsilon})}\right\} \\ \leq& e^{-v_{p_i}(y_{n+p,\epsilon}-y_{n+p-1,\epsilon})} + e^{-v_{p_i}(y_{n+p-1,\epsilon}-y_{n,\epsilon})} \\ &\vdots \\ \leq& \left[\left(\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)^n + \cdots + \left(\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)^{n+p-1}\right]\ e^{-v_{p_i}(A_\epsilon y_{0,\epsilon}-y_{0,\epsilon})} \\ \leq& \frac{\left(\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)^n}{1-\frac{e^{-\lambda}}{1-e^{-\lambda}}} e^{-v_{p_i}(A_\epsilon y_{0,\epsilon}-y_{0,\epsilon})}. \end{aligned}$$

From the first property in the theorem we have $e^{-v_{p_i}(A_\epsilon y_{0,\epsilon}-y_{0,\epsilon})}$ is finite, the right hand side of the last inequality tends to zero as $n \to +\infty$. Then, $(y_n)_n$ is a Cauchy sequence in $\mathcal{G}_E$ which is complete by Proposition 4.2, and thus there is σ in $\mathcal{G}_E$ such that $y_n \to 0$ as $n \to +\infty$.

Now, let us prove that σ is a fixed point of the mapping A, we have

$$\begin{aligned} e^{-v_{p_i}(A_\epsilon \sigma_\epsilon - \sigma_\epsilon)} &\leq e^{\sup\left\{-v_{p_i}(A_\epsilon\sigma_\epsilon-y_{n,\epsilon}),-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)\right\}} \\ &\leq e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-y_{n,\epsilon})} + e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)}. \end{aligned}$$

On the other hand

$$\begin{aligned} e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)} &\leq e^{-\lambda}\left(e^{-\min\left\{v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon}),v_{p_i}(\sigma_\epsilon-A_\epsilon\sigma_\epsilon)\right\}}\right)\\ &\leq e^{-\lambda}\sup\left\{(e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})}, e^{-v_{p_i}(\sigma_\epsilon-A_\epsilon\sigma_\epsilon)}\right\}. \end{aligned}$$

Hence,

$$e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)} \leq e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)} + e^{-\lambda}e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})} + e^{-\lambda}e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)}.$$

Thus,

$$\left(1-e^{-\lambda}\right)e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)} \leq e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)} + e^{-\lambda}\, e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})}.$$

By passing to the limit as $n \to +\infty$, we obtain $e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)} = 0$, this implies

$$v_{p_i}\left(A_\epsilon\sigma_\epsilon - \sigma_\epsilon\right) = +\infty,$$

which means that $\left(A_\epsilon\sigma_\epsilon - \sigma_\epsilon\right)_\epsilon \in \mathcal{N}_E$. Then, $A\sigma = \sigma$. It now remains to demonstrate the uniqueness. Assume that there is another fixed point ϱ of A, $A\varrho = \varrho$, such that $\sigma \neq \varrho$.

So, we can write

$$\begin{aligned} e^{-v_{p_i}(\sigma_\epsilon-\varrho_\epsilon)} &\leq e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-A_\epsilon\varrho_\epsilon)}\\ &\leq e^{-\lambda}\, e^{\sup\left\{(-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon),-v_{p_i}(A_\epsilon\varrho_\epsilon-\varrho_\epsilon)\right\}}\\ &\leq e^{-\lambda}\sup\left\{e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)}, e^{-v_{p_i}(A_\epsilon\varrho_\epsilon-\varrho_\epsilon)}\right\}\\ &\leq e^{-\lambda}\left(e^{-v_{p_i}(A_\epsilon\sigma_\epsilon-\sigma_\epsilon)} + e^{-v_{p_i}(A_\epsilon\varrho_\epsilon-\varrho_\epsilon)}\right). \end{aligned}$$

Since $A\sigma = \sigma$ and $A\varrho = \varrho$, then

$$v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon) = v_{p_i}(A_\epsilon\varrho_\epsilon - \varrho_\epsilon) = +\infty,$$

which implies that $e^{-v_{p_i}(\sigma_\epsilon-\varrho_\epsilon)} = 0$. Thus, $v_{p_i}(\sigma_\epsilon - \varrho_\epsilon) = +\infty$. Then, $(\sigma_\epsilon - \varrho_\epsilon)_\epsilon \in \mathcal{N}_E$.

