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NECESSARY AND SUFFICIENT CONDITION FOR OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, necessary and sufficient conditions are obtained for oscillatory and asymptotic behaviour of solutions of second-order neutral delay differential equations of the form

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}[x(t)+p(t)x(\tau(t))]\right]+q(t)G\left(x(\sigma(t))\right)=0, \quad \text{for } t\geq t_0,$$

under the assumption $\int^{\infty}\frac{1}{r(\eta)}d\eta=\infty$ for various ranges of the bounded neutral coefficient p. Our main tools are Lebesgue's dominated convergence theorem and Banach's contraction mapping principle. Further, an illustrative example showing the applicability of the new results is included.

1. Introduction

Consider a class of nonlinear neutral delay differential equations of the form:

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\Big[x(t)+p(t)x(\tau(t))\Big]\right]+q(t)G\Big(x(\sigma(t))\Big)=0, \tag{1.1}$$

where

$(A1)$ $r,q,\tau,\sigma\in C(\mathbb{R}_+,\mathbb{R}_+)$, $p\in C(\mathbb{R}_+,\mathbb{R})$ such that $\tau(t)\leq t$, $\sigma(t)\leq t$ for $t\geq t_0$, $\tau(t)\to\infty$, $\sigma(t)\to\infty$ as $t\to\infty$, with invertible τ when necessary;

$(A2)$ $G\in C(\mathbb{R},\mathbb{R})$ is nondecreasing with satisfying the property $uG(u)>0$ for $u\neq 0$

and

$(A3)$ $R(t)=\int_0^t\frac{d\eta}{r(\eta)}\to+\infty$ as $t\to\infty$.

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Baculikova *et al.* [3] have studied the linear counterpart of (1.1),

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}[x(t)+p(t)x(\tau(t))]\right]+q(t)x(\sigma(t))=0, \tag{1.2}$$

when $0 \le p(t) \le p_0 < \infty$ and $(A3)$ holds. The authors have obtained sufficient conditions for oscillation of solutions of (1.2) through some comparison results, where the comparison results are unpredictable. In [6], Džurina have studied (1.2) when $0 \le p(t) \le p_0 < \infty$ and $(A3)$ holds true. He has established sufficient condition for oscillation of solutions of (1.2) by comparison techniques. In [16], under various ranges of p, Santra studied oscillatory behaviour of the solutions of the following neutral differential equations

$$\frac{d}{dt}[x(t)+p(t)x(t-\tau)]+q(t)G\left(x(t-\sigma)\right)=0$$

and

$$\frac{d}{dt}[x(t)+p(t)x(t-\tau)]+q(t)G\left(x(t-\sigma)\right)=f(t). \tag{1.3}$$

Also, sufficient conditions are obtained for existence of bounded positive solutions of (1.3). Tripathy *et al.* [18] have studied and obtained the sufficient conditions for oscillation, nonoscillation and asymptotic behavior of solutions of (1.1) provided G could be linear or nonlinear. The motivation of the present work come from the above studies. Hence, in this work, an attempt is made to study the more general form of (1.2) without making any comparison. It seems that this method is the next alternative to the works [3, 6] when p is bounded.

The neutral differential equations find numerous applications in natural sciences and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines (see, for e.g., [8]). In this paper, we restrict our attention to study (1.1), which includes a class of nonlinear functional differential equations of neutral type. In this direction we refer the reader to some of the works (see [1, 4, 5, 10, 13, 19, 20]) and the references cited therein.

By a solution to equation (1.1), we mean a function $x \in \mathrm{C}([T_x,\infty),\mathbb{R})$, where $T_x \ge t_0$, such that $rz' \in \mathrm{C}^1([T_x,\infty),\mathbb{R})$, where

$$z(t) := x(t)+p(t)x(\tau(t)), \quad \text{for } t \ge T_x, \tag{1.4}$$

and satisfies (1.1) on the interval $[T_x,\infty)$. A solution x of (1.1) is said to be proper if x is not identically zero eventually, i.e., $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is said to be nonoscillatory. (1.1) itself is said to be oscillatory if all of its solutions are oscillatory.

Remark 1.1. When the domain is not specified explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all t large enough.

2. Main Results

In this section, necessary and sufficient conditions are obtained for oscillatory and asymptotic behaviour of solutions of second order nonlinear neutral differential equations of the form (1.1).

Lemma 2.1. *Assume that* $(A1)$-$(A3)$ *hold. If* x *is an eventually positive solution of* (1.1) *such that the companion function* z *defined by* (1.4) *is also eventually positive, then* z *satisfies*

$$z'(t) > 0 \quad \textit{and} \quad (rz')'(t) < 0, \quad \textit{for all large } t. \tag{2.1}$$

Proof. Suppose that $x(t) > 0$ and $z(t) > 0$ for $t \geq t_1$, where $t \geq t_0$. By $(A1)$, we may assume without loss of generality that $x\big(\sigma(t)\big) > 0$ for $t \geq t_1$. From (1.1) and $(A2)$, it follows that

$$(rz')'(t) = -q(t)G\left(x(\sigma(t))\right) < 0, \quad \text{for } t \geq t_1. \tag{2.2}$$

