Kragujevac Journal of Mathematics
Volume 50(9) (2026), Pages 1433–1444.

ON $F(p,q,s)$ SPACES AND WEIGHTED DIFFERENTIATION COMPOSITION OPERATORS ON THEM

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Abstract. In this work, we first consider weighted differentiation composition operator from a general family of analytic functions into Bloch–type space and find an approximation for the essential norm. Then, we characterize the boundedness of the operator into Zygmund–type space and finally, essential norm of this operator is estimated.

1. Introduction

Let f be an analytic function on the open unit disk in the complex plane and φ be an analytic self–map of the unit disk. The well–known composition operator denoted by C_φ is defined by

$$C_\varphi f = f \circ \varphi.$$

This operator can also be defined on the space of measurable functions. Composition operators are used in the study of isometries, backward shifts of all multiplicities, commutates of multiplication operators and more general operators and De Branges' proof of the Bieberbach conjecture. It is an interesting problem to describe the operator properties of C_φ in terms of the function properties of the symbol φ when the operator C_φ acts on several spaces of analytic functions on the unit disk. It is clear that C_φ is linear and bounded by Littlewood's subordination principle on the classical Hardy and Bergman spaces, see [5, 8].

Key words and phrases. Boundedness, compactness, essential norm, Zygmund type space.
2020 *Mathematics Subject Classification.* Primary: 47B38, 30H30.
DOI
Received: October 17, 2024.
Accepted: January 26, 2025.

The composition operators can be generalized through multiplication to the weighted composition operators and more generally weighted differentiation composition operators. This type of operators contain many other operators that are well known in this field. We are going to investigate the properties of weighted differentiation composition operators on the general space of analytic functions $F(p,q,s)$ to weighted Bloch and Zygmund spaces. The space $F(p,q,s)$ includes several classical spaces of analytic functions such as Bergman, Bloch, Hardy, BMOA and Q_p. So the results of the paper can be stated for several operators and spaces.

Information about $F(p,q,s)$ can be found in [12, 13]. Also for some equivalent norms in the case of the unit ball, one can refer to [11]. Weighted composition operators from $F(p,q,s)$ to H_μ^∞ have been studied by Zhu in [15]. Yang investigated generalized weighted composition operators from $F(p,q,s)$ space to the Bloch-type space in [10]. Volterra-type operators from $F(p,q,s)$ space to Bloch–Orlicz and Zygmund–Orlicz spaces were characterized in [4]. Recently, the essential norm of t-generalized composition operators from $F(p,q,s)$ to iterated weighted-type Banach space has been approximated in [1]. For further research on this subject, see [2,3,6,7,9].

2. Background Material

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disk $\mathbb{D}$ and $\mathcal{S}(\mathbb{D})$ be the set of all analytic self–maps of $\mathbb{D}$. Let $u \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$. The generalized weighted composition operator, weighted differentiation composition operator, is given by

$$D^n_{u,\varphi} f(z) = u(z) f^{(n)}(\varphi(z)), \quad f \in \mathcal{H}(\mathbb{D}), z \in \mathbb{D},$$

where $n \in \mathbb{N}_0$. This type of operators include composition operator C_φ, multiplication operator M_u, DC_φ, $C_\varphi D$ and some other classical operators. While the operator $D^n_{u,\varphi}$ acts on some subspaces of $\mathcal{H}(\mathbb{D})$, it is a natural question to find conditions under which the operator has operator properties such as boundedness, compactness, and others. Instead of using classical subspaces individually, we deal with a general function space $F(p,q,s)$. Let $0 < p < +\infty$, $0 \le s < +\infty$ and $-2 < q < +\infty$. The space $F(p,q,s)$ is the space of all analytic functions in $\mathbb{D}$ for which

$$\|f\|^p_{p,q,s} = |f(0)|^p + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < +\infty.$$

Here dA is normalized area measure on $\mathbb{D}$, $g(z,a) = \log|\frac{1}{\varphi_a(z)}|$, $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. The above space was introduced in [12] and becomes a Banach space with $\|.\|_{p,q,s}$. For the properties and the other spaces included in it see [13]. We bring here some spaces.

- When $s > 1$, $F(p,q,s) = \mathcal{B}^{(q+2)/p}$, the Bloch space.
- $F(2,0,s) = Q_s$ and $F(2,0,1) = BMOA$.
- When $0 < p < +\infty$ and $-1 < q < +\infty$, $F(p,q,0) = D^p_q$, Dirichlet space and if $1 < p < +\infty$, then $F(p,p-2,0) = B_p$, the analytic Besov space.
- $F(2,1,0) = H^2$, Hardy space.
- When $0 < p < +\infty$, $F(p,p+q,0) = A^p_q$, the Bergman space.

