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EXISTENCE AND STABILITY ANALYSIS OF SEQUENTIAL COUPLED SYSTEM OF HADAMARD-TYPE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study existence, uniqueness and Hyers-Ulam stability for a sequential coupled system consisting of fractional differential equations of Hadamard type, subject to nonlocal Hadamard fractional integral boundary conditions. The existence of solutions is derived from Leray-Schauder's alternative, whereas the uniqueness of solution is established by Banach contraction principle. An example is also presented which illustrate our results.

1. Introduction

In this paper, we study the sequential coupled system of Hadamard fractional differential equations with nonlocal Hadamard fractional integral boundary conditions of the following form:

$$
\begin{cases}
(D^q + kD^{q-1})u(t) = f(t, u(t), v(t)), \quad k > 0,\ 1 < q \leq 2,\ t \in (1, e),\\
(D^p + kD^{p-1})v(t) = g(t, u(t), v(t)), \quad k > 0,\ 1 < p \leq 2,\ t \in (1, e),\\
u(1) = 0, \quad \displaystyle\sum_{i=1}^{m} \lambda_i I^{\alpha_i} u(\eta_i) = \sum_{j=1}^{n} \mu_j (I^{\beta_j} u(e) - I^{\beta_j} u(\xi_j)),\\
v(1) = 0, \quad \displaystyle\sum_{i=1}^{m} \rho_i I^{\gamma_i} v(\theta_i) = \sum_{j=1}^{n} \kappa_j (I^{\delta_j} v(e) - I^{\delta_j} v(\zeta_j)),
\end{cases} \tag{1.1}
$$

where $D^{(\cdot)}$ denotes the Hadamard fractional derivative of order p and q, $f, g : [1, e] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and $\eta_i, \theta_i, \xi_j, \zeta_j \in (1, e)$ and $\lambda_i, \rho_i, \mu_j, \kappa_j \in \mathbb{R}$,

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$i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and $I^{(\phi)}$ is the Hadamard fractional integral of order $\phi > 0$, $\phi = \alpha_i, \gamma_i, \beta_j, \delta_j$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Fractional calculus is the field of mathematical analysis, which deals with the investigation and applications of integrals and derivatives of an arbitrary order. Fractional differential equations (FDEs) have played a significant role in many engineering and scientific disciplines e.g. as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron analytical chemistry, biology, control theory, fitting of experimental data and so forth [5, 14, 15]. FDEs also serve as an excellent tool for the description of hereditary properties of various materials and processes [18].

The theory of fractional order differential equations, involving different kinds of boundary conditions has been a field of interest in pure and applied sciences. Nonlocal conditions are used to describe certain features of applied mathematics and physics such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics and so on [1, 6, 17].

In the classical text [19], it has been mentioned that Hadamard in 1892 [8] suggested a concept of fractional integro–differentiation in terms of the fractional power of the type $(x\frac{d}{dx})^p$ in contrast to its Riemann-Liouville counterpart of the form $(\frac{d}{dx})^p$. The kind of derivative, introduced by Hadamard contains logarithmic function of arbitrary exponent in the kernel of the integral appearing in its definition. Hadamard's construction is invariant in relation to dilation and is well suited to the problems containing half axes.

Coupled systems of FDEs have also been investigated by many authors. Such systems appear naturally in many real world situations. Some recent results on the topic can be found in a series of papers [2, 3, 9, 12, 20, 25].

Another aspect of FDEs which has very recently got attentions from the researchers is the Ulam type stability analysis of the aforesaid equations. The mentioned stability was first pointed out by Ulam [26] in 1940, which was further explained by Hyers [10], over Banach space. Latter on, many researchers done valuable work on the same task and interesting results were formed for linear and nonlinear integral and differential equations, for detail see [4, 24, 27, 28, 30–32, 35]. This stability analysis is very useful in many applications, such as numerical analysis, optimization, etc., where finding the exact solution is quite difficult. For detailed study of Ulam-type stability with different approaches, we recommend papers [13, 21–24, 29, 33, 34, 36–39].

In addition, the inspiration of this paper comes from the following two problems [11, 16]. In [16], Thiramanus et al. investigated the existence and uniqueness of solutions for a fractional boundary value problem involving Hadamard-type fractional differential equations and nonlocal fractional integral boundary conditions:

$$
\begin{cases} D^q x(t) = f(t, x(t)), \quad 1 < q \leq 2,\ t \in (1, e), \\ x(1) = 0, \\ \sum_{i=1}^{m} \lambda_i I^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n} \mu_j (I^{\beta_j} x(e) - I^{\beta_j} x(\xi_j)), \end{cases} \tag{1.2}
$$

where D^q denotes the Hadamard fractional derivative of order q, $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\eta_i, \xi_j \in (1, e)$, $\lambda_i, \mu_j \in \mathbb{R}$ for all $i = 1, 2, \dots, m$, $j = 1, 2, \cdots, n$, $\eta_1 < \eta_2 < \cdots < \eta_m$, $\xi_1 < \xi_2 < \cdots < \xi_n$ and $I\phi$ is the Hadamard fractional integral of order $\phi > 0$, $\phi = \alpha_i, \beta_j$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

In [11], the authors discussed existence results for Hadamard type fractional functional integro-differential equations with integral boundary conditions:

$$
\begin{cases} D^q y(t) = f(t, y(t), (T_1 y)(t), (T_2 y)(t)), \quad 1 < q \leq 2,\ t \in (1, e), \\ y(1) = 0, \\ \sum_{i=1}^{m} \lambda_i I^{\alpha_i} y(\eta_i) = \sum_{j=1}^{n} \mu_j (I^{\beta_j} x(e) - I^{\beta_j} x(\xi_j)). \end{cases} \tag{1.3}
$$

We show the existence of solutions for problem (1.1) by applying Leray-Schauder alternative criterion while uniqueness of solutions for (1.1) relies on Banach contraction mapping principle.

The rest of the paper is organized as follows: In Section 2, we recall some preliminary concepts which we need in the sequel. Section 3 contains the main results for problem (1.1). In Section 4, we present the Hyers-Ulam stability for problem (1.1).

2. Preliminaries and Background Materials

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs, and then we prove an auxiliary lemma for the linear modification of problem (1.1).