Conclusion is $\sigma = \varrho$ in $\mathcal{G}_E$. ☐

The following theorem is based on the theorem in the classical case, of Chatterjia.

Theorem 4.5. *Let $A : (\mathcal{G}_E, (\mathcal{P}_i)_{i\in I}) \to (\mathcal{G}_E, (\mathcal{P}_i)_{i\in I})$ be a mapping from an algebra of generalized functions equipped with a family of ultra pseudoseminorms $(\mathcal{P}_i)_{i\in I}$ into itself, and satisfying the two following conditions:*

1. *there exists $\sigma_0 \in \mathcal{G}_E$, $p_i(A_\epsilon\sigma_{0,\epsilon} - \sigma_{0,\epsilon}) = O_{\epsilon\to 0}(\epsilon^k)$, with $k > 0$;*
2. *for any $i \in I$, $p_i(A_\epsilon\sigma_\epsilon - A_\epsilon v_\epsilon) \leq M\epsilon^\beta\Big[p_i(A_\epsilon\sigma_\epsilon - \varrho_\epsilon) + p_i(A_\epsilon v_\epsilon - \sigma_\epsilon)\Big]$, for all $(\sigma_\epsilon)_\epsilon, (\varrho_\epsilon)_\epsilon \in \mathcal{M}_E$, $\beta > \ln(2)$, $M > 0$.*

Then A has a unique fixed point in $\mathcal{G}_E$.

Proof. Let us consider the following sequence

$$\begin{cases} y_{n+1} = Ay_n,\\ y_0 = \sigma_0, \end{cases} \quad \text{is equivalent to} \quad \begin{cases} y_{n+1,\epsilon} = Ay_{n,\epsilon} + n_\epsilon,\\ y_{0,\epsilon} = \sigma_{0,\epsilon} + m_\epsilon, \end{cases}$$

where $(n_\epsilon)_\epsilon, (m_\epsilon)_\epsilon \in \mathcal{N}_E$. We have $e^{-v_{p_i}(y_{n+1,\epsilon}-y_{n,\epsilon})} = e^{-v_{p_i}(A_\epsilon y_{n+1,\epsilon}-A_\epsilon y_{n-1,\epsilon})}$. By the definition of the valuation function v_{p_i} we can write

$$p_i\left(A_\epsilon y_{n,\epsilon} - A_\epsilon y_{n-1,\epsilon}\right) \le M\epsilon^\beta \epsilon^{\min\left\{v_{p_i}(A_\epsilon y_{n,\epsilon}-y_{n,\epsilon}),v_{p_i}(A_\epsilon y_{n-1,\epsilon}-y_{n-1,\epsilon})\right\}}.$$

So,

$$v_{p_i}\left(A_\epsilon y_{n+1,\epsilon} - A_\epsilon y_{n-1,\epsilon}\right) \ge \beta + \min\left\{v_{p_i}\left(A_\epsilon y_{n,\epsilon} - y_{n-1,\epsilon}\right), v_{p_i}\left(A_\epsilon y_{n-1,\epsilon} - y_{n,\epsilon}\right)\right\}.$$

Thus,

$$\begin{aligned} e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-A_\epsilon y_{n-1,\epsilon})} &\le e^{-\beta} e^{-\min\left\{v_{p_i}(A_\epsilon y_{n,\epsilon}-y_{n-1,\epsilon}),v_{p_i}(A_\epsilon y_{n-1,\epsilon}-y_{n,\epsilon})\right\}} \\ &\le e^{-\beta}\sup\left\{e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-y_{n-1,\epsilon})}, e^{-v_{p_i}(A_\epsilon y_{n-1,\epsilon}-y_{n,\epsilon})}\right\}. \end{aligned}$$