If $q+s \leq -1$, then $F(p,q,s)$ is the space of constant functions, see [12]. For $0 < p, s < +\infty$, $-2 < q < +\infty$ and $q+s > -1$, if $f \in F(p,q,s)$, then

$$|f^{(m)}(z)| \leq C \frac{\|f\|_{p,q,s}}{(1-|z|^2)^{\frac{2+q-p}{p}+m}}, \quad m \in \mathbb{N}, \tag{2.1}$$

see [1]. Here we use weighted Bloch and Zygmund spaces. A positive continuous function μ on $\mathbb{D}$ is called a weight. For a weight μ which is radial, $\mu(z) = \mu(|z|)$, the weighted Zygmund space $\mathcal{Z}_\mu$, Zygmund-type space, is the space of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)| < +\infty.$$

The above statement is a semi–norm and it will be a norm if we add $|f(0)| + |f'(0)|$. So,

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)|.$$

With this norm, $\mathcal{Z}_\mu$ is a Banach space. The weighted Bloch space $\mathcal{B}_\mu$ is defined by

$$\mathcal{B}_\mu = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f'(z)| < +\infty\}.$$

Note that if $\mu(z) = (1-|z|^2)^\alpha$, $\alpha > 0$, then we obtain the Zygmund space $\mathcal{Z}^\alpha$ and the Bloch space $\mathcal{B}^\alpha$.

Throughout the paper we set $\gamma = \frac{q+2}{p}$ and

$$A(u,t) = \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^t}.$$

All positive constants will be denoted by C which may vary from one occurrence to another. By $A \succeq B$ we mean there exists a constant C such that $A \geq CB$ and $A \approx B$ means that $A \succeq B \succeq A$.

3. Main Results

Lemma 3.1. *Let $0 < p, s < +\infty$, $-2 < q < +\infty$, $q+s > -1$, $\varphi \in \mathcal{S}(\mathbb{D})$, $u \in \mathcal{H}(\mathbb{D})$ and μ be a radial weight. If $\rho = \|\varphi\|_\infty < 1$ and $D^n_{u,\varphi} : F(p,q,s) \to \mathcal{B}_\mu$ is bounded, then $D^n_{u,\varphi} : F(p,q,s) \to \mathcal{B}_\mu$ is compact.*

Proof. We need to show that for any bounded sequence $\{f_i\}$ in $F(p,q,s)$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$, $\|D^n_{u,\varphi} f_i\|_{\mathcal{B}_\mu}$ converges to 0 as $i \to +\infty$.

The boundedness of the operator implies that for any $f \in F(p,q,s)$, $\|D^n_{u,\varphi} f\|_{\mathcal{B}_\mu} \leq C\|f\|_{p,q,s}$. Apply this to the functions z^n and z^{n+1} and using triangle inequality we obtain

$$\begin{aligned} C_1 &= \sup_{z\in\mathbb{D}} \mu(|z|)|u'(z)| < +\infty, \\ C_2 &= \sup_{z\in\mathbb{D}} \mu(|z|)|u(z)\varphi'(z)| < +\infty. \end{aligned} \tag{3.1}$$

Therefore,

$$\begin{aligned}\|D^n_{u,\varphi}f_i\|_{\mathcal{B}_\mu} =&|(D^n_{u,\varphi}(f_i))(0)|+\sup_{z\in\mathbb{D}}\mu(|z|)|(D^n_{u,\varphi}(f_i))'(z)|\\ \le&|u(0)||f_i^{(n)}(\varphi(0))|+\sup_{z\in\mathbb{D}}\mu(|z|)|u'(z)||f_i^{(n)}(\varphi(z))|\\ &+\sup_{z\in\mathbb{D}}\mu(|z|)|u(z)\varphi'(z)||f_i^{(n+1)}(\varphi(z))|\\ \le&|u(0)||f_i^{(n)}(\varphi(0))|+\sup_{|\varphi(z)|\le\rho}\mu(|z|)|u'(z)||f_i^{(n)}(\varphi(z))|\\ &+\sup_{|\varphi(z)|\le\rho}\mu(|z|)|u(z)\varphi'(z)||f_i^{(n+1)}(\varphi(z))|\\ \le&|u(0)||f_i^{(n)}(\varphi(0))|+C_1\sup_{|\varphi(z)|\le\rho}|f_i^{(n)}(\varphi(z))|+C_2\sup_{|\varphi(z)|\le\rho}|f_i^{(n+1)}(\varphi(z))|.\end{aligned}$$