Definition 2.1. The Hadamard derivative of fractional order q for a function $f \in C^n[1, \infty)$, is defined as

$$
D^q f(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt} \right)^n \int_0^t \left(\log \frac{t}{s} \right)^{n-q-1} \frac{f(s)}{s} ds, \quad n - 1 < q < n,
$$

where $n = [q] + 1$, $[q]$ denotes the integer part of q and $\log(\cdot) = \log_e(\cdot)$ provided that integral exists.

Definition 2.2. The Hadamard fractional integral of order q for a function $f : [1, \infty) \to \mathbb{R}$ is defined by

$$
I^q f(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} \frac{f(s)}{s} ds, \quad q > 0,
$$

provided that integral exists, and $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(q) = \int_0^\infty e^{-s} t^{q-1} ds, \quad q > 0.$$

Lemma 2.1. *Let ϕ, ψ are continuous functions from $[1, e]$ to $\mathbb{R}$. Then the solution for the linear system of sequential fractional differential equations:*

$$\begin{cases} (D^q + kD^{q-1})u(t) = \phi(t), & k > 0,\ 1 < q \leq 2,\ t \in (1, e), \\ (D^p + kD^{p-1})v(t) = \psi(t), & 1 < p \leq 2,\ t \in (1, e), \\ u(1) = 0, & \sum_{i=1}^{m} \lambda_i I^{\alpha_i} u(\eta_i) = \sum_{j=1}^{n} \mu_j (I^{\beta_j} u(e) - I^{\beta_j} u(\xi_j)), \\ v(1) = 0, & \sum_{i=1}^{m} \rho_i I^{\gamma_i} v(\theta_i) = \sum_{j=1}^{n} \kappa_j (I^{\delta_j} v(e) - I^{\delta_j} v(\zeta_j)), \end{cases} \tag{2.1}$$

is

$$\begin{aligned} u(t) =& \frac{1}{A_1}\left(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2} ds\right)\Bigg[\sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha_i)\Gamma(q-1)} \\ &\times \int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i - 1} s^{-k-1}\left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2}\frac{\phi(m)}{m}dm\right)dr\right)ds \\ &+ \sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1} s^{-k-1} \\ &\times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2}\frac{\phi(m)}{m}dm\right)dr\right)ds \\ &- \frac{e^{-k}}{\Gamma(q-1)}\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\beta_j)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_j-1} s^{-1} \\ &\times \left(\int_1^e r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2}\frac{\phi(m)}{m}dm\right)dr\right)ds\Bigg] \\ &+ t^{-k}\int_1^t s^{k-1} I^{q-1}\phi(s) ds \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} v(t) =& \frac{1}{A_2}\left(t^{-k}\int_1^t s^{k-1}(\log s)^{p-2} ds\right)\Bigg[\sum_{i=1}^{m} \frac{\rho_i}{\Gamma(\gamma_i)\Gamma(p-1)} \\ &\times \int_1^{\theta_i}\left(\log\frac{\theta_i}{s}\right)^{\gamma_i - 1} s^{-k-1}\left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{p-2}\frac{\psi(m)}{m}dm\right)dr\right)ds \\ &+ \sum_{j=1}^{n}\frac{\kappa_j}{\Gamma(\delta_j)\Gamma(p-1)}\int_1^{\zeta_j}\left(\log\frac{\zeta_j}{s}\right)^{\delta_j-1} s^{-k-1} \\ &\times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{p-2}\frac{\psi(m)}{m}dm\right)dr\right)ds \end{aligned}$$

$$
\begin{aligned}
&- \frac{e^{-k}}{\Gamma(p-1)} \sum_{j=1}^{n} \frac{\kappa_j}{\Gamma(\delta_j)} \int_1^e \left(\log \frac{e}{s}\right)^{\delta_j - 1} s^{-1} \\
&\times \left(\int_1^e r^{k-1} \left(\int_1^r \left(\log \frac{r}{m} \right)^{p-2} \frac{\psi(m)}{m} dm \right) dr \right) ds \Bigg] \\
&+ t^{-k} \int_1^t s^{k-1} I^{p-1} \psi(s) ds, \qquad (2.3)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 = & \sum_{j=1}^{n} \frac{e^{-k}\mu_j}{\Gamma(\beta_j)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta_j - 1} s^{-1} \left(\int_1^e r^{k-1} (\log r)^{q-2} dr \right) ds \\
& - \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\beta_j)} \int_1^{\xi_j} \left(\log \frac{\xi_j}{s}\right)^{\beta_j - 1} s^{-k-1} \left(\int_1^r r^{k-1} (\log r)^{q-2} dr \right) ds \\
& - \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha_i)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha_i - 1} s^{-k-1} \left(\int_1^r r^{k-1} (\log r)^{q-2} dr \right) ds \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
A_2 = & \sum_{j=1}^{n} \frac{e^{-k}\kappa_j}{\Gamma(\delta_j)} \int_1^e \left(\log \frac{e}{s}\right)^{\delta_j - 1} s^{-1} \left(\int_1^e r^{k-1} (\log r)^{p-2} dr \right) ds \\
& - \sum_{j=1}^{n} \frac{\kappa_j}{\Gamma(\delta_j)} \int_1^{\zeta_j} \left(\log \frac{\zeta_j}{s}\right)^{\delta_j - 1} s^{-k-1} \left(\int_1^r r^{k-1} (\log r)^{p-2} dr \right) ds \\
& - \sum_{i=1}^{m} \frac{\rho_i}{\Gamma(\gamma_i)} \int_1^{\theta_i} \left(\log \frac{\theta_i}{s}\right)^{\gamma_i - 1} s^{-k-1} \left(\int_1^r r^{k-1} (\log r)^{p-2} dr \right) ds. \qquad (2.5)
\end{aligned}
$$