Since, $\left(A_\epsilon y_{n-1,\epsilon} - y_{n,\epsilon}\right)_\epsilon \in \mathcal{N}_E$, it follows $v_{p_i}(A_\epsilon y_{n-1,\epsilon} - y_{n,\epsilon}) = +\infty$. Then,

$$e^{-v_{p_i}(A_\epsilon y_{n-1,\epsilon}-y_{n,\epsilon})} = 0.$$

We can write the last inequality as follow

$$e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-A_\epsilon y_{n-1,\epsilon})} \le e^{-\beta}\, e^{-v_{p_i}(y_{n+1,\epsilon}-y_{n-1,\epsilon})},$$

and by the properties of the valuation function we can conclude

$$\begin{aligned} v_{p_i}\left(y_{n+1,\epsilon} - y_{n-1,\epsilon}\right) &= v_{p_i}\left(\left(y_{n+1,\epsilon} - y_{n,\epsilon}\right) + \left(y_{n,\epsilon} - y_{n-1,\epsilon}\right)\right) \\ &\ge \min\left\{v_{p_i}\left(y_{n+1,\epsilon} - y_{n,\epsilon}\right), v_{p_i}\left(y_{n,\epsilon} - y_{n-1,\epsilon}\right)\right\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-A_\epsilon y_{n-1,\epsilon})} &\le e^{-\beta}\left(e^{-v_{p_i}(y_{n+1,\epsilon}-y_{n,\epsilon})} + e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})}\right) \\ &\le e^{-\beta} e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-A_\epsilon y_{n-1,\epsilon})} + e^{-\beta} e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})}. \end{aligned}$$

Then,

$$e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-A_\epsilon y_{n-1,\epsilon})} \le \left(\frac{e^{-\beta}}{1-e^{-\beta}}\right)\, e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})}.$$

By induction we can conclude that

$$e^{-v_{p_i}(A_\epsilon y_{n,\epsilon}-A_\epsilon y_{n-1,\epsilon})} \le \left(\frac{e^{-\beta}}{1-e^{-\beta}}\right)^n\, e^{-v_{p_i}(A_\epsilon y_{0,\epsilon}-y_{0,\epsilon})}.$$

Now let us prove that $(y_n)_n$ is a Cauchy sequence in $\mathcal{G}_E$

$$\begin{aligned} e^{-v_{p_i}(y_{n+p,\epsilon}-y_{n,\epsilon})} =& e^{-v_{p_i}(y_{n+p,\epsilon}-y_{n,\epsilon})} \\ \le& e^{\max\left\{-v_{p_i}(y_{n+p,\epsilon}-y_{n+p-1,\epsilon}),-v_{p_i}(y_{n+p-1,\epsilon}-y_{n,\epsilon})\right\}} \\ \le& \max\left\{e^{-v_{p_i}(y_{n+p,\epsilon}-y_{n+p-1,\epsilon})}, e^{-v_{p_i}(y_{n+p-1,\epsilon}-y_{n,\epsilon})}\right\} \\ \le& e^{-v_{p_i}(y_{n+p,\epsilon}-y_{n+p-1,\epsilon})} + e^{-v_{p_i}(y_{n+p-1,\epsilon}-y_{n,\epsilon})} \\ & \vdots \end{aligned}$$

$$\leq \left[\left(\frac{e^{-\beta}}{1-e^{-\beta}}\right)^{n} + \cdots + \left(\frac{e^{-\beta}}{1-e^{-\beta}}\right)^{n+p-1}\right] e^{-v_{p_i}(A_\epsilon y_{0,\epsilon} - y_{0,\epsilon})}$$
$$\leq \frac{\left(\frac{e^{-\beta}}{1-e^{-\beta}}\right)^{n}}{1-\left(\frac{e^{-\beta}}{1-e^{-\beta}}\right)} e^{-v_{p_i}(A_\epsilon y_{0,\epsilon} - y_{0,\epsilon})}.$$

Thus $(y_n)_n$ is a Cauchy sequence in $\mathcal{G}_E$ which is complete by Proposition 4.2, so there is $\sigma \in \mathcal{G}_E$ such that $y_n \to \sigma$ as $n \to +\infty$. We show that σ is a fixed point of the mapping A. We have

$$\begin{aligned} e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon)} &\leq e^{\sup\left\{-v_{p_i}(A_\epsilon\sigma_\epsilon - y_{n,\epsilon}), -v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)\right\}} \\ &\leq e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - y_{n,\epsilon})} + e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)} &\leq e^{-\lambda}\left(e^{-\min\left\{v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon}), v_{p_i}(\sigma_\epsilon - A_\epsilon\sigma_\epsilon)\right\}}\right) \\ &\leq e^{-\lambda}\sup\left\{e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})}, e^{-v_{p_i}(\sigma_\epsilon - A_\epsilon\sigma_\epsilon)}\right\}. \end{aligned}$$