All the three terms above tend to 0 and $\|D^n_{u,\varphi}f_i\|_{\mathcal{B}_\mu}$ tends to 0 and so the proof is completed. □

Theorem 3.1. *Suppose that $\varphi\in\mathcal{S}(\mathbb{D})$ with $\|\varphi\|_\infty=1$, $u\in\mathcal{H}(\mathbb{D})$ and μ is a normal weight on $[0,1)$ as in [10]. Also suppose that $0<p,s<+\infty$, $-2<q<+\infty$ and $q+s>-1$. If $D^n_{u,\varphi}:F(p,q,s)\to\mathcal{B}_\mu$ is bounded, then*

$$\|D^n_{u,\varphi}\|_e\approx\limsup_{|\varphi(z)|\to1}A(u',\gamma+n-1)+\limsup_{|\varphi(z)|\to1}A(u\varphi',\gamma+n). \tag{3.2}$$

Proof. Define the sequences of functions as follows

$$g_i(z)=\frac{1-|\varphi(z_i)|}{(1-\overline{\varphi(z_i)}z)^\gamma},\quad f_i(z)=\frac{1-|\varphi(z_i)|}{(1-\overline{\varphi(z_i)}z)^\gamma}-\frac{2+q}{2+q+pn}\cdot\frac{(1-|\varphi(z_i)|)^2}{(1-\overline{\varphi(z_i)}z)^{\gamma+1}}.$$

Here $\{z_i\}\subset\mathbb{D}$ is a sequence that $|\varphi(z_i)|\to1$. By performing a simple calculation on the derivatives of the g_i and f_i, one can see that

$$g_i^{(n)}(\varphi(z_i))=\beta\frac{\overline{\varphi(z_i)}^n}{(1-|\varphi(z_i)|^2)^{\gamma+n-1}},\quad g_i^{(n+1)}(\varphi(z_i))=\beta(\gamma+n)\frac{\overline{\varphi(z_i)}^{n+1}}{(1-|\varphi(z_i)|^2)^{\gamma+n}},$$

$$f_i^{(n)}(\varphi(z_i))=0,\quad f_i^{(n+1)}(\varphi(z_i))=\beta\frac{\overline{\varphi(z_i)}^{n+1}}{(1-|\varphi(z_i)|^2)^{\gamma+n}},$$

where $\beta=\gamma(\gamma+1)\cdots(\gamma+n-1)$. From [10], we see that both sequences are bounded in $F(p,q,s)$ and converge to zero uniformly on compact subsets of $\mathbb{D}$. So for any compact operator K from $F(p,q,s)$ to $\mathcal{B}_\mu$, $\|Kg_i\|_{\mathcal{B}_\mu}\to0$ and $\|Kf_i\|_{\mathcal{B}_\mu}\to0$, as $i\to+\infty$. Noting to the definition of the essential norm we get

$$\|D^n_{u,\varphi}\|_{e,F(p,q,s)\to\mathcal{B}_\mu}=\inf_K\|D^n_{u,\varphi}-K\|_{F(p,q,s)\to\mathcal{B}_\mu}. \tag{3.3}$$

So, it will be sufficient to compute the operator norm of $D^n_{u,\varphi}-K:F(p,q,s)\to\mathcal{B}_\mu$. Therefore,

$$\|D^n_{u,\varphi}-K\|_{F(p,q,s)\to\mathcal{B}_\mu}=\sup_{\|f\|_{p,q,s}\le1}\|(D^n_{u,\varphi}-K)f\|_{\mathcal{B}_\mu}\succeq\|(D^n_{u,\varphi}-K)f_i\|_{\mathcal{B}_\mu}. \tag{3.4}$$

Then,

$$\begin{aligned}
\|D^n_{u,\varphi}-K\|_{F(p,q,s)\to\mathcal{B}_\mu} &\succeq \limsup_{i\to+\infty}\|(D^n_{u,\varphi}-K)f_i\|_{\mathcal{B}_\mu}=\limsup_{i\to\infty}\|D^n_{u,\varphi}f_i\|_{\mathcal{B}_\mu}\\
&\geq \limsup_{i\to+\infty}\sup_{z\in\mathbb{D}}\mu(|z|)|(D^n_{u,\varphi}f_i)'(z)|\\
&\geq \limsup_{i\to+\infty}\mu(|z_i|)|(D^n_{u,\varphi}f_i)'(z_i)|\\
&=\limsup_{i\to+\infty}\beta\frac{\mu(|z_i|)|u(z_i)\varphi'(z_i)||\varphi(z_i)|^{n+1}}{(1-|\varphi(z_i)|^2)^{\gamma+n}}\\
&=\limsup_{i\to+\infty}\beta\frac{\mu(|z_i|)|u(z_i)\varphi'(z_i)|}{(1-|\varphi(z_i)|^2)^{\gamma+n}}.
\end{aligned}$$