Proof. As argued in [14], the general solution of the system (2.1) can be written as

$$u(t) = c_0 t^{-k} + c_1 t^{-k} \int_1^t s^{k-1} (\log s)^{q-2} ds + t^{-k} \int_1^t s^{k-1} I^{q-1} \phi(t) ds \qquad (2.6)$$

and

$$v(t) = d_0 t^{-k} + d_1 t^{-k} \int_1^t s^{k-1} (\log s)^{p-2} ds + t^{-k} \int_1^t s^{k-1} I^{p-1} \psi(t) ds, \qquad (2.7)$$

where c_i, d_i, $i = 0, 1$, are unknown arbitrary constants. The conditions $u(1) = 0$ and $v(1) = 0$ in (2.6) and (2.7) imply that $c_0 = 0$ and $d_0 = 0$, which leads to

$$u(t) = c_1 t^{-k} \int_1^t s^{k-1} (\log s)^{q-2} ds + t^{-k} \int_1^t s^{k-1} I^{q-1} \phi(t) ds \qquad (2.8)$$

and

$$v(t) = d_1 t^{-k} \int_1^t s^{k-1} (\log s)^{p-2} ds + t^{-k} \int_1^t s^{k-1} I^{p-1} \psi(t) ds. \qquad (2.9)$$

Now, using the coupled integral boundary conditions given by (1.1), in (2.8) and (2.9), we obtain

$$c_1 = \frac{J_1}{A_1}, \quad d_1 = \frac{J_2}{A_2}, \tag{2.10}$$

where A_1 and A_2 are respectively given by (2.4) and (2.5), and

$$\begin{aligned}
J_1 =& \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha_i)\Gamma(q-1)} \int_1^{\eta_i} \left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1} s^{-k-1} \\
&\times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2} \frac{\phi(m)}{m} dm\right) dr\right) ds \\
&+ \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\beta_j)\Gamma(q-1)} \int_1^{\xi_j} \left(\log\frac{\xi_j}{s}\right)^{\beta_j-1} s^{-k-1} \\
&\times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2} \frac{\phi(m)}{m} dm\right) dr\right) ds \\
&- \sum_{j=1}^{n} \frac{e^{-k}\mu_j}{\Gamma(\beta_j)} \int_1^e \left(\log\frac{e}{s}\right)^{\beta_j-1} s^{-1}\left(\int_1^e r^{k-1} I^{q-1}\phi(r) dr\right) ds,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
J_2 =& \sum_{i=1}^{m} \frac{\rho_i}{\Gamma(\gamma_i)\Gamma(p-1)} \int_1^{\theta_i} \left(\log\frac{\theta_i}{s}\right)^{\gamma_i-1} s^{-k-1} \\
&\times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{p-2} \frac{\psi(m)}{m} dm\right) dr\right) ds \\
&+ \sum_{j=1}^{n} \frac{\kappa_j}{\Gamma(\delta_j)\Gamma(p-1)} \int_1^{\zeta_j} \left(\log\frac{\zeta_j}{s}\right)^{\delta_j-1} s^{-k-1} \\
&\times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{p-2} \frac{\psi(m)}{m} dm\right) dr\right) ds \\
&- \sum_{j=1}^{n} \frac{e^{-k}\kappa_j}{\Gamma(\delta_j)} \int_1^e \left(\log\frac{e}{s}\right)^{\delta_j-1} s^{-1}\left(\int_1^e r^{k-1} I^{p-1}\psi(r) dr\right) ds.
\end{aligned} \tag{2.12}$$

Substituting the values of c_1 and c_2 in (2.8) and (2.9) we obtained the solutions (2.2) and (2.3). □

3. Main Results

Let $\mathcal{C} = C([1,e],\mathbb{R})$ denotes the Banach space of all continuous functions from $[1,e]$ to $\mathbb{R}$. Let us introduce the space $X = \{u(t) : u(t) \in C^1([1,e])\}$ endowed with the norm $\|u\| = \sup\{|u(t)| : t \in [1,e]\}$. Obviously, $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{v(t) : v(t) \in C^1([1,e])\}$ be endowed with the norm $\|v\| = \sup\{|v(t)| : t \in [1,e]\}$. Obviously the product space $(X \times Y, \|(u,v)\|)$ is a Banach space with norm $\|(u,v)\| =$

$\|u\| + \|v\|$. In view of Lemma 2.1, we define an operator $\mathcal{T} : X \times Y \to X \times Y$ by

$$\mathcal{T}(u,v)(t) = \begin{pmatrix} \mathcal{T}_1(u,v)(t) \\ \mathcal{T}_2(u,v)(t) \end{pmatrix},$$

where

$$\begin{aligned}
\mathcal{T}_1(u,v)(t) =& \frac{1}{A_1}\left(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\right) \\
&\left[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha_i)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1}s^{-k-1}\left(\int_1^s r^{k-1}\left(I^{q-1}f(r,u(r),v(r))\right)dr\right)ds\right. \\
&+\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\beta_j)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1}s^{-k-1}\left(\int_1^s r^{k-1}\left(I^{q-1}f(r,u(r),v(r))\right)dr\right)ds \\
&\left.-\sum_{j=1}^{n}\frac{e^{-k}\mu_j}{\Gamma(\beta_j)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\beta_j-1}s^{-1}\left(\int_1^e r^{k-1}I^{q-1}f(r,u(r),v(r))dr\right)ds\right] \\
&+t^{-k}\int_1^t s^{k-1}I^{q-1}f(s,u(s),v(s))ds
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\mathcal{T}_2(u,v)(t) =& \frac{1}{A_2}\left(t^{-k}\int_1^t s^{k-1}(\log s)^{p-2}ds\right) \\
&\left[\sum_{i=1}^{m}\frac{\rho_i}{\Gamma(\gamma_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{s}\right)^{\gamma_i-1}s^{-k-1}\left(\int_1^s r^{k-1}\left(I^{p-1}g(r,u(r),v(r))\right)dr\right)ds\right. \\
&+\sum_{j=1}^{n}\frac{\kappa_j}{\Gamma(\delta_j)}\int_1^{\zeta_j}\left(\log\frac{\zeta_j}{s}\right)^{\delta_j-1}s^{-k-1}\left(\int_1^s r^{k-1}\left(I^{p-1}g(r,u(r),v(r))\right)dr\right)ds \\
&\left.-\sum_{j=1}^{n}\frac{e^{-k}\kappa_j}{\Gamma(\delta_j)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\delta_j-1}s^{-1}\left(\int_1^e r^{k-1}I^{p-1}g(r,u(r),v(r))dr\right)ds\right] \\
&+t^{-k}\int_1^t s^{k-1}I^{p-1}g(s,u(s),v(s))ds.
\end{aligned} \tag{3.2}$$

