Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 473–483.

CERTAIN PROPERTIES ON MEROMORPHIC FUNCTIONS DEFINED BY A NEW LINEAR OPERATOR INVOLVING THE MITTAG-LEFFLER FUNCTION

AQEEL KETAB AL-KHAFAJI[1] AND ABBAS KAREEM WANAS[2]

Abstract. Our paper introduces a new linear operator using the convolution between a Mittag–Leffler Function and basic hypergeometric function. Use of the linear operator creates a new class of meromorphic functions defined in the punctured open unit disk. Consequently, the paper examines different aspects Apps and assets like, extreme points, coefficient inequality, growth and distortion. In conclusion, the work discusses modified Hadamard product and closure theorems.

1. Introduction

Let Σ indicate the class of type functions

$$h(z) = z^{-1} + \sum_{j=1}^{\infty} a_j z^j, \quad j \in \mathbb{N} = \{1, 2, 3, \dots\}, \tag{1.1}$$

which are analytic in the punctured open unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$.

Denote by $\Sigma_i(\delta)$ and $\Sigma^*(\delta)$ the subclasses of Σ that are meromorphically convex function of order δ, and meromorphically starlike of order δ, respectively. Function $h \in \Sigma$ of the type (1.1), is in the class $\Sigma_i(\delta)$, if it meets

$$\operatorname{Re}\left\{-\left(1 + \frac{zh''(z)}{h'(z)}\right)\right\} > \delta, \quad z \in U^*,$$

Key words and phrases. Linear operator, basic hypergeometric function, meromorphic function, Mittag–Leffler function, coefficient inequality, convex linear combination, growth and distortion theorem, extreme points, Hadamard product.

2010 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 30C50.

DOI

Received: November 06, 2020.

Accepted: May 31, 2021.

and h is in the class $\Sigma^*(\delta)$, if it meets

$$\operatorname{Re}\left\{-\frac{zh'(z)}{h(z)}\right\} > \delta, \quad z \in U^*.$$

The Hadamard product (or convolution) $h * k$ for two analytic functions h given by (1.1) in U^* and

$$k(z) = z^{-1} + \sum_{j=1}^{\infty} b_j z^j,$$

is define by

$$(h * k)(z) = z^{-1} + \sum_{j=1}^{\infty} a_j b_j z^j.$$

For complex components $q, b_k, a_i,\ b_k \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$, $k = 1,\dots,r$, $i = 1,\dots,m$, the basic hypergeometric function or (q-hypergeometric function) ψ_r^m is defined by:

(1.2)

$$\psi_r^m(a_1,\dots,a_m;b_1,\dots,b_r;q,z) = \sum_{j=0}^{\infty} \frac{(a_1,q)_j \cdots (a_m,q)_j}{(q,q)_j (b_1,q)_j \cdots (b_r,q)_j} \left[(-1)^j q^{\frac{j(1-j)}{2}}\right]^{1+r-m} z^j,$$

where $q \neq 0$, when $m > r+1$, $m, r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $(a,q)_j$ is q-analogue of the Pochhammer symbol $(a)_j$ is defined by

$$(a,q)_j = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{j-1}), & j = 1,2,3,\dots, \\ 1, & j = 0. \end{cases}$$

Initially, the function ψ_r^m given by (1.2), was introduced and referred to by Heine in 1846, as the series of Heine. For readers to refer to further q-theory information can be found in (see [9] and [11]).

Now, for $|q| < 1$, $m = r+1$ and $z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, the q-hypergeometric function [25] defined in Equation (1.2), takes the form below

$$\psi_r^m(a_1,\dots,a_m;b_1,\dots,b_r;q,z) = \sum_{j=0}^{\infty} \frac{(a_1,q)_j \cdots (a_m,q)_j}{(q,q)_j (b_1,q)_j \cdots (b_r,q)_j} z^j$$

that absolutely converges in the open unit disk $\mathbb{U}$.

With regard to the function $\psi_r^m(a_1,\dots,a_m;b_1,\dots,b_r;q,z)$, for meromorphic function $h \in \Sigma$ that includes functions in shape of (1.1) (see work of [1] and [18]), which is shown below, have successfully introduced the q-analogue of the Liu–Srivastava operator

$$\begin{aligned} \mathcal{G}_r^m(a_1,\dots,a_m;b_1,\dots,b_r;q,z)\,h(z) &= z_l^{-1}\psi_r^m(a_1,\dots,a_m;b_1,\dots,b_r;q,z) * h(z) \\ &= z^{-1} + \sum_{j=1}^{\infty} \frac{\prod_{i=1}^{m}(a_i,q)_{j+1}}{(q,q)_{j+1}\prod_{k=1}^{r}(b_k,q)_{j+1}} a_j z^j. \end{aligned}$$

Before we continue moving on, the Mittag-Leffler function $E_\delta(z)$, suggested by Mittag-Leffler (see [16] and [17]) and defined by

$$E_\alpha(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+1)},\quad z\in\mathbb{U},\alpha\in\mathbb{C},\mathrm{Re}\,(\alpha)>0,$$

where $\Gamma(\delta)$ denotes the Gamma function.