From (3.3) and (3.4) we have

$$\|D^n_{u,\varphi}\|_{e,F(p,q,s)\to\mathcal{B}_\mu}\succeq\beta\limsup_{|\varphi(z)|\to1}\frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}}. \tag{3.5}$$

Now we consider the sequence $\{g_i\}$ and get to the following

$$\begin{aligned}
\|D^n_{u,\varphi}\|_{e,F(p,q,s)\to\mathcal{B}_\mu} &\succeq \limsup_{i\to+\infty}\|D^n_{u,\varphi}g_i\|_{\mathcal{B}_\mu}\\
&\geq \limsup_{i\to+\infty}\sup_{z\in\mathbb{D}}\mu(|z|)|(D^n_{u,\varphi}g_i)'(z)|\\
&\geq \limsup_{i\to+\infty}\mu(|z_i|)|(D^n_{u,\varphi}g_i)'(z_i)|\\
&\geq \limsup_{i\to+\infty}\mu(|z_i|)|u'(z_i)||g_i^{(n)}(\varphi(z_i))|\\
&\quad-\limsup_{i\to+\infty}\mu(|z_i|)|u(z_i)\varphi'(z_i)||g_i^{(n+1)}(\varphi(z_i))|\\
&=\beta\limsup_{i\to+\infty}\frac{\mu(|z_i|)|u'(z_i)||\varphi(z_i)|^n}{(1-|\varphi(z_i)|^2)^{\gamma+n-1}}\\
&\quad-\beta(\gamma+n)\limsup_{i\to+\infty}\frac{\mu(|z_i|)|u(z_i)\varphi'(z_i)||\varphi(z_i)|^{n+1}}{(1-|\varphi(z_i)|^2)^{\gamma+n}}\\
&=\beta\limsup_{|\varphi(z)|\to1}\frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}}\\
&\quad-\beta(\gamma+n)\limsup_{|\varphi(z)|\to1}\frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}}\\
&\succeq\beta\limsup_{|\varphi(z)|\to1}\frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}}-(\gamma+n)\|D^n_{u,\varphi}\|_{e,F(p,q,s)\to\mathcal{B}_\mu},
\end{aligned}$$

where we use (3.5) in the last line. Then,

$$(\gamma+n+1)\|D^n_{u,\varphi}\|_{e,F(p,q,s)\to\mathcal{B}_\mu}\succeq\beta\limsup_{|\varphi(z)|\to1}\frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}}. \tag{3.6}$$

Now (3.5) and (3.6) imply that

$$\|D^n_{u,\varphi}\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \succeq \limsup_{|\varphi(z)|\to 1} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}},$$

and the lower estimate for the essential norm is obtained. Now, suppose that $\{r_j\}$ is a sequence in $(0,1)$ and $r_j \to 1$ such that $j \to +\infty$. Since $D^n_{u,\varphi} : F(p,q,s) \to \mathcal{B}_\mu$ is bounded, then from Theorem 1 of [10] we obtain

$$\sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} < +\infty, \quad \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}} < +\infty.$$

So,

$$\sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u'(z)|}{(1-|r_j\varphi(z)|^2)^{\gamma+n-1}} < +\infty, \quad \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|r_j\varphi(z)|^2)^{\gamma+n}} < +\infty.$$

Again Theorem 1 of [10] implies that $D^n_{u,r_j\varphi} : F(p,q,s) \to \mathcal{B}_\mu$ is bounded. Also, since $\|r_j\varphi\|_\infty < 1$, then using Lemma 3.1, $D^n_{u,r_j\varphi} : F(p,q,s) \to \mathcal{B}_\mu$ is a compact operator for any j. So,

$$\|D^n_{u,\varphi}\|_{e:F(p,q,s)\to\mathcal{B}_\mu} \le \|D^n_{u,\varphi} - D^n_{u,r_j\varphi}\| = \sup_{\|f\|_{p,q,s}\le 1} \|(D^n_{u,\varphi} - D^n_{u,r_j\varphi})f\|_{\mathcal{B}_\mu}, \quad j \in \mathbb{N}.$$