For the sake of convenience, we set

$$\begin{aligned}
M_1 =& \frac{1}{|A_1|}\left(\sum_{i=1}^{m}\frac{|\lambda_i|(\log\eta_i)^{\alpha_i+q}}{(q-1)\Gamma(q+1)\Gamma(\alpha_i+1)}+\sum_{j=1}^{n}\frac{|\mu_j|(\log\xi_j)^{\beta_j+q}}{(q-1)\Gamma(q+1)\Gamma(\beta_j+1)}\right) \\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{(q-1)\Gamma(q+1)\Gamma(\beta_j+1)},
\end{aligned} \tag{3.3}$$

$$M_2 = \frac{1}{|A_2|}\left(\sum_{i=1}^{m}\frac{|\rho_i|(\log\theta_i)^{\gamma_i+p}}{(p-1)\Gamma(p+1)\Gamma(\gamma_i+1)}+\sum_{j=1}^{n}\frac{|\kappa_j|(\log\zeta_j)^{\delta_j+p}}{(p-1)\Gamma(p+1)\Gamma(\delta_j+1)}\right)$$

$$+\sum_{j=1}^{n} \frac{|\kappa_j|}{(p-1)\Gamma(p+1)\Gamma(\delta_j+1)} \tag{3.4}$$

and

$$M_0 = \min\{1-(M_1k_1+M_2\lambda_1), 1-(M_1k_2+M_2\lambda_2\}, \quad k_i, \lambda_i \leq 0, \ i=1,2. \tag{3.5}$$

The first result is concerned with the existence and uniqueness of solution for the problem (1.1) and is based on Banach contraction mapping principle.

Theorem 3.1. *Assume that $f, g : [1,e] \times \mathbb{R}^{\not{k}} \to \mathbb{R}$ are continuous functions and there exist constants $m_i, n_i, i = 1,2$ such that for all $t \in [1,e]$ and $u_i, v_i \in \mathbb{R}$, $i = 1,2$, such that*

$$|f(t,u_2,v_2) - f(t,u_1,v_1)| \leq m_1|u_2-u_1| + m_2|v_2-v_1|$$

and

$$|g(t,u_2,v_2) - g(t,u_1,v_1)| \leq n_1|u_2-u_1| + n_2|v_2-v_1|.$$

In addition, assume that

$$M_1(m_1+m_2) + M_2((n_1+n_2) < 1,$$

where M_i, $i=1,2$, are given by (3.3) and (3.4). Then the boundary value problem (1.1) has a unique solution.

Proof. Define $\sup_{t\in[1,e]} f(t,0,0) = N_1 < \infty$ and $\sup_{t\in[1,e]} g(t,0,0) = N_2 < \infty$ such that

$$r > \frac{M_1N_1 + M_2N_2}{1 - M_1(m_1+m_2) + M_2(n_1+n_2)}.$$

We show that $\mathcal{T}\mathbf{B_r} \subset \mathbf{B_r}$, where $\mathbf{B_r} = \{(u,v) \in X \times Y : \|(u,v)\| < r\}$. For $(u,v) \in \mathbf{B_r}$, we have

$$\begin{aligned}
|\mathcal{T}_1(u,v)(t)| = \sup_{t\in[1,e]} &\left\{ \frac{1}{A_1}\left(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\right)\left[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha_i)\Gamma(q-1)}\right.\right. \\
&\times \int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1} s^{-k-1}\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\right.\right. \\
&\times \left.\left.\frac{f(m,u(m),v(m))}{m}dm\right)dr\right)ds + \sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\beta_j)\Gamma(q-1)} \\
&\times \int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1} s^{-k-1}\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\right.\right. \\
&\left.\left.\frac{f(m,u(m),v(m))}{m}dm\right)dr\right)ds - \sum_{j=1}^{n}\frac{e^{-k}\mu_j}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_j-1} s^{-1} \\
&\times \left.\left(\int_1^e r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\frac{f(m,u(m),v(m))}{m}dm\right)dr\right)ds\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\left(\int_1^s\left(\log\frac{s}{r}\right)^{q-2}\frac{f(r,u(r),v(r))}{r}dr\right)ds\Bigg\}\\
\leq&\frac{1}{|A_1|}\left(\int_1^t s^{k-1}(\log s)^{q-2}ds\right)\Bigg[\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\alpha_i)\Gamma(q-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1}s^{-k-1}\\
&\times\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\right.\right.\\
&\times\left.\left.\frac{(|f(m,u(m),v(m))-f(m,0,0)|+|f(m,0,0)|)}{m}dm\right)dr\right)ds\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1}s^{-k-1}\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\right.\right.\\
&\times\left.\left.\frac{(|f(m,u(m),v(m))-f(m,0,0)|+|f(m,0,0)|)}{m}dm\right)dr\right)ds\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\beta_j-1}s^{-1}\left(\int_1^e r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\right.\right.\\
&\times\left.\left.\frac{(|f(m,u(m),v(m))-f(m,0,0)|+|f(m,0,0)|)}{m}dm\right)dr\right)ds\Bigg]\\
&+\frac{1}{\Gamma(q-1)}\int_1^t s^{k-1}\left(\int_1^s\left(\log\frac{s}{r}\right)^{q-2}\right.\\
&\times\left.\frac{(|f(r,u(r),v(r))-f(r,0,0)|+|f(r,0,0)|)}{r}dr\right)ds\\
\leq&(m_1\|u\|+m_2\|v\|+N_1)\\
&\times\Bigg[\frac{1}{|A_1|}\Bigg(\sum_{i=1}^{m}\frac{|\lambda_i|(\log\eta_i)^{\alpha_i+q}}{(q-1)\Gamma(\alpha_i+1)\Gamma(q+1)}\\
&+\sum_{j=1}^{n}\frac{|\mu_j|(\log\xi_j)^{\beta_j+q}}{(q-1)\Gamma(\beta_j+1)\Gamma(q+1)}\Bigg)\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{(q-1)\Gamma(\beta_j+1)\Gamma(q+1)}\Bigg]\\
=&M_1[m_1\|u\|+m_2\|v\|+N_1]\\
=&M_1[(m_1+m_2)r+N_1].
\end{aligned}
$$