Hence,

$$e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon)} \leq e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)} + e^{-\beta}e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})} + e^{-\beta}e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon)}.$$

So,

$$(1-e^{-\beta})e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon)} \leq e^{-v_{p_i}(y_{n,\epsilon}-\sigma_\epsilon)} + e^{-\beta}\, e^{-v_{p_i}(y_{n,\epsilon}-y_{n-1,\epsilon})}.$$

By passing to the limit as $n \to +\infty$ we obtained $e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon)} = 0$, which implies $v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon) = +\infty$, and thus $(A_\epsilon\sigma_\epsilon - \sigma_\epsilon)_\epsilon \in \mathcal{N}_E$. Then, $A\sigma = \sigma$.

Now, let us prove the uniqueness of fixed point. Assume that there is another fixed point ϱ of A, $A\varrho = \varrho$, such that $\sigma \neq \varrho$. So, we can write

$$\begin{aligned} e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} &= e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - A_\epsilon\varrho_\epsilon)} \\ &\leq e^{-\beta}e^{\sup\left\{-v_{p_i}(A_\epsilon\sigma_\epsilon - \varrho_\epsilon), -v_{p_i}(A_\epsilon\sigma_\epsilon - \varrho_\epsilon)\right\}} \\ &\leq e^{-\beta}\sup\left\{e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - \varrho_\epsilon)}, e^{-v_{p_i}(A_\epsilon\varrho_\epsilon - \varrho_\epsilon)}\right\} \\ &\leq e^{-\beta}\left(e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon)} + e^{-\beta}e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)}\right) \\ &\leq \left(\frac{e^{-\beta}}{1-e^{-\beta}}\right)e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon)}. \end{aligned}$$

Since, $A\sigma = \sigma$. Then, $v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon) = +\infty$, that implies $e^{-v_{p_i}(A_\epsilon\sigma_\epsilon - \sigma_\epsilon)} = 0$. Thus, $e^{-v_{p_i}(\sigma_\epsilon - \varrho_\epsilon)} = 0$, then $v_{p_i}(\sigma_\epsilon - \varrho_\epsilon) = +\infty$, which meas that $(\sigma_\epsilon - \varrho_\epsilon)_\epsilon$ is a negligible element. Finally $\sigma = v$ in $\mathcal{G}_E$. □

5. Application to an Evolution Problem

We consider the standard Cauchy problem

$$\begin{cases} \sigma'(t) = f(t, \sigma(t)), \quad t \in \mathbb{R}^+, \\ \sigma(0) = \sigma_0 \in \widetilde{\mathbb{R}}, \end{cases} \tag{5.1}$$

where $f : \mathbb{R} \times \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}$ and $\sigma \in \mathcal{G}_{\mathbb{R}}$.

Proposition 5.1. *If f_ϵ satisfies the following condition*

$$|f_\epsilon(s, \sigma_\epsilon) - f_\epsilon(s, \varrho_\epsilon)| \le k_\epsilon(s)\, |\sigma_\epsilon(s) - \varrho_\epsilon(s)|, \quad \textit{for all } \sigma, \varrho \in \mathcal{G}_{\mathbb{R}}, \tag{5.2}$$

where $k_\epsilon(s) = M(s)\epsilon^\lambda$ with $0 < M(s) < 1$ and $\lambda > 0$.

Then, the problem (5.1) *has a unique generalized solution.*

Proof. $\widetilde{u}$ is a solution of the problem (5.1) if and only if it is a fixed point of the mapping

$$\begin{cases} A : \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}, \\ x \mapsto A\sigma = \sigma_0 + \displaystyle\int_0^t f(s, \sigma(s))ds. \end{cases} \tag{5.3}$$