Let $f \in F(p,q,s)$ and $\|f\|_{p,q,s} \le 1$. Then,

$$\begin{aligned}
\|(D^n_{u,\varphi} - D^n_{u,r_j\varphi})f\|_{\mathcal{B}_\mu} =& |((D^n_{u,\varphi} - D^n_{u,r_j\varphi})f)(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|((D^n_{u,\varphi} - D^n_{u,r_j\varphi})f)'(z)| \\
\le& |u(0)f^{(n)}(\varphi(0)) - u(0)f^{(n)}(r_j\varphi(0))| \\
&+ \sup_{z\in\mathbb{D}} \mu(|z|)|u'(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_j\varphi(z))| \\
&+ \sup_{z\in\mathbb{D}} \mu(|z|)|u(z)\varphi'(z)||f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}(r_j\varphi(z))| \\
=& E + F + G. \qquad (3.7)
\end{aligned}$$

Then, $E \to 0$ as $j \to +\infty$. For F and G we have

$$\begin{aligned}
F \le& \sup_{|\varphi(z)|\le\delta} \mu(|z|)|u'(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_j\varphi(z))| \\
&+ \sup_{|\varphi(z)|>\delta} \mu(|z|)|u'(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_j\varphi(z))| = F_1 + F_2, \\
G \le& \sup_{|\varphi(z)|\le\delta} \mu(|z|)|u(z)\varphi'(z)||f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}(r_j\varphi(z))| \\
&+ \sup_{|\varphi(z)|>\delta} \mu(|z|)|u(z)\varphi'(z)||f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}(r_j\varphi(z))| = G_1 + G_2,
\end{aligned}$$

where $\delta \in (0,1)$ is fixed. Hence $F_1, G_1 \to 0$ as $j \to +\infty$, due to the compactness of the set $\{|\varphi(z)| \le \delta\}$. Therefore, we get from (2.1) that

$$F_2 = \sup_{|\varphi(z)|>\delta} \mu(|z|)|u'(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_j\varphi(z))|$$

$$\preceq \sup_{|\varphi(z)|>\delta} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} + \sup_{|\varphi(z)|>\delta} \frac{\mu(|z|)|u'(z)|}{(1-|r_j\varphi(z)|^2)^{\gamma+n-1}}$$
$$\leq 2 \sup_{|\varphi(z)|>\delta} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}}.$$

Also, for G_2 we get

$$G_2 \preceq 2 \sup_{|\varphi(z)|>\delta} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}}.$$

From (3.7) and above discussion we have

$$\|(D^n_{u,\varphi} - D^n_{u,r_j\varphi})f\|_{\mathcal{B}_\mu} \preceq \sup_{|\varphi(z)|>\delta} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} + \sup_{|\varphi(z)|>\delta} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}},$$

and letting $\delta \to 1$ and using the definition of essential norm, we obtain

$$\|D^n_{u,\varphi}\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \preceq \limsup_{|\varphi(z)|\to 1} \frac{\mu(|z|)|u'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(|z|)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}}.$$

□

In the following theorem we find a necessary and sufficient condition for the operator $D^n_{u,\varphi} : F(p,q,s) \to \mathcal{Z}_\mu$ to be bounded.

Theorem 3.2. *Suppose that $\varphi \in \mathcal{S}(\mathbb{D})$, $u \in \mathcal{H}(\mathbb{D})$ and μ is a radial weight. Also suppose that $0 < p, s < +\infty$, $-2 < q < +\infty$ and $q+s > -1$. Then, $D^n_{u,\varphi} : F(p,q,s) \to \mathcal{Z}_\mu$ is bounded if and only if the functions $A(u'', \gamma+n-1)$, $A(2u'\varphi' + u\varphi'', \gamma+n)$ and $A(u\varphi'^2, \gamma+n+1)$ are bounded on $\mathbb{D}$.*

Proof. Suppose that the above functions are bounded on $\mathbb{D}$. To prove that the operator is bounded, we compute its norm. To do that, we have