Consequently, rz' is nonincreasing on $[t_1, \infty)$ and thus either $z'(t) < 0$ or $z'(t) > 0$ for $t \geq t_2$, where $t_2 \geq t_1$. If $z'(t) < 0$, then there exists $\varepsilon > 0$ such that $r(t)z'(t) \leq -\varepsilon$ for $t \geq t_2$, which yields upon integration over $[t_2, t) \subset [t_2, \infty)$ after dividing through by r that

$$z(t) \leq z(t_2) - \varepsilon \int_{t_2}^{t} \frac{1}{r(\eta)} d\eta, \quad \text{for } t \geq t_2. \tag{2.3}$$

In view of $(A3)$, letting $t \to \infty$ in (2.3) yields $z(t) \to -\infty$, which is a contradiction. Therefore, $z'(t) > 0$ for $t \geq t_2$. This completes the proof. ☐

Remark 2.1. It follows from Lemma 2.1 that $\lim_{t\to\infty} z(t) > 0$, i.e., there exists $\varepsilon > 0$ such that $z(t) \geq \varepsilon$ for all large t.

Lemma 2.2. *Assume that* $(A1)$-$(A3)$ *hold. If* x *is an eventually positive solution of* (1.1) *such that the companion function* z *defined by* (1.4) *is bounded, then* z *satisfies* (2.1) *for all large* t.

Theorem 2.1. *Assume that* $(A1)$-$(A3)$ *hold and* $-1 < -a \leq p(t) \leq 0$, $a \geq 0$ *for* $t \in \mathbb{R}_+$. *Furthermore, assume that*

$(A4)$ *G is strictly sublinear, that is,* $\frac{G(u)}{u^\beta} \geq \frac{G(v)}{v^\beta}$, $0 < u \leq v$, $\beta < 1$,

holds. Then every unbounded solution of (1.1) *oscillates if and only if*

$(A5)$ $\int_T^\infty q(\eta)G\Big(\varepsilon R(\sigma(\eta))\Big)d\eta = +\infty$, $T > 0$ *for every* $\varepsilon > 0$.

Proof. Suppose the contrary that x is a nonoscillatory solution of (1.1). Then, there exists $t_1 \geq t_0$ such that either $x(t) > 0$ or $x(t) < 0$ for $t \geq t_1$. Assume that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see rz' is nonincreasing and z is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. We have the following two possible cases.

Case 1. Let $z(t) < 0$ for $t \geq t_2$. As x is unbounded, there exists $T \geq t_2$ such that $x(T) = \max\{x(\eta) : \ t_2 \leq \eta \leq T\}$. Then, from (1.4), we have $x(T) \leq z(T) + x(\tau(T)) < x(T)$, which is a contradiction.

Case 2. Let $z(t) > 0$ for $t \geq t_2$. By Lemma 2.1, (2.1) holds for $t \geq t_3$. Note that $\lim_{t\to\infty} r(t)z'(t)$ exists. Upon using $z(t) \leq x(t)$ in (2.2) and then integrating the final inequality from t to $+\infty$, we obtain

$$\int_t^\infty q(\eta)G\Big(z(\sigma(\eta))\Big)d\eta \leq r(t)z'(t),$$

that is,

$$z'(t) \geq \frac{1}{r(t)} \int_t^\infty q(\eta)G\Big(z(\sigma(\eta))\Big)d\eta, \tag{2.4}$$

for $t \geq t_3$. Let $t_4 > t_3$ be a point such that

$$R(t) - R(t_4) \geq \frac{1}{2}R(t), \quad t \geq t_4.$$

Then integrating (2.4) from t_4 to $t(> t_4)$, we get

$$\begin{aligned} z(t) - z(t_4) &\geq \int_{t_4}^t \frac{1}{r(\eta)} \int_\eta^\infty q(\zeta)G\Big(z(\sigma(\zeta))\Big)d\zeta d\eta \\ &\geq \int_{t_4}^t \frac{1}{r(\eta)} \int_t^\infty q(\zeta)G\Big(z(\sigma(\zeta))\Big)d\zeta d\eta, \end{aligned}$$

that is,

$$\begin{aligned} z(t) \geq& \Big(R(t) - R(t_4)\Big) \int_t^\infty q(\zeta)G\Big(z(\sigma(\zeta))\Big)d\zeta \\ \geq& \frac{1}{2}R(t) \int_t^\infty q(\zeta)G\Big(z(\sigma(\zeta))\Big)d\zeta, \quad t \geq t_4. \end{aligned} \tag{2.5}$$

Using the fact that $r(t)z'(t)$ is nonincreasing on $[t_4, \infty)$, we can find a constant $\varepsilon > 0$ and $t_5 > t_4$ such that $r(t)z'(t) \leq \varepsilon$ for $t \geq t_5$ and hence $z(t) \leq \varepsilon R(t)$, $t \geq t_5$. On the otherhand, $(A3)$ implies that

$$\begin{aligned} G\Big(z(\sigma(\zeta))\Big) &= \frac{G\Big(z(\sigma(\zeta))\Big)}{z^\beta\Big(\sigma(\zeta)\Big)} z^\beta\Big(\sigma(\zeta)\Big) \\ &\geq \frac{G\Big(\varepsilon R(\sigma(\zeta))\Big)}{\varepsilon^\beta R^\beta\Big(\sigma(\zeta)\Big)} z^\beta\Big(\sigma(\zeta)\Big). \end{aligned}$$