Let

$$\begin{cases} A_\epsilon : \mathbb{R} \to \mathbb{R}, \\ x_\epsilon \mapsto A_\epsilon\sigma_\epsilon = \sigma_{0\epsilon} + \displaystyle\int_0^t f_\epsilon(s, \sigma_\epsilon(s))ds, \end{cases} \tag{5.4}$$

be a representative of A. Since f satisfies condition (5.2) and we defined the following ultra pseudo-seminorms on $\widetilde{\mathbb{R}}$ by $\mathcal{P}_T(\sigma) = e^{-v_{p_T}(\sigma_\epsilon)}$, where $p_T(\sigma_\epsilon) = \sup_{t\in[0,T]} |\sigma_\epsilon(t)|$ and T is a non negative real number. We have

$$\begin{aligned} |A_\epsilon\sigma_\epsilon - A_\epsilon\varrho_\epsilon| &= \left|\int_0^t [f_\epsilon(s, \sigma_\epsilon(s)) - f_\epsilon(s, \varrho_\epsilon(s))]ds\right| \\ &\leqslant \int_0^t |f_\epsilon(s, \sigma_\epsilon(s)) - f_\epsilon(s, \varrho_\epsilon(s))|ds \\ &\leqslant \int_0^t M\, \epsilon^k |\sigma_\epsilon(s) - \varrho_\epsilon(s)|ds \quad \left(M = \sup_{t\in[0,T]} |M(t)|\right) \\ &\le T\, M\, \epsilon^\lambda\, p_T(\sigma_\epsilon - \sigma_\epsilon) \\ &\le C\, \epsilon^\lambda\, p_T(\sigma_\epsilon - \sigma_\epsilon). \end{aligned}$$

So, $p_T(A_\epsilon\sigma_\epsilon - A_\epsilon\sigma_\epsilon) \le C\,\epsilon^\lambda p_T(\sigma_\epsilon - \sigma_\epsilon)$. Then, the second condition of first Theorem 4.3 is satisfied. Moreover, we have

$$p_T(A_\epsilon\sigma_{0\epsilon} - \sigma_{0\epsilon}) = p_T\left(\int_0^t f_\epsilon(s, \sigma_{0\epsilon})ds\right) \le T\, p_T(f_\epsilon) = O_{\epsilon\to 0}(\epsilon^c), \quad c \in \mathbb{R}.$$

According to our first theorem there is a generalized solution for the abstract Cauchy problem. We have to prove now the uniqueness, assume that there is another fixed point ϱ of A, $A\varrho = \varrho$, such that $\sigma \neq \varrho$. Then we can write

$$e^{-v_{p_T}(\sigma_\epsilon - \varrho_\epsilon)} = \mathcal{P}_T(\sigma - \varrho) = \mathcal{P}_T(A\sigma - A\varrho) = e^{-v_{p_T}(A_\epsilon\sigma_\epsilon - A_\epsilon\varrho_\epsilon)} \leq e^{-k}\, e^{-v_{p_T}(\sigma_\epsilon - \varrho_\epsilon)}.$$

Hence, $e^{-v_{p_T}(\sigma_\epsilon - \varrho_\epsilon)}(1 - e^{-k}) \leq 0$, which gives, $e^{-v_{P_T}(\sigma_\epsilon - \varrho_\epsilon)} = 0$, in other words, $v(\sigma_\epsilon - \varrho_\epsilon) = +\infty$. Therefore, $(\sigma_\epsilon - \varrho_\epsilon)_\epsilon \in \mathcal{N}_E$. Finally, $\sigma = \varrho$ in $\widetilde{\mathbb{R}}$. And the solution is unique in $\widetilde{\mathbb{R}}$. □

Example 5.1. Let's consider the example that inspired the fixed point theorem in Colombeau algebra. Consider the following problem from [9, 15]:

$$\begin{cases} \partial^2\sigma(t) = h(\sigma(t))\delta(t) + g(t), \\ \sigma(-1) = \sigma_0, \\ \sigma'(-1) = \sigma_1, \end{cases} \tag{5.5}$$

where δ the Dirac distribution and $g, h \in \mathcal{C}^\infty(\mathbb{R})$.

It is a significant differential equation which comes from physics having a product of the distributions in the first equation, initial conditions are singular generalized numbers σ_0, σ_1 and does not allow to use the classical tools to have a solution. In the references mentioned above we find proof of moderation, other nontrivial steps implying classical results. Let α be positive constant, $L = \int_{-2}^{1}\int_{-2}^{1}|g(\tau)|d\tau ds$ and k is a Lipschitz constant of h on a compact subset of $\mathbb{R}$ containing $\Omega =]-1-\frac{\alpha}{2}, \frac{\alpha}{2}[$. The equation (5.5) can be reformulated as the Cauchy problem (5.1) with $f = [(f)_\epsilon]$, $f(t,\cdot) = h(\cdot)\rho_\epsilon(t) + g(t)$ is a smooth function and $\delta = [(\delta * \rho)_\epsilon]$ is the embedding of the Dirac measure in $\mathcal{G}_E$, $E = \mathcal{C}^\infty$, where $\rho_\epsilon(t) = \frac{1}{\epsilon}\rho(\frac{t}{\epsilon})$ and $\rho \in \mathcal{C}^\infty(\mathbb{R})$, $\int_{\mathbb{R}}\rho(t)dt = 1$, $\rho(t) \geq 0$. We defined the following norm on $\mathcal{G}_E$ by