$$\begin{aligned}\|D^n_{u,\varphi}\|_{F(p,q,s)\to\mathcal{Z}_\mu} = &\sup_{\|f\|_{p,q,s}\leq 1} \|D^n_{u,\varphi} f\|_{\mathcal{Z}_\mu}\\
=&|(D^n_{u,\varphi}f)(0)| + |(D^n_{u,\varphi}f)'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|(D^n_{u,\varphi}f)''(z)|\\
=&|u(0)f^{(n)}(\varphi(0))| + |u'(0)f^{(n)}(\varphi(0)) + u(0)\varphi'(0)f^{(n+1)}(\varphi(0))|\\
&+ \sup_{z\in\mathbb{D}} \mu(|z|)|u''(z)f^{(n)}(\varphi(z)) + (2u'(z)\varphi'(z)\\
&+ u(z)\varphi''(z))f^{(n+1)}(\varphi(z)) + u(z)\varphi'^2(z)f^{(n+2)}(\varphi(z))|\\
\preceq&\frac{|u(0)|}{(1-|\varphi(0)|^2)^{\gamma+n-1}} + \frac{|u'(0)|}{(1-|\varphi(0)|^2)^{\gamma+n-1}} + \frac{|u(0)\varphi'(0)|}{(1-|\varphi(0)|^2)^{\gamma+n}}\\
&+ \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u''(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} + \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}}\\
&+ \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\gamma+n+1}} < +\infty.\end{aligned}$$

So, $D^n_{u,\varphi} : F(p,q,s) \to \mathcal{Z}_\mu$ is bounded. Now suppose that $D^n_{u,\varphi} : F(p,q,s) \to \mathcal{Z}_\mu$ is bounded. Then there exists a positive constant C such that for every $f \in F(p,q,s)$, $\|D^n_{u,\varphi} f\|_{\mathcal{Z}_\mu} \leq C\|f\|_{p,q,s}$. Applying this to the polynomials z^n, z^{n+1}, z^{n+2} and using some basic techniques, the followings are obtained

$$\sup_{z\in\mathbb{D}} A(u'',0) < +\infty, \quad \sup_{z\in\mathbb{D}} A(2u'\varphi' + u\varphi'',0) < +\infty, \quad \sup_{z\in\mathbb{D}} A(u\varphi'^2,0) < +\infty. \tag{3.8}$$

By [3] and Lemma 3 in [9], there exist functions $g_{i,w} \in F(p,q,s)$, $i \in \{n, n+1, n+2\}$ and $w \in \mathbb{D}$, such that

$$g^{(j)}_{i,w}(w) = \frac{\delta_{i,j}\overline{w}^j}{(1-|w|^2)^{\gamma+j-1}}, \quad j \in \{n, n+1, n+2\}.$$

Also, $\sup_{w\in\mathbb{D}} \|g_{i,w}\|_{p,q,s} < +\infty$. Here $\delta_{i,j}$ is the Kronecker delta. We prove that $A(u'', \gamma+n-1)$ is bounded, the others are similar. Noting that by the boundeness of the operator we have

$$\begin{aligned} C \geq & \|D^n_{u,\varphi} g_{n,\varphi(w)}\|_{\mathcal{Z}_\mu} \geq \sup_{z\in\mathbb{D}} \mu(|z|)|(D^n_{u,\varphi} g_{n,\varphi(w)})''(z)| \geq \mu(|w|)|(D^n_{u,\varphi} g_{n,\varphi(w)})''(w)| \\ = & \frac{\mu(|w|)|u''(w)||\varphi(w)|^n}{(1-|\varphi(w)|^2)^{\gamma+n-1}}. \end{aligned}$$

Then,

$$\sup_{|\varphi(w)|>\delta} \frac{\mu(|w|)|u''(w)|}{(1-|\varphi(w)|^2)^{\gamma+n-1}} < +\infty, \tag{3.9}$$

where $\delta \in (0,1)$ is fixed. Also

$$\sup_{|\varphi(w)|\leq\delta} \frac{\mu(|w|)|u''(w)|}{(1-|\varphi(w)|^2)^{\gamma+n-1}} < \frac{1}{(1-\delta^2)^{\gamma+n-1}} \sup_{w\in\mathbb{D}} A(u'',0) < +\infty. \tag{3.10}$$

From the above discussion and using the fact that w is arbitrary, we get

$$A(u'', \gamma+n-1) = \sup_{w\in\mathbb{D}} \frac{\mu(w)|u''(w)|}{(1-|\varphi(w)|^2)^{\gamma+n-1}} < +\infty.$$

Applying the functions $g_{n+1,\varphi(w)}$ and $g_{n+2,\varphi(w)}$, the boundedness of the functions $A(2u'\varphi' + u\varphi'', \gamma+n)$ and $A(u\varphi'^2, \gamma+n+1)$ is obtained. □

The proof of the following lemma is analogues to the proof of Lemma 3.1.