Consequently, (2.5) becomes

$$z(t) \geq \frac{R(t)}{2} \int_t^\infty \frac{q(\zeta)G\Big(\varepsilon R(\sigma(\zeta))\Big)z^\beta(\sigma(\zeta))}{\varepsilon^\beta R^\beta(\sigma(\zeta))} d\zeta,$$

for $t \geq t_5$. If we define

$$w(t) = \frac{1}{2} \int_t^\infty \frac{q(\zeta) G\Big(\varepsilon R(\sigma(\zeta))\Big) z^\beta(\sigma(\zeta))}{\varepsilon^\beta R^\beta(\sigma(\zeta))} d\zeta,$$

then $z(t) \geq R(t) w(t)$ for $t \geq t_5$. Now,

$$\begin{aligned} w'(t) \leq & -\frac{1}{2} \frac{q(t) G\Big(\varepsilon R(\sigma(t))\Big) z^\beta(\sigma(t))}{\varepsilon^\beta R^\beta(\sigma(t))} \\ \leq & -\frac{1}{2} \frac{q(t) G\Big(\varepsilon R(\sigma(t))\Big)}{\varepsilon^\beta} w^\beta(\sigma(t)) \leq 0, \quad t \geq t_5, \end{aligned}$$

implies that $w(t)$ is nonincreasing on $[t_5, \infty)$ and $\lim_{t \to \infty} w(t)$ exists. It is easy to verify that

$$\begin{aligned} \Big[w^{1-\beta}(t)\Big]' &\leq -\frac{(1-\beta)}{2} w^{-\beta}(t) \frac{q(t) G\Big(\varepsilon R(\sigma(t))\Big)}{\varepsilon^\beta} w^\beta\Big(\sigma(t)\Big) \\ &\leq -\frac{(1-\beta)}{2} w^{-\beta}(t) \frac{q(t) G\Big(\varepsilon R(\sigma(t))\Big)}{\varepsilon^\beta} w^\beta(t) \\ &\leq -\frac{(1-\beta)}{2\varepsilon^\beta} q(t) G\Big(\varepsilon R(\sigma(t))\Big), \end{aligned} \tag{2.6}$$

for $t \geq t_5$. Integrating (2.6) from t_5 to $t (> t_5)$, we obtain

$$\frac{(1-\beta)}{2\varepsilon^\beta} \int_{t^5}^t q(\eta) G\Big(\varepsilon R(\sigma(\eta))\Big) d\eta \quad \leq \quad -\Big[w^{1-\beta}(\eta)\Big]_{t_5}^t < w^{1-\beta}(t_5) < \infty,$$

a contradiction to $(A5)$.

If $x(t) < 0$ for $t \geq t_1$, then we set $y(t) := -x(t)$ for $t \geq t_1$ in (1.1). Using $(A2)$, we find

$$\frac{d}{dt}\left[r(t) \frac{d}{dt}[y(t) + p(t) y(\tau(t))]\right] + q(t) H\left(y(\sigma(t))\right) = 0, \quad \text{for } t \geq t_1,$$

where $H(u) := -G(-u)$ for $u \in \mathbb{R}$. Clearly, H also satisfies $(A2)$. Then, proceeding as above, we find the same contradiction.

Next, we suppose that $(A5)$ does not hold. For $\varepsilon > 0$, let us assume that

$$\int_T^\infty q(\eta) G\Big(\varepsilon R(\sigma(\eta))\Big) d\eta \leq \frac{\varepsilon}{3}.$$

Consider

$$\begin{aligned} M = \Big\{ & x : x \in C([t_0, \infty), \mathbb{R}), x(t) = 0 \text{ for } t \in [t_0, T] \text{ and} \\ & \frac{\varepsilon}{3}[R(t) - R(T)] \leq x(t) \leq \varepsilon[R(t) - R(T)] \Big\}, \end{aligned}$$

and define

$$(\Phi x)(t) = \begin{cases} (\Phi x)(T), & t \in [t_0, T], \\ -p(t)x\Big(\tau(t)\Big) + \int_T^t \frac{1}{r(\eta)}\left[\frac{\varepsilon}{3} + \int_\eta^\infty q(\zeta)G\Big(x(\sigma(\zeta))\Big)d\zeta\right]d\eta, & t \geq T. \end{cases}$$