Hence,

$$|\mathcal{T}_1(u,v)(t)|\leq M_1[(m_1+m_2)r+N_1].$$

In the same way, we can obtain that

$$|\mathcal{T}_2(u,v)(t)|\leq M_2[(n_1+n_2)r+N_2].$$

Consequently, $|\mathcal{T}(u,v)(t)| \leq r$. Now, for $(u_1,v_1),(u_2,v_2) \in X\times Y$, and for any $t\in[1,e]$, we get

$$
\begin{aligned}
&|\mathcal{T}_1(u_2,v_2)(t)-\mathcal{T}_1(u_1,v_1)(t)|\\
\leq&\frac{1}{|A_1|}\left(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\right)\Bigg[\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\alpha_i)\Gamma(q-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1}s^{-k-1}\\
&\times\Bigg(\int_1^s r^{k-1}\Bigg(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\\
&\frac{|f(m,u_2(m),v_2(m))-f(m,u_1(m),v_1(m))|}{m}dm\Bigg)dr\Bigg)ds\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1}s^{-k-1}\\
&\times\Bigg(\int_1^s r^{k-1}\Bigg(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\\
&\frac{|f(m,u_2(m),v_2(m))-f(m,u_1(m),v_1(m))|}{m}dm\Bigg)dr\Bigg)ds\\
&\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\beta_j-1}s^{-1}\\
&\times\Bigg(\int_1^e r^{k-1}\Bigg(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\\
&\frac{|f(m,u_2(m),v_2(m))-f(m,u_1(m),v_1(m))|}{m}dm\Bigg)dr\Bigg)ds\Bigg]\\
&+\frac{1}{\Gamma(q-1)}\int_1^t s^{k-1}\Bigg(\int_1^s\left(\log\frac{s}{r}\right)^{q-2}\\
&\frac{|f(r,u_2(r),v_2(r))-f(r,u_1(r),v_1(r))|}{r}dr\Bigg)ds\\
\leq&M_1[m_1\|u_2-u_1\|+m_2\|v_2-v_1\|]\\
\leq&M_1(m_1+m_2)(\|u_2-u_1\|+\|v_2-v_1\|),
\end{aligned}
$$

and consequently, we obtain

$$|\mathcal{T}_1(u_2,v_2)(t)-\mathcal{T}_1(u_1,v_1)(t)| \leq M_1(m_1+m_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{3.6}$$

Similarly,

$$|\mathcal{T}_2(u_2,v_2)(t)-\mathcal{T}_2(u_1,v_1)(t)| \leq M_2(n_1+n_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$|\mathcal{T}(u_2,v_2)(t)-\mathcal{T}(u_1,v_1)(t)| \leq[M_1(m_1+m_2)+M_2(n_1+n_2)]$$

$$\times (\|u_2 - u_1\| + \|v_2 - v_1\|). \tag{3.8}$$

Since $M_1(m_1 + m_2) + M_2(n_1 + n_2) < 1$, therefore, $\mathcal{T}$ is a contraction operator. So, by Banach fixed point theorem, the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of problem (1.1). This completes the proof. □

In the next result, we prove the existence of solutions for the problem (1.1) by applying Leray-Schauder alternative.

Lemma 3.1 (Leray-Schauder alternative [7])**.** *Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\aleph(F) = \{x \in E \mid x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\aleph(F)$ is unbounded or F has at least one fixed point.

Theorem 3.2. *Assume that there exist real constants $k_i, \lambda_i > 0$, $i = 1, 2$, and $k_0 > 0$, $\lambda_0 > 0$ such that for all $x_i \in \mathbb{R}$, $i = 1, 2$, we have*

$$|f(t, x_1, x_2)| \le k_0 + k_1|x_1| + k_2|x_2|$$

and

$$|g(t, x_1, x_2)| \le \lambda_0 + \lambda_1|x_1| + \lambda_2|x_2|.$$

In addition, also assume that

$$(M_1k_1 + M_2\lambda_1) \le 1 \quad \text{and} \quad (M_1k_2 + M_2\lambda_2) \le 1,$$

where M_i, $i = 1, 2$, are given by (3.3) *and* (3.4)*. Then there exists at least one solution for the boundary value problem* (1.1)*.*

Proof. First we show that the operator $\mathcal{T} : X \times Y \to X \times Y$ is completely continuous. By continuity of functions f and g the operator $\mathcal{T}$ is continuous.

Let $\Theta \subset X \times Y$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t, u(t), v(t))| \le L_1, \quad |g(t, u(t), v(t))| \le L_2, \quad \text{for all } (u, v) \in \Theta.$$

For any $(u, v) \in \Theta$, we have

$$\begin{aligned}
\|\mathcal{T}_1(u,v)(t)\| \le & \frac{1}{|A_1|}\left(\int_1^t s^{k-1}(\log s)^{q-2}ds\right)\left[\sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\alpha_i)\Gamma(q-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1} s^{-k-1}\right. \\
& \times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2}\frac{|f(m,u(m),v(m))|}{m}dm\right)dr\right)ds \\
& + \sum_{j=1}^n \frac{|\mu_j|}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1} s^{-k-1} \\
& \times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2}\frac{|f(m,u(m),v(m))|}{m}dm\right)dr\right)ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_j-1}s^{-1}\\
&\times\left(\int_1^e r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\frac{(|f(m,u(m),v(m))|}{m}dm\right)dr\right)ds\Bigg]\\
&+\frac{1}{\Gamma(q-1)}\int_1^t s^{k-1}\left(\int_1^s\left(\log\frac{s}{r}\right)^{q-2}\frac{(|f(r,u(r),v(r))|}{r}dr\right)ds\\
\le&\frac{L_1}{|A_1|}\left(\int_1^t s^{k-1}(\log s)^{q-2}ds\right)\Bigg[\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\alpha_i)\Gamma(q-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1}s^{-k-1}\\
&\times\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\frac{1}{m}dm\right)dr\right)ds\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1}s^{-k-1}\\
&\times\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\frac{1}{m}dm\right)dr\right)ds\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_j-1}s^{-1}\\
&\times\left(\int_1^e r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\frac{1}{m}dm\right)dr\right)ds\Bigg]\\
&+\frac{L_1}{\Gamma(q-1)}\int_1^t s^{k-1}\left(\int_1^s\left(\log\frac{s}{r}\right)^{q-2}\frac{1}{r}dr\right)ds,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\mathcal{T}_1(u,v)(t)\|\le&L_1\Bigg\{\frac{1}{|A_1|}\Bigg(\sum_{i=1}^{m}\frac{|\lambda_i|(\log\eta_i)^{\alpha_i+q}}{(q-1)\Gamma(\alpha_i+1)\Gamma(q+1)}\\
&+\sum_{j=1}^{n}\frac{|\mu_j|(\log\xi_j)^{\beta_j+q}}{(q-1)\Gamma(\beta_j+1)\Gamma(q+1)}\Bigg)\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{(q-1)\Gamma(\beta_j+1)\Gamma(q+1)}\Bigg\}\\
=&L_1M_1.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
\|\mathcal{T}_2(u,v)(t)\|\le&L_2\Bigg\{\frac{1}{|A_2|}\Bigg(\sum_{i=1}^{m}\frac{|\rho_i|(\log\theta_i)^{\gamma_i+p}}{(p-1)\Gamma(\gamma_i+1)\Gamma(p+1)}\\
&+\sum_{j=1}^{n}\frac{|\kappa_j|(\log\zeta_j)^{\delta_j+p}}{(p-1)\Gamma(\delta_j+1)\Gamma(p+1)}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\frac{|\kappa_j|}{(p-1)\Gamma(\delta_j+1)\Gamma(p+1)}\Bigg\}\\
=&L_2M_2.
\end{aligned}
$$