Lemma 3.2. *Let $0 < p, s < +\infty$, $-2 < q < +\infty$, $q + s > -1$, $\varphi \in \mathcal{S}(\mathbb{D})$, $u \in \mathcal{H}(\mathbb{D})$ and μ is a radial weight. If $\|\varphi\|_\infty < 1$ and $D^n_{u,\varphi} : F(p,q,s) \to \mathcal{Z}_\mu$ be bounded, then $D^n_{u,\varphi} : F(p,q,s) \to \mathcal{Z}_\mu$ is compact.*

Theorem 3.3. *Suppose that $\varphi \in \mathcal{S}(\mathbb{D})$, $u \in \mathcal{H}(\mathbb{D})$ and μ is a radial weight. Also suppose that $0 < p, s < +\infty$, $-2 < q < +\infty$ and $q + s > -1$. If $D^n_{u,\varphi} : F(p,q,s) \to \mathcal{Z}_\mu$ is bounded, then*

$$\|D^n_{u,\varphi}\|_e \approx \limsup_{|\varphi(z)|\to 1} \left\{ A(u'', \gamma+n-1) + A(2u'\varphi' + u\varphi'', \gamma+n) + A(u\varphi'^2, \gamma+n+1) \right\}.$$

Proof. Let $\{w_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(w_k)| \to 1$, as $k \to +\infty$. By [3] and Lemma 3 in [9], there exist functions $g_{i,w_k} \in F(p,q,s)$, $i \in \{n, n+1, n+2\}$ such that

$$g_{i,w_k}^{(j)}(w_k) = \frac{\delta_{i,j}\overline{w_k}^j}{(1-|w_k|^2)^{\gamma+j-1}}, \quad j \in \{n, n+1, n+2\}.$$

Also $\sup_{w_k\in\mathbb{D}} \|g_{i,w_k}\|_{p,q,s} < +\infty$. Here $\delta_{i,j}$ is the Kronecker delta. Moreover $g_{i,w_k} \to 0$ uniformly on compact subsets of $\mathbb{D}$, as $k \to +\infty$. So, for any compact operator $T: F(p,q,s) \to \mathcal{Z}_\mu$, $\|Tg_{i,w_k}\|_{\mathcal{Z}_\mu} \to 0$. If we use the sequence $\{g_{n,w_k}\}$, then we have

$$\begin{aligned}
\|D^n_{u,\varphi} - T\|_{F(p,q,s)\to\mathcal{Z}_\mu} &\succeq \limsup_{k\to+\infty} \|(D^n_{u,\varphi} - T)g_{n,w_k}\|_{\mathcal{Z}_\mu} = \limsup_{k\to+\infty} \|D^n_{u,\varphi} g_{n,w_k}\|_{\mathcal{Z}_\mu} \\
&\geq \limsup_{k\to+\infty} \sup_{z\in\mathbb{D}} \mu(|z|)|(D^n_{u,\varphi} g_{n,w_k})''(z)| \\
&\geq \limsup_{k\to+\infty} \mu(|w_k|)|(D^n_{u,\varphi} g_{n,w_k})''(w_k)| \\
&= \limsup_{k\to+\infty} \frac{\mu(|w_k|)|u''(w_k)||\varphi(w_k)|^n}{(1-|\varphi(w_k)|^2)^{\gamma+n-1}} \\
&= \limsup_{k\to+\infty} \frac{\mu(|w_k|)|u''(w_k)|}{(1-|\varphi(w_k)|^2)^{\gamma+n-1}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|D^n_{u,\varphi}\|_{e,F(p,q,s)\to\mathcal{Z}_\mu} &= \inf_{T \text{ compact}} \|D^n_{u,\varphi} - T\|_{F(p,q,s)\to\mathcal{Z}_\mu} \\
&\succeq \|D^n_{u,\varphi} - T\|_{F(p,q,s)\to\mathcal{Z}_\mu} \succeq \limsup_{|\varphi(z)|\to1} \frac{\mu(|z|)|u''(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} \\
&= \limsup_{|\varphi(z)|\to1} A(u'', \gamma+n-1).
\end{aligned}$$

In the same way, using the sequences $\{g_{n+1,w_k}\}$ and $\{g_{n+2,w_k}\}$, the following is obtained

$$\begin{aligned}
\|D^n_{u,\varphi}\|_{e,F(p,q,s)\to\mathcal{Z}_\mu} &\succeq \limsup_{|\varphi(z)|\to1} A(2u'\varphi' + u\varphi'', \gamma+n), \\
\|D^n_{u,\varphi}\|_{e,F(p,q,s)\to\mathcal{Z}_\mu} &\succeq \limsup_{|\varphi(z)|\to1} A(u\varphi'^2, \gamma+n+1),
\end{aligned}$$

and the lower estimate for the essential norm is obtained.