$$\mathcal{P}_T(\sigma) = e^{-v_{p_T}(\sigma_\epsilon)},$$

where $p_T(\sigma_\epsilon) = \sup_{t\in\Omega}|\sigma_\epsilon(t)|$ and let

$$\begin{cases} A_\epsilon : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R}), \\ \sigma_\epsilon \mapsto A_\epsilon\sigma_\epsilon(t) = \sigma_{0\epsilon} + (t+1)\sigma_{1\epsilon} + \int_{-1}^{1}\int_{-1}^{s} f(\sigma_\epsilon(\tau))\rho_\epsilon(\tau)d\tau ds + \int_{-1}^{t}\int_{-1}^{s} g(\tau)d\tau ds. \end{cases}$$

We have

$$\begin{aligned} A_\epsilon\sigma_{0\epsilon} - \sigma_{0\epsilon} =& (t+1)\sigma_{1\epsilon} + \int_{-1}^{1}\int_{-1}^{s} f(\sigma_\epsilon(\tau))\rho_\epsilon(\tau)d\tau ds + \int_{-1}^{t}\int_{-1}^{s} g(\tau)d\tau ds \\ \leq& (|t|+1)|\sigma_{1\epsilon}| + \int_{-1}^{1}\int_{-1}^{s} |f(\sigma_\epsilon(\tau))\rho_\epsilon(\tau)|d\tau ds + \int_{-1}^{t}\int_{-1}^{s} |g(\tau)|d\tau ds \\ \leq& |\sigma_{1\epsilon}|(\alpha/2+1) + \|h\|_\infty\|\rho_\epsilon\|_\infty\frac{\alpha}{4}(\alpha+2) + L := M_\epsilon. \end{aligned}$$

So,

$$p_T(A_\epsilon\,\sigma_{0\epsilon} - \sigma_{0\epsilon}) \leq\ p_T(M_\epsilon) = O_{\epsilon\to 0}(\epsilon^c), \quad c \in \mathbb{R}.$$

On the other hand, we have

$$\begin{aligned} A_\epsilon\sigma_\epsilon(t) - A_\epsilon\varrho_\epsilon(t) =& \int_{-1}^{1}\int_{-1}^{s} f(\sigma_\epsilon(\tau))\rho_\epsilon(\tau)d\tau ds - \int_{-1}^{1}\int_{-1}^{s} f(\varrho_\epsilon(\tau))\rho_\epsilon(\tau)d\tau ds \\ =& \int_{-1}^{1}\int_{-1}^{s} [f(\sigma_\epsilon(\tau)) - f(\varrho_\epsilon(\tau))]\rho_\epsilon(\tau)d\tau ds \\ \leq& k\|\rho_\epsilon\|_\infty \int_{-1}^{1}\int_{-1}^{s} |\sigma_\epsilon(\tau) - \varrho_\epsilon(\tau)|d\tau ds \\ \leq& k\|\rho_\epsilon\|_\infty \frac{\alpha}{4}(\alpha+2)p_T(\sigma_\epsilon - \varrho_\epsilon) \\ \leq& k_\epsilon p_T(\sigma_\epsilon - \varrho_\epsilon). \end{aligned}$$

Thus, $p_T(A_\epsilon\sigma_\epsilon - A_\epsilon\varrho_\epsilon) \leq k_\epsilon p_T(\sigma_\epsilon - \varrho_\epsilon)$. It follows that the mapping A is Lipschitz, since it is continuous in $\mathcal{G}_E$. Moreover, from Theorem 4.3, A has a fixed point which is a solution for (5.5). Once the Theorem 4.3 is applied.

Acknowledgements. The authors would like to thank to the editor and reviewers due to their helpful comments and suggestions for improving paper.

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