Thus, it follows from the above inequalities that the operator $\mathcal{T}$ is uniformly bounded.

Next, we show that $\mathcal{T}$ is equicontinuous. Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$. Then we have

$$
\begin{aligned}
&|\mathcal{T}_1(u(t_2), v(t_2)) - \mathcal{T}_1(u(t_1), v(t_1))|\\
\leq&\frac{L_1}{|A_1|}\left(\frac{|t_1^k - t_2^k|}{t_1^k t_2^k}\int_1^{t_1} s^{k-1}(\log s)^{q-2}ds + t_2^{-k}\int_{t_1}^{t_2} s^{k-1}(\log s)^{q-2}ds\right)\\
&\times\left[\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\alpha_i)\Gamma(q-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1} s^{-k-1}\right.\\
&\times\left(\int_1^{s} r^{k-1}\left(\int_1^{r}\left(\log\frac{r}{m}\right)^{q-2}\frac{1}{m}dm\right)dr\right)ds\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1} s^{-k-1}\\
&\times\left(\int_1^{s} r^{k-1}\left(\int_1^{r}\left(\log\frac{r}{m}\right)^{q-2}\frac{1}{m}dm\right)dr\right)ds\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\beta_j-1} s^{-1}\\
&\left.\times\left(\int_1^{e} r^{k-1}\left(\int_1^{r}\left(\log\frac{r}{m}\right)^{q-2}\frac{1}{m}dm\right)dr\right)ds\right]\\
&+\frac{L_1}{\Gamma(q-1)}\left[\frac{|t_1^k - t_2^k|}{t_1^k t_2^k}\int_1^{t_1} s^{k-1}\left(\int_1^{s}\left(\log\frac{s}{r}\right)^{q-2}\frac{1}{r}dr\right)ds\right.\\
&\left.+t_2^{-k}\int_{t_1}^{t_2} s^{k-1}\left(\int_1^{s}\left(\log\frac{s}{r}\right)^{q-2}\frac{1}{r}dr\right)ds\right],
\end{aligned}
$$

which implies that

$$
|\mathcal{T}_1(u(t_2), v(t_2)) - \mathcal{T}_1(u(t_1), v(t_1))| \to 0 \quad \text{as} \quad t_2 - t_1 \to 0. \tag{3.9}
$$

Analogously, we can obtain

$$
|\mathcal{T}_2(u(t_2), v(t_2)) - \mathcal{T}_2(u(t_1), v(t_1))| \to 0 \quad \text{as} \quad t_2 - t_1 \to 0. \tag{3.10}
$$

From (3.9) and (3.10), it is obvious that $\mathcal{T}$ is equicontinuous and thus is completely continuous.

Finally, it will be verified that the set $\aleph = \{(u, v) \in X \times Y : (u, v) = \lambda\mathcal{T}(u, v),\ 0 \leq \lambda \leq 1\}$ is bounded. Let $(u, v) \in \aleph$, then $(u, v) = \lambda\mathcal{T}(u, v)$. For any $t \in [1, e]$, we have

$$
u(t) = \lambda\mathcal{T}_1(u, v)(t), \quad v(t) = \lambda\mathcal{T}_2(u, v)(t).
$$

Then

$$\begin{aligned}|u(t)| \leq &(k_0 + k_1\|u\| + k_2\|v\|)\Bigg[\frac{1}{|A_1|}\Bigg(\sum_{i=1}^{m}\frac{|\lambda_i|(\log\eta_i)^{\alpha_i+q}}{(q-1)\Gamma(\alpha_i+1)\Gamma(q+1)}\\
&+\sum_{j=1}^{n}\frac{|\mu_j|(\log\xi_j)^{\beta_j+q}}{(q-1)\Gamma(\beta_j+1)\Gamma(q+1)}\Bigg)+\sum_{j=1}^{n}\frac{|\mu_j|}{(q-1)\Gamma(\beta_j+1)\Gamma(q+1)}\Bigg]\end{aligned}$$

and

$$\begin{aligned}|v(t)| \leq &(\lambda_0 + \lambda_1\|u\| + \lambda_2\|v\|)\Bigg[\frac{1}{|A_2|}\Bigg(\sum_{i=1}^{m}\frac{|\rho_i|(\log\theta_i)^{\gamma_i+p}}{(p-1)\Gamma(\gamma_i+1)\Gamma(p+1)}\\
&+\sum_{j=1}^{n}\frac{|\kappa_j|(\log\zeta_j)^{\delta_j+p}}{(p-1)\Gamma(\delta_j+1)\Gamma(p+1)}\Bigg)+\sum_{j=1}^{n}\frac{|\kappa_j|}{(p-1)\Gamma(\delta_j+1)\Gamma(p+1)}\Bigg].\end{aligned}$$

Hence, we have

$$\|u(t)\| \leq (k_0 + k_1\|u\| + k_2\|v\|)M_1$$

and

$$\|v(t)\| \leq (\lambda_0 + \lambda_1\|u\| + \lambda_2\|v\|)M_2,$$

which implies that

$$\|u(t)\| + \|v(t)\| \leq (M_1k_0 + M_2\lambda_0) + (M_1k_1 + M_2\lambda_1)\|u\| + (M_1k_2 + M_2\lambda_2)\|v\|.$$