Let $\{r_k\}$ be a sequence in $(0,1)$ such that $\lim_{k\to+\infty} r_k = 1$. Using the boundedness of $D^n_{u,\varphi}$ and Theorem 3.2, the operator $D^n_{u,r_k\varphi}: F(p,q,s) \to \mathcal{Z}_\mu$ is bounded and then it is compact by Lemma 3.2, since $\|r_k\varphi\|_\infty < 1$. Therefore,

$$\|D^n_{u,\varphi}\|_e \leq \|D^n_{u,\varphi} - D^n_{u,r_k\varphi}\| = \sup_{\|f\|_{p,q,s}\leq1} \|(D^n_{u,\varphi} - D^n_{u,r_k\varphi})f\|_{\mathcal{Z}_\mu}. \tag{3.11}$$

Noting that $f - f_{r_k} \to 0$ uniformly on compact subsets of $\mathbb{D}$, $f_r(z) = f(rz)$, and employing (3.8) we have

$$\limsup_{k\to+\infty} |u(0)f^{(n)}(\varphi(0)) - u(0)f^{(n)}(r_k\varphi(0))| = 0,$$

$$\limsup_{k\to+\infty} |u'(0)(f^{(n)}(\varphi(0)) - f^{(n)}(r_k\varphi(0)))| = 0,$$
$$\limsup_{k\to+\infty} |u(0)\varphi'(0)(f^{(n+1)}(\varphi(0)) - r_k f^{(n+1)}(r_k\varphi(0)))| = 0,$$
$$\limsup_{k\to+\infty} \sup_{|\varphi(z)|\le\delta} \mu(|z|)|u''(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_k\varphi(z))| = 0,$$
$$\limsup_{k\to+\infty} \sup_{|\varphi(z)|\le\delta} \mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)||f^{(n+1)}(\varphi(z)) - r_k f^{(n+1)}(r_k\varphi(z))| = 0,$$
$$\limsup_{k\to+\infty} \sup_{|\varphi(z)|\le\delta} \mu(|z|)|u(z)\varphi'^2(z)||f^{(n+2)}(\varphi(z)) - r_k^2 f^{(n+2)}(r_k\varphi(z))| = 0,$$

where $\delta \in (0,1)$ is fixed. Also, (2.1) implies that for any f with $\|f\|_{p,q,s} \le 1$,

$$\begin{aligned}
&\sup_{\delta<|\varphi(z)|<1} \mu(|z|)|u''(z)||f^{(n)}(\varphi(z)) - f^{(n)}(r_k\varphi(z))| \\
\le &\sup_{\delta<|\varphi(z)|<1} \mu(|z|)|u''(z)||f^{(n)}(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \mu(|z|)|u''(z)||f^{(n)}(r_k\varphi(z))| \\
\preceq &\sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|u''(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} + \sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|u''(z)|}{(1-|r_k\varphi(z)|^2)^{\gamma+n-1}} \\
\le &2 \sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|u''(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\sup_{\delta<|\varphi(z)|<1} \mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)||f^{(n+1)}(\varphi(z)) - r_k f^{(n+1)}(r_k\varphi(z))| \\
\preceq &\sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{\delta<|\varphi(z)|<1} \mu(|z|)|u(z)\varphi'^2(z)||f^{(n+2)}(\varphi(z)) - r_k^2 f^{(n+2)}(r_k\varphi(z))| \\
\preceq &\sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\gamma+n+1}}.
\end{aligned}$$

We get from the above discussion that

$$\begin{aligned}
\sup_{\|f\|_{p,q,s}\le 1} \|(D^n_{u,\varphi} - D^n_{u,r_k\varphi})f\|_{\mathcal{Z}_\mu} \preceq &\sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|u''(z)|}{(1-|\varphi(z)|^2)^{\gamma+n-1}} \\
&+ \sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\gamma+n}} \\
&+ \sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\gamma+n+1}}.
\end{aligned}$$

Now it follows from (3.11) along with $\delta \to 1$ that

$$\|D^n_{u,\varphi}\|_e \preceq \limsup_{|\varphi(z)|\to 1}\left\{A(u'', \gamma+n-1) + A(2u'\varphi' + u\varphi'', \gamma+n) + A(u\varphi'^2, \gamma+n+1)\right\}.$$

□

Remark 3.1. As we noted in the introduction, the operators we use here contain some other well–known operators and also $F(p,q,s)$ spaces include several classical spaces of analytic functions. So some corollaries can be stated here which we do not bring here.

Acknowledgements. The authors would like to thank the referees for their useful comments.

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