For every $x \in M$,

$$\begin{aligned}(\Phi x)(t) \geq & \int_T^t \frac{1}{r(\eta)}\left[\frac{\varepsilon}{3} + \int_\eta^\infty q(\zeta)G\Big(x(\sigma(\zeta))\Big)d\zeta\right]d\eta \\ \geq & \frac{\varepsilon}{3}\int_T^t \frac{d\eta}{r(\eta)} = \frac{\varepsilon}{3}[R(t) - R(T)]\end{aligned}$$

and $x(t) \leq \varepsilon R(t)$ implies that

$$\begin{aligned}(\Phi x)(t) \leq & -p(t)x\Big(\tau(t)\Big) + \frac{2\varepsilon}{3}\int_T^t \frac{d\eta}{r(\eta)} \\ \leq & a\varepsilon\Big[R(\tau(t)) - R(T)\Big] + \frac{2\varepsilon}{3}\Big[R(t) - R(T)\Big] \\ \leq & a\varepsilon\Big[R(t) - R(T)\Big] + \frac{2\varepsilon}{3}\Big[R(t) - R(T)\Big] \\ = & \left(a + \frac{2}{3}\right)\varepsilon\Big[R(t) - R(T)\Big] \\ \leq & \varepsilon\Big[R(t) - R(T)\Big]\end{aligned}$$

implies that $(\Phi x)(t) \in M$. Define $u_n : [t_0, +\infty) \to \mathbb{R}$ by the recursive formula

$$u_n(t) = \Big(\Phi u_{n-1}\Big)(t), \quad n \geq 1,$$

with the initial condition

$$u_0(t) = \begin{cases} 0, & t \in [t_0, T], \\ \frac{\varepsilon}{3}[R(t) - R(T)], & t \geq T. \end{cases}$$

Inductively it is easy to verify that

$$\frac{\varepsilon}{3}\Big[R(t) - R(T)\Big] \leq u_{n-1}(t) \leq u_n(t) \leq \varepsilon\Big[R(t) - R(T)\Big],$$

for $t \geq T$. Therefore, for $t \geq t_0$, $\lim_{n\to\infty} u_n(t)$ exists. By the Lebesgue's dominated convergence theorem, $u \in M$ and $(\Phi u)(t) = u(t)$, where $u(t)$ is a solution of (1.1) such that $u(t) > 0$. Hence, $(A5)$ is necessary. This completes the proof of the theorem. □

Theorem 2.2. *Assume that* $(A1)$*-*$(A3)$ *hold and* $-1 < -a \leq p(t) \leq 0$, $a > 0$ *for* $t \in \mathbb{R}_+$*. Then every unbounded solution of* (1.1) *oscillates if and only if* $(A5)$ *holds for every* $\varepsilon > 0$*.*

Proof. Without loss of generality, suppose the contrary that x is an eventually positive unbounded solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x\Big(\tau(t)\Big) > 0$ and $x\Big(\sigma(t)\Big) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see rz' is nonincreasing and z is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. We have the following two possible cases.

Case 1. Let $z(t) < 0$ for $t \geq t_2$. The case is same as in proof of Theorem 2.1.

Case 2. Let $z(t) > 0$ for $t \geq t_2$. By Lemma 2.1, (2.1) holds for $t \geq t_3$. Since $z(t)$ is unbounded and monotonic increasing, then it follows that

$$\lim_{t\to\infty} \frac{z(t)}{R(t)} = \lim_{t\to\infty} \frac{z'(t)}{R'(t)} = \lim_{t\to\infty} r(t)z'(t) = \alpha < \infty.$$

If $\alpha = 0$, then $\lim_{t\to\infty} R(t) = +\infty$ implies that $\lim_{t\to\infty} z(t) < +\infty$, which is absurd (because of unbounded $z(t)$). Hence $\alpha \neq 0$. Therefore, there exists a constant $\varepsilon > 0$ and a $t_2 > t_1$ such that $z(t) \geq \varepsilon R(t)$ for $t \geq t_2$. Consequently, $x(t) \geq z(t) \geq \varepsilon R(t)$ for $t \geq t_2$. Using $x(t) \geq \varepsilon R(t)$ in (2.2) and then integrating from t_2 to $+\infty$, we obtain a contradiction to $(A5)$ for every $\varepsilon > 0$.

The case where x is eventually negative unbounded solution is very similar and we omit it here.

The necessary part is same as in Theorem 2.1. This completes the proof of the theorem. □

Theorem 2.3. *Assume that* $(A1)$-$(A4)$ *hold and* $-1 < -a \leq p(t) \leq 0$, *where* $a > 0$, $t \in \mathbb{R}_+$. *Then every solution of* (1.1) *oscillates or converges to zero if and only if* $(A5)$ *holds for every* $\varepsilon > 0$.

Proof. Without loss of generality, suppose the contrary that x is an eventually positive solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see rz' is nonincreasing and, rz' and z is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. By Lemma 2.1, we have the following three possible cases.

Case 1. Let $z(t) < 0$, $r(t)z'(t) < 0$ for $t \geq t_2$. Since $z(t) < 0$ implies $z(t)$ is bounded due to Theorem 2.1 and $r(t)z'(t) < 0$ implies that $z(t)$ is unbounded due to Lemma 2.1, a contradiction.

Case 2. Assume that $z(t) < 0$, $r(t)z'(t) > 0$ holds for $t \geq t_2$. Therefore,

$$\begin{aligned} 0 \geq \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty} z(t) \\ &\geq \limsup_{t\to\infty} \Big(x(t) - ax(\tau(t))\Big) \\ &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty} \Big(-ax(\tau(t))\Big) \\ &= (1-a) \limsup_{t\to\infty} x(t), \end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$.