Consequently,

$$\|(u,v)\| \quad \leq \quad \frac{M_1k_0 + M_2\lambda_0}{M_0},$$

for any $t \in [1,e]$, where M_0 is defined by (3.5), which proves that $\aleph$ is bounded. Thus, by Lemma 3.1 the operator $\mathcal{T}$ has at least one fixed point. Hence, the boundary value problem (1.1) has at least one solution. The proof is complete. ☐

4. Hyers-Ulam Stability of System (1.1)

This section is devoted to the investigation of the Hyers-Ulam stability of our proposed system. Let $\varepsilon_1, \varepsilon_2 > 0$ such that:

$$\begin{cases} |(D^q + kD^{q-1})u(t) - f(t,u(t),v(t))| \leq \varepsilon_1, & t \in [1,e],\\ |(D^p + kD^{p-1})v(t) - g(t,u(t),v(t))| \leq \varepsilon_2, & t \in [1,e]. \end{cases} \tag{4.1}$$

Definition 4.1. Problem (1.1) is said to be Hyers-Ulam stable if there exist $M_i > 0$, $i = 1,2$, such that, for given $\varepsilon_1, \varepsilon_2 > 0$ and for each solution $(u,v) \in C([1,e]\times\mathbb{R}^{\nVdash},\mathbb{R})$ of inequality (4.1), there exists a solution $(u^*,v^*) \in C([1,e]\times\mathbb{R}^{\nVdash},\mathbb{R})$ of problem (1.1) with

$$\begin{cases} |u(t) - u^*(t)| \leq M_1\varepsilon_1, & t \in [1,e],\\ |u(t) - v^*(t)| \leq M_2\varepsilon_2, & t \in [1,e]. \end{cases} \tag{4.2}$$

Remark 4.1. A (u,v) is a solution of inequality (4.1) if there exist functions $Q_i \in C([1,e],\mathbb{R})$, $i = 1,2$ which depend on u,v respectively such that

- $|Q_1(t)| \leq \varepsilon_1$, $|Q_2(t)| \leq \varepsilon_2$, $t \in [1, e]$, and
-

$$\begin{cases} (D^q + kD^{q-1})u(t) = f(t, u(t), v(t)) + Q_1(t), & t \in [1, e], \\ (D^p + kD^{p-1})v(t) = g(t, u(t), v(t)) + Q_2(t), & t \in [1, e]. \end{cases} \tag{4.3}$$

Remark 4.2. If (x, y) represents a solution of inequality (4.1), then (x, y) is a solution of following inequality:

$$\begin{cases} |x(t) - x^*(t)| \leq M_1\varepsilon_1, & t \in [1, e], \\ |y(t) - y^*(t)| \leq M_2\varepsilon_2, & t \in [1, e]. \end{cases} \tag{4.4}$$

As from Remark 4.1, we have

$$\begin{cases} (D^q + kD^{q-1})u(t) = f(t, u(t), v(t)) + Q_1(t), & t \in [1, e], \\ (D^p + kD^{p-1})v(t) = g(t, u(t), v(t)) + Q_2(t), & t \in [1, e]. \end{cases} \tag{4.5}$$

With the help of Definition 4.1 and Remark 4.1, we verified Remark 4.2, in the following lines

$$\begin{aligned}
&\Bigg| u(t) - \frac{1}{A_1}\left(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\right)\Bigg[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha_i)\Gamma(q-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1}s^{-k-1} \\
&\times\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\frac{f(m,u(m),v(m))}{m}dm\right)dr\right)ds \\
&+\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1}s^{-k-1} \\
&\times\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\frac{f(m,u(m),v(m))}{m}dm\right)dr\right)ds \\
&-\sum_{j=1}^{n}\frac{e^{-k}\mu_j}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\beta_j-1}s^{-1} \\
&\times\left(\int_1^e r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\frac{f(m,u(m),v(m))}{m}dm\right)dr\right)ds\Bigg] \\
&-\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\left(\int_1^s\left(\log\frac{s}{r}\right)^{q-2}\frac{f(r,u(r),v(r))}{r}dr\right)ds\Bigg| \\
=&\Bigg|\frac{1}{A_1}\left(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\right)\Bigg[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha_i)\Gamma(q-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1}s^{-k-1} \\
&\times\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\frac{[f(m,u(m),v(m))+Q_1(m)]}{m}dm\right)dr\right)ds \\
&+\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1}s^{-k-1}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2}\frac{[f(m,u(m),v(m))+Q_1(m)]}{m}dm\right)dr\right)ds\\
&-\sum_{j=1}^{n}\frac{e^{-k}\mu_j}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_j-1}s^{-1}\\
&\times \left(\int_1^e r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2}\frac{[f(m,u(m),v(m))+Q_1(m)]}{m}dm\right)dr\right)ds\Bigg]\\
&+\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\left(\int_1^s\left(\log\frac{s}{r}\right)^{q-2}\frac{[f(r,u(r),v(r))+Q_1(r)]}{r}dr\right)ds\\
&-\frac{1}{A_1}\left(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\right)\Bigg[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha_i)\Gamma(q-1)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha_i-1}s^{-k-1}\\
&\times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2}\frac{f(m,u(m),v(m))}{m}dm\right)dr\right)ds\\
&+\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{s}\right)^{\beta_j-1}s^{-k-1}\\
&\times \left(\int_1^s r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2}\frac{f(m,u(m),v(m))}{m}dm\right)dr\right)ds\\
&-\sum_{j=1}^{n}\frac{e^{-k}\mu_j}{\Gamma(\beta_j)\Gamma(q-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_j-1}s^{-1}\\
&\times \left(\int_1^e r^{k-1}\left(\int_1^r \left(\log\frac{r}{m}\right)^{q-2}\frac{f(m,u(m),v(m))}{m}dm\right)dr\right)ds\Bigg]\\
&-\frac{t^{-k}}{\Gamma(q-1)}\int_1^t s^{k-1}\left(\int_1^s\left(\log\frac{s}{r}\right)^{q-2}\frac{f(r,u(r),v(r))}{r}dr\right)ds\Bigg|\\
\le&\varepsilon_1\Bigg[\frac{1}{|A_1|}\left(\sum_{i=1}^{m}\frac{|\lambda_i|(\log\eta_i)^{\alpha_i+q}}{(q-1)\Gamma(\alpha_i+1)\Gamma(q+1)}+\sum_{j=1}^{n}\frac{|\mu_j|(\log\xi_j)^{\beta_j+q}}{(q-1)\Gamma(\beta_j+1)\Gamma(q+1)}\right)\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{(q-1)\Gamma(\beta_j+1)\Gamma(q+1)}\Bigg]\\
&+\varepsilon_2\Bigg[\frac{1}{|A_2|}\left(\sum_{i=1}^{m}\frac{|\rho_i|(\log\theta_i)^{\gamma_i+p}}{(p-1)\Gamma(\gamma_i+1)\Gamma(p+1)}+\sum_{j=1}^{n}\frac{|\kappa_j|(\log\zeta_j)^{\delta_j+p}}{(p-1)\Gamma(\delta_j+1)\Gamma(p+1)}\right)\\
&+\sum_{j=1}^{n}\frac{|\kappa_j|}{(p-1)\Gamma(\delta_j+1)\Gamma(p+1)}\Bigg]\\
=&M_1\varepsilon_1. \qquad (4.6)
\end{aligned}
$$