Case 3. Let $z(t) > 0$, $r(t)z'(t) > 0$ for $t \geq t_2$. The case follows from Theorem 2.1.

Hence, $(A5)$ is a sufficient condition. The case where x is negative solution is similar and we omit it here.

The necessary part is same as in the Theorem 2.1. Thus, the proof of the theorem is complete. ☐

Theorem 2.4. *Assume that* $(A1)$-$(A3)$ *hold and* $-1 < -a \leq p(t) \leq 0$ *such that* $r(t) \geq r\big(\sigma(t)\big)$ *for* $a > 0$, $t \in \mathbb{R}_+$. *Furthermore, assume that*

$(A6)$ *G is strictly superlinear, that is,* $\frac{G(u)}{u^\beta} \geq \frac{G(v)}{v^\beta}$, $u \geq v > 0$, $\beta > 1$,

holds. Then every solution of (1.1) *either oscillates or converges to zero if and only if*

$(A7)$ $\int_0^\infty \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)d\zeta\right] d\eta = +\infty.$

Proof. For the sufficient part, we use the same type of argument as in the proof of Theorem 2.3 for first two cases of the pair $z(t)$ and $r(t)z'(t)$. Let us consider the **Case 3** for $t \geq t_1$. By Remark 2.1, there exists a constant $\varepsilon > 0$ and $t_2 > t_1$ such that $z\big(\sigma(t)\big) \geq \varepsilon$ for $t \geq t_2$. Consequently,

$$G\Big(z(\sigma(t))\Big) = \frac{G\Big(z(\sigma(t))\Big)}{z^\beta\big(\sigma(t)\big)} z^\beta\big(\sigma(t)\big) \geq \frac{G(\varepsilon)}{\varepsilon^\beta} z^\beta\big(\sigma(t)\big),$$

for $t \geq t_2$. Therefore, (2.4) becomes

$$\begin{aligned} r(t)z'(t) \geq& \frac{G(\varepsilon)}{\varepsilon^\beta} \int_t^\infty q(\eta) z^\beta\big(\sigma(\eta)\big) d\eta, \\ \geq& \frac{G(\varepsilon)}{\varepsilon^\beta} \left[\int_t^\infty q(\eta)d\eta\right] z^\beta\big(\sigma(t)\big), \end{aligned}$$

that is,

$$r\big(\sigma(t)\big) z'\big(\sigma(t)\big) \geq \frac{G(\varepsilon)}{\varepsilon^\beta} \left[\int_t^\infty q(\eta)d\eta\right] z^\beta\big(\sigma(t)\big),$$

for $t \geq t_2$, implies that

$$\begin{aligned} z'\big(\sigma(t)\big) \geq& \frac{G(\varepsilon)}{\varepsilon^\beta r\big(\sigma(t)\big)} \left[\int_t^\infty q(\eta)d\eta\right] z^\beta\big(\sigma(t)\big) \\ \geq& \frac{G(\varepsilon)}{\varepsilon^\beta} \frac{z^\beta\big(\sigma(t)\big)}{r(t)} \left[\int_t^\infty q(\eta)d\eta\right]. \end{aligned}$$

Integrating the last inequality from t_2 to $+\infty$, we get

$$\frac{G(\varepsilon)}{\varepsilon^\beta} \int_{t_2}^\infty \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)d\zeta\right] d\eta \leq \int_{t_2}^\infty \frac{z'\big(\sigma(\eta)\big)}{z^\beta\big(\sigma(\eta)\big)} d\eta < \infty,$$

which is a contradiction to $(A7)$.

The case where x is eventually negative solution is omitted since it can be dealt similarly.

Next, we show that $(A7)$ is necessary. Assume that $(A7)$ fails to hold and let

$$G(\varepsilon)\int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \le \frac{\varepsilon}{3}, \quad T \ge T^*,$$

where $\varepsilon > 0$ is a constant. Consider

$$M = \left\{x \in C([t_0,\infty),\mathbb{R}) : x(t) = \frac{\varepsilon}{3},\ t \in [t_0,T],\ \frac{\varepsilon}{3} \le x(t) \le \varepsilon,\ \text{for } t \ge T\right\},$$

and define

$$(\Phi x)(t) = \begin{cases} \dfrac{\varepsilon}{3}, & t \in [t_0,T], \\ -p(t)x\big(\tau(t)\big) + \dfrac{\varepsilon}{3} + \displaystyle\int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G\big(x(\sigma(\zeta))\big)d\zeta\right]d\eta, & t \ge T, \end{cases}$$

for every $x \in M$, $(\Phi x)(t) \ge \frac{\varepsilon}{3}$ and

$$\begin{aligned}(\Phi x)(t) \le& a\varepsilon + \frac{\varepsilon}{3} + G(\varepsilon)\int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \\ \le& a\varepsilon + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \left(a + \frac{2}{3}\right)\varepsilon \\ \le& \varepsilon,\end{aligned}$$

implies that $\Phi x \in M$. The rest of the proof follows from Theorem 2.1. This completes the proof of the theorem. □