By the same method, we can obtain that

$$|y(t) - y^*(t)| \leq M_2 \varepsilon_2, \tag{4.7}$$

where M_i, $i = 1, 2$, are given by (3.3) and (3.4). Hence, Remark 4.2 is verified, with the help of (4.4) and (4.5). Thus the nonlinear sequential coupled system of Hadamard fractional differential equations is Hyers-Ulam stable and consequently, the system (1.1) is Hyers-Ulam stable.

Example 4.1. Consider the following coupled system of Hadamard fractional differential equation:

(4.8)
$$\begin{cases} (D^{3/2} + 3D^{1/2})u(t) = \dfrac{|u(t)|}{(t+3)^4 + (1 + |u(t)|)} + \dfrac{1}{27(1 + |v(t)|)} + \dfrac{1}{81}, \quad t \in [1, e], \\ (D^{3/2} + 3D^{1/2})v(t) = \dfrac{\sin(2\pi |u(t)|)}{40\pi} + \dfrac{1}{10\sqrt{t+4}} + \dfrac{|v(t)|}{60(1 + |v(t)|)}, \quad t \in [1, e], \\ u(1) = 0, \\ 2I^{1/4}u(5/4) + \frac{1}{5}I^{3/2}u(9/5) + 3I^{1/2}u(15/7) \\ = I^{2/3}u(e) - I^{2/3}u(10/7) + 5(I^{9/7}u(e) - I^{9/7}u(2)) - 2(I^{11/4}u(e) - I^{11/4}u(9/4)), \\ v(1) = 0, \\ \frac{1}{4}I^{7/6}v(7/3) - \frac{2}{3}I^{1/2}v(7/5) - 2I^{5/2}v(2) \\ = 4(I^5 v(e) - I^5 v(11/5)) + \frac{11}{4}(I^{3/4}v(e) - I^{3/4}v(13/16)) - \frac{1}{2}(I^{7/4}v(e) \\ -I^{7/4}v(1/3)). \end{cases}$$

Here, $q = p = 3/2$, $n = 3$, $m = 3$, $k = 3$, $\lambda_1 = 2$, $\lambda_2 = 1/5$, $\lambda_3 = 3$, $\alpha_1 = 1/4$, $\alpha_2 = 3/2$, $\alpha_3 = 1/2$, $\eta_1 = 5/4$, $\eta_2 = 9/5$, $\eta_3 = 15/7$, $\mu_1 = 1$, $\mu_2 = 5$, $\mu_3 = -2$, $\beta_1 = 2/3$, $\beta_2 = 9/7$, $\beta_3 = 11/4$, $\xi_1 = 10/7$, $\xi_2 = 2$, $\xi_3 = 9/4$, $\rho_1 = 1/4$, $\rho_2 = -2/3$, $\rho_3 = -2$, $\gamma_1 = 7/6$, $\gamma_2 = 1/2$, $\gamma_3 = 5/2$, $\theta_1 = 7/3$, $\theta_2 = 7/5$, $\theta_3 = 2$, $\kappa_1 = 4$, $\kappa_2 = 11/4$, $\kappa_3 = -1/2$, $\delta_1 = 5$, $\delta_2 = 3/4$, $\delta_3 = 7/4$, $\zeta_1 = 11/5$, $\zeta_2 = 13/16$, $\zeta_3 = 1/3$. Thus,

$$f(t, u, v) = (u(t))/((t+3)^4 + (1 + |u(t)|)) + 1/(27(1 + |v(t)|)) + 1/81$$

and

$$g(t, u, v) = (\sin(2\pi |u(t)|))/40\pi + 1/(10\sqrt{t+4}) + (|v(t)|)/(60(1 + |v(t)|)),$$

which implies

$$|f(t, u_2, v_2) - f(t, u_1, v_1)| \leq (1/81)|u_2 - u_1| + (1/27)|v_2 - v_1|$$

and

$$|g(t, u_2, v_2) - g(t, u_1, v_1)| \leq (1/20)|u_2 - u_1| + (1/60)|v_2 - v_1|.$$

Clearly, $m_1 = 1/81$, $m_2 = 1/27$, $n_1 = 1/20$, $n_2 = 1/60$, $M_1 \simeq 5.695$, $M_2 \simeq 6.785$, and

$$[M_1(m_1 + m_2) + M_2(n_1 + n_2)] \simeq 0.732 < 1.$$

Thus, all the conditions of Theorem 3.1 are satisfied. Therefore, by Theorem 3.1, the problem (4.8) has a unique solution on $[1, e]$. Further, it is also straightforward to prove that the problem (4.8) is Hyers-Ullam stable.

5. Conclusion

We have discussed the existence and Hyers-Ulam stability for a sequential coupled system consisting of fractional differential equations of Hadamard type, subjected to nonlocal Hadamard fractional integral boundary conditions. The existence and uniqueness of solutions rely on Banach's contraction principle, while the existence of solutions is established by applying Leray-Schauder's alternative. As an application, an example is presented to illustrate the main results.

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