Theorem 2.5. *Assume that* $(A1)$-$(A3)$, $(A6)$ *hold and* $0 \le p(t) \le a < 1$ *such that* $r(t) \ge r\big(\sigma(t)\big)$ *for* $t \in \mathbb{R}_+$. *Furthermore, assume that* G *is Lipschitzian on the interval of the form* $[c,d]$, $0 < c < d < \infty$. *Then every solution of* (1.1) *oscillates if and only if* $(A7)$ *holds.*

Proof. Suppose the contrary that x is a nonoscillatory solution of (1.1). Then, there exists $t_1 \ge t_0$ such that either $x(t) > 0$ or $x(t) < 0$ for $t \ge t_1$. Assume that $x(t) > 0$, $x\big(\tau(t)\big) > 0$ and $x\big(\sigma(t)\big) > 0$ for $t \ge t_1$. Clearly, z defined by (2.1) is positive on $[t_1,\infty)$. By Lemma 2.1 and Remark 2.1, there exists $\varepsilon > 0$ such that $z(t) \ge \varepsilon$ for $t \ge t_2$, where $t_2 \ge t_1$. On the other hand, z being increasing implies that

$$\begin{aligned}(1-a)z(t) \le& \big(1-p(t)\big)z(t) \le z(t) - p(t)z(\tau(t)) \\ =& x(t) - p(t)p(\tau(t))x(\tau(\tau(t))) \le x(t),\end{aligned}$$

for $t \ge t_3$, where $t_3 \ge t_2$. Consequently, (1.1) becomes

$$\big(r(t)z'(t)\big)' + q(t)G\big((1-a)z(\sigma(t))\big) \le 0,$$

for $t \geq t_3$. Using $(A6)$ it follows that

$$\begin{aligned} G\Big((1-a)z(\sigma(t))\Big) =& \frac{G\Big((1-a)z(\sigma(t))\Big)}{(1-a)^{\beta}z^{\beta}\Big(\sigma(t)\Big)}(1-a)^{\beta}z^{\beta}\Big(\sigma(t)\Big) \\ \geq & \frac{G\Big(\varepsilon(1-a)\Big)}{\varepsilon^{\beta}(1-a)^{\beta}}(1-a)^{\beta}z^{\beta}\Big(\sigma(t)\Big). \end{aligned}$$

The remaining portion of the sufficient part follows from Theorem 2.4.

Conversely, suppose that $(A7)$ fails to hold. Then there exists $T \geq T^*$ such that

$$\int_T^{\infty} \frac{1}{r(\eta)}\left[\int_{\eta}^{\infty} q(\zeta)d\zeta\right]d\eta < \frac{1-a}{5K},$$

where $K = \max\{K_1, G(1)\}$ and K_1 is the Lipschitz constant of G on $\left[\frac{7(1-a)}{10}, 1\right]$ for $t \geq t_0$. Let $X = BC([t_0,\infty),\mathbb{R})$ be the space of real valued continuous functions on $[t_0,\infty)$. Indeed, X is a Banach space with respect to sup norm defined by

$$\|x\| = \sup\{|x(t)| : t \ \geq t_0\}.$$

Define

$$S = \left\{u \in X : \frac{7(1-a)}{10} \leq u(t) \leq 1, \ t \geq t_0\right\}.$$

We notice that S is a closed convex subspace of X. Let $\Phi : S \to S$ be such that

$$(\Phi x)(t) = \begin{cases} (\Phi x)(T), & t \in [t_0, T], \\ -p(t)x\Big(\tau(t)\Big) + \dfrac{9+a}{10} - \displaystyle\int_t^{\infty} \frac{1}{r(\eta)}\left[\int_{\eta}^{\infty} q(\zeta)G\Big(x(\sigma(\zeta))\Big)d\zeta\right]d\eta, & t \geq T. \end{cases}$$

For every $x \in X$, $(\Phi x)(t) \leq \frac{9+a}{10} \leq 1$ and

$$(\Phi x)(t) \geq -a + \frac{9+a}{10} - \frac{1-a}{5} = \frac{7}{10}(1-a),$$

implies that $\Phi(x) \in S$. Now for $x_1, x_2 \in S$, we have

$$\begin{aligned} |(\Phi x_1)(t) - (\Phi x_2)(t)| \leq & a|x_1\Big(\tau(t)\Big) - x_2\Big(\tau(t)\Big)| \\ & + \int_t^{\infty} \frac{1}{r(\eta)}\left[\int_{\eta}^{\infty} q(\zeta)|G\Big(x_1(\sigma(\zeta))\Big) - G\Big(x_2(\sigma(\zeta))\Big)|d\zeta\right]d\eta, \end{aligned}$$

that is,

$$\begin{aligned} |(\Phi x_1)(t) - (\Phi x_2)(t)| \leq & a\|x_1 - x_2\| + \|x_1 - x_2\| K_1 \int_t^{\infty} \frac{1}{r(\eta)}\left[\int_{\eta}^{\infty} q(\zeta)d\zeta\right]d\eta \\ \leq & \left(a + \frac{1-a}{5}\right)\|x_1 - x_2\| \\ = & \frac{1+4a}{5}\|x_1 - x_2\|. \end{aligned}$$

Therefore, $\|\Phi x_1 - \Phi x_2\| \leq \frac{1+4a}{5}\|x_1 - x_2\|$ implies that Φ is a contraction. By using Banach's contraction mapping principle, it follows that Φ has a unique fixed point $x(t)$ in $\left[\frac{7(1-a)}{10}, 1\right]$. Hence, $(A7)$ is the necessary condition for oscillation of (1.1). This completes the proof of the theorem. $\square$

Theorem 2.6. *Assume that* $(A1)$-$(A3)$ *hold and* $0 \leq p(t) \leq a < 1$ *for* $t \in \mathbb{R}_+$. *Furthermore, assume that* G *be Lipschitzian on intervals of the form* $[c, d]$, $0 < c < d < \infty$. *Then every bounded solutions of* (1.1) *oscillates if and only if* $(A7)$ *holds.*

Proof. Proceeding as in proof of the Theorem 2.5 we have obtained $x(t) \geq (1-a)z(t) \geq (1-a)\varepsilon = \varepsilon_1$. Consequently, (1.1) becomes

$$\Big(r(t)z'(t)\Big)' + q(t)G(\varepsilon_1) \leq 0.$$

Twice integration on last inequality yields a contradiction to $(A7)$. The necessary part is same as in the proof of Theorem 2.5. Hence the details are omitted. Thus the proof of theorem is complete. $\square$

Theorem 2.7. *Assume that* $(A1)$-$(A3)$ *hold and* $-\infty < -a_1 \leq p(t) \leq -a_2 < -1$ *such that* $3a_2 > a_1$ *for* $t \in \mathbb{R}_+$ *where* $a_1, a_2 > 0$. *Let* G *be Lipschitzian on intervals of the form* $[c, d]$, $0 < c < d < \infty$. *Then every bounded solution of* (1.1) *oscillates or tends to zero if and only if* $(A7)$ *holds.*

Proof. Without loss of generality, suppose the contrary that x is an eventually positive solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see rz' is nonincreasing and, rz' and z is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. Since $x(t)$ is bounded, then by (1.4), $z(t)$ is bounded and hence $\lim_{t\to\infty} z(t)$ exists. It is easy to see that the case $z(t) < 0$, $r(t)z'(t) < 0$ is not possible. Using the proof of Lemma 2.2, we conclude that the case $z(t) > 0$, $r(t)z'(t) < 0$ does not arise. Therefore, we have following two cases.

Case 1. Let $z(t) > 0$, $r(t)z'(t) > 0$ for $[t_3, \infty)$, $t_3 > t_2$. Then we can find a constant $\varepsilon > 0$ and $t_4 > t_3$ such that $z\Big(\sigma(t)\Big) \geq \varepsilon$ for $t \geq t_4$, that is, $x\Big(\sigma(t)\Big) \geq z\Big(\sigma(t)\Big) \geq \varepsilon$ for $t \geq t_4$. Hence, (1.1) becomes

$$\Big(r(t)z'(t)\Big)' + G(\varepsilon)q(t) \leq 0,\ t \geq t_4.$$

Twice integration on last inequality gives a contradiction to $(A7)$.

Case 2. Let $z(t) < 0$, $r(t)z'(t) > 0$ for $[t_3, \infty)$, $t_3 > t_2$. We claim that $\lim_{t\to\infty} z(t) = 0$. If not, there exist $\alpha < 0$ and $t_4 > t_3$ such that $z\Big(\tau^{-1}(\sigma(t))\Big) < \alpha$ for $t \geq t_4$. Hence, $z(t) \geq -a_1 x\Big(\tau(t)\Big)$ implies that $x(t) \geq -a_1^{-1} z\Big(\tau^{-1}(t)\Big)$, that is, $x\Big(\sigma(t)\Big) \geq -a_1^{-1} z\Big(\tau^{-1}(\sigma(t))\Big) \geq -a_1^{-1}\alpha$ for $t \geq t_4$. Consequently, (1.1) reduces to

$$\Big(r(t)z'(t)\Big)' + G\Big(-a_1^{-1}\alpha\Big)q(t) \leq 0,$$

for $t \geq t_4$. Using the same type of argument as in the former case, we get a contradiction to $(A7)$. Thus, our claim holds and hence

$$\begin{aligned}
0 = \lim_{t\to\infty} z(t) =& \liminf_{t\to\infty}\Big(x(t) + p(t)x(\tau(t))\Big)\\
\leq& \liminf_{t\to\infty}\Big(x(t) - a_2x(\tau(t))\Big)\\
\leq& \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}\Big(-a_2x(\tau(t))\Big)\\
=&(1-a_2)\limsup_{t\to\infty} x(t),
\end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ [$\because 1 - a_2 < 0$]. Therefore, $\lim_{t\to\infty} x(t) = 0$.

The case where x is negative bounded solution is very similar and we omit it here.

For the necessary part, it is possible to find $T \geq T^*$ such that

$$\int_T^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta < \frac{a_2-1}{3K},$$

where $K = \max\{K_1, G(1)\}$ and K_1 is the Lipschitz constants of G on $[a,1]$, where $a = \frac{(a_2-1)(3a_2-a_1)}{3a_1a_2}$. Let $X = BC([t_0,\infty),\mathbb{R})$ be the space of real valued continuous functions defined on $[t_0,\infty)$. Indeed, X is a Banach space with the sup norm defined by

$$\|x\| = \sup\{|x(t)| : t \ \geq t_0\}.$$

Define

$$S = \{u \in X : a \leq u(t) \leq 1,\ t \geq t_0\}$$

and we note that S is a closed convex subspace of X. Let $\Phi : S \to S$ be such that

$$(\Phi x)(t) = \begin{cases} (\Phi x)(T), & t \in [t_0,T],\\ -\dfrac{x\Big(\tau^{-1}(t)\Big)}{p\Big(\tau^{-1}(t)\Big)} - \dfrac{a_2-1}{p\Big(\tau^{-1}(t)\Big)} & \\ +\dfrac{1}{p\Big(\tau^{-1}(t)\Big)}\displaystyle\int_T^{\tau^{-1}(t)}\frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G\Big(x(\sigma(\zeta))\Big)d\zeta\right]d\eta, & t \geq T.\end{cases}$$

For every $x \in S$,

$$(\Phi x)(t) \leq -\frac{x\Big(\tau^{-1}(t)\Big)}{p\Big(\tau^{-1}(t)\Big)} - \frac{a_2-1}{p\Big(\tau^{-1}(t)\Big)} \leq \frac{1}{a_2} + \frac{a_2-1}{a_2} = 1$$

and

$$
\begin{aligned}
(\Phi x)(t) \geq & -\frac{a_2-1}{p\Big(\tau^{-1}(t)\Big)}+\frac{1}{p\Big(\tau^{-1}(t)\Big)}\int_T^{\tau^{-1}(t)}\frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G\Big(x(\sigma(\zeta))\Big)d\zeta\right]d\eta\\
\geq & -\frac{a_2-1}{a_1}+\frac{G(1)}{p\Big(\tau^{-1}(t)\Big)}\int_T^{\tau^{-1}(t)}\frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta\\
\geq & -\frac{a_2-1}{a_1}-\frac{G(1)}{a_2}\int_T^\infty\frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta\\
\geq & -\frac{a_2-1}{a_1}-\frac{a_2-1}{3a_2}=a,
\end{aligned}
$$

implies that $\Phi x\in S$. Now for $x_1,x_2\in S$, we have

$$
\begin{aligned}
|(\Phi x_1)(t)-(\Phi x_2)(t)| \leq & \frac{1}{\Big|p\Big(\tau^{-1}(t)\Big)\Big|}|x_1\Big(\tau^{-1}(t)\Big)-x_2\Big(\tau^{-1}(t)\Big)|+\frac{K_1}{\Big|p\Big(\tau^{-1}(t)\Big)\Big|}\\
& \times\int_T^{\tau^{-1}(t)}\frac{1}{r(\eta)}\left[\int_\eta^\infty |x_1\Big(\sigma(\zeta)\Big)-x_2\Big(\sigma(\zeta)\Big)|q(\zeta)d\zeta\right]d\eta\\
\leq & \frac{1}{a_2}\|x_1-x_2\|+\frac{a_2-1}{3a_2}\|x_1-x_2\|\\
= & \gamma\|x_1-x_2\|,
\end{aligned}
$$

implies that

$$\|\Phi x_1-\Phi x_2\|\leq\gamma\|x_1-x_2\|,$$

where $\gamma=\frac{1}{a_2}(1+\frac{a_2-1}{3})<1$. Therefore, Φ is a contraction. Hence by the Banach's contraction mapping principle Φ has a unique fixed point $x\in S$. It is easy to see that $\lim_{t\to\infty}x(t)\neq 0$. This completes the proof of the theorem. □

3. Discussion and Example

It is worth observation that we could succeed partially to establish the oscillation of all solutions of the nonlinear equation (1.1), when $|p(t)|<\infty$. We failed to obtain the necessary and sufficient conditions in the range $1\leq p(t)<\infty$ and $p(t)\equiv -1$. Therefore, the undertaken problem is incomplete for all range of $p(t)$.

Remark 3.1. In Theorems 2.2, 2.6 and 2.7, G could be linear, sublinear or superlinear.

We conclude this section with the following examples to illustrate our main results:

Example 3.1. Consider the delay differential equations

$$\frac{d}{dt}\left[t\frac{d}{dt}[x(t)-3x(e^{-\pi}t)]\right]+\frac{4}{t}x(t)=0,\quad\text{for } t\geq 1,\tag{3.1}$$

where $r(t):=t$, $p(t):\equiv -3$, $\tau(t):=e^{-\pi}t$, $q(t):=\frac{4}{t^2}$, $\sigma(t):=t$ and $G(u):=u$ for $t\geq 1$ and $u\in\mathbb{R}$. It can be easily shown that Theorem 2.7 applies to (3.1). Thus, every bounded solution oscillates or converges to zero asymptotically. Obviously, $x(t) = \sin(\ln(t^2))$ for $t \geq 1$ is an oscillating solution.

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