Also, Wiman [26], studied another function $E_{\delta,\mu}(z)$ have numerous similarities of $E_\delta(z)$, and given by

$$E_{\alpha,\mu}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\mu)},\quad z\in\mathbb{U},\ \alpha,\mu\in\mathbb{C},\ \mathrm{Re}\,(\alpha)>0,\ \mathrm{Re}\,(\mu)>0. \tag{1.3}$$

In recent years, there has been growing interest in Mittag-Leffler for application problems including, electric network, fluid flow, probability, statistical distribution theory, etc. (see [2, 4, 8, 12, 15, 19, 22–24] and [27] for more information about this function and its applications). Bansal and Prajapat recently investigated geometric characteristics in [5] for the function $E_{\alpha,\mu}(z)$, like starlikeness, convexity and closed to convex. In addition, certain results were obtained in [21] for the partial sum of the Mettag-Leffler function.

We note that, the function given by (1.3), is not part of class Σ. Therefore, the function $E_{\alpha,\mu}(z)$, is then normalized on the basis of the following:

$$\mathcal{E}_{\alpha,\mu}(z)=\Gamma(\mu)\,z^{-1}E_{\alpha,\mu}(z)=z^{-1}+\sum_{j=1}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\alpha(j+1)+\mu)}z^j. \tag{1.4}$$

Application of the function $\mathcal{E}_{\alpha,\mu}(z)$ defined by (1.4), a new operator $\mathfrak{J}_{\alpha,\mu}:\Sigma\to\Sigma$, is defined in terms of Hadamard product as follows

$$\begin{aligned}\mathfrak{J}_{\alpha,\mu}h(z)&=\mathcal{E}_{\alpha,\mu}(z)*\mathcal{G}_r^m(a_1,\dots,a_m;b_1,\dots,b_r;q,z)\,h(z)\\&=z^{-1}+\sum_{j=1}^{\infty}\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)a_jz^j,\end{aligned}$$

where

$$\delta_{(j+1,\alpha,\mu)}(a_m,b_r,q)=\frac{\prod_{i=1}^{m}(a_i,q)_{j+1}}{(q,q)_{j+1}\prod_{k=1}^{r}(b_k,q)_{j+1}}\left(\frac{\Gamma(\mu)}{\Gamma(\alpha(j+1)+\mu)}\right).$$

Remark 1.1. You can see that when the parameters are defined r, m, α, μ, q, $a_1,\dots,a_m$ and $b_1,\dots,b_r$, it's here noted that the operator defined $\mathfrak{J}_{\alpha,\mu}h(z)$, performs different operators. For further explanation, examples are given.

(a) For $\alpha=0$, $\mu=1$, $a_i=q^{a_i}$, $b_k=q^{b_k}$, $a_i>0$, $b_k>0$, $i=1,\dots,m$, $k=1,\dots,r$, $m=r+1$ and $q\to1$, we obtain the operator defined in [14].

(b) For $m=2$, $r=1$, $\alpha=0$, $\mu=1$, $a_2=q$ and $q\to1$, we obtain the operator defined in [13].

(c) For $m=1$, $r=0$, $\alpha=0$, $\mu=1$, $a_1=\lambda+1$ and $q\to1$, we obtain the operator defined in [10], and it was then generalized through [29].

Some other authors have studied various classes of meromorphic univalent functions, such as, see [3, 6, 7, 20, 28] and [30]). Such works encouraged us to create the new class $\mathcal{T}^{\tau}_{\alpha,\mu}(a_m, b_r, d)$ of Σ, that includes the operator $\mathfrak{J}_{\alpha,\mu}h(z)$, and it is presented as follows.

Definition 1.1. For $d \geq 1$, $\tau > 0$, the function $h \in \Sigma$ is in the class $\mathcal{T}^{\tau}_{\alpha,\mu}(a_m, b_r, d)$ if it satisfies the inequality

$$\left| \frac{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)} - 1}{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)} + d} \right| < \tau. \tag{1.5}$$

Denote by Σ^* the subclass of Σ composed of the form functions

$$h(z) = z^{-1} + \sum_{j=1}^{\infty} |a_j| z^j. \tag{1.6}$$

Define the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$ by

$$\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d) = \mathcal{T}^{\tau}_{\alpha,\mu}(a_m, b_r, d) \cap \Sigma^*.$$

2. Main Results

This section introduces work to obtain sufficient conditions for the function h given by (1.6), in the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$, it also shows that for functions belonging to this class, this requirement is necessary, as well as growth and distortion bounds, extreme points and linear combinations are submitted for the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$.

Theorem 2.1. *A function h given by* (1.6) *is in the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$ if and only if*

$$\sum_{j=1}^{\infty} \left[j^2(1-\tau) - (1+\tau d)\right] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) |a_j| \leq \tau(1+d). \tag{2.1}$$

Proof. Assume that the inequality (1.6) holds true. We have

$$\begin{aligned}
&\left| \frac{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)} - 1}{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)} + d} \right| \\
=&\left| \frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))' - \mathfrak{J}_{\alpha,\mu}h(z)}{\tau\left[z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'+d\mathfrak{J}_{\alpha,\mu}h(z)\right]} \right| \\
=&\left| \frac{\sum_{j=1}^{\infty}[j^2-1]\Delta_{(j+1,\alpha,\mu)}(a_m, b_r)|a_j| z^j}{(1+d) + \sum_{j=1}^{\infty}[j^2+d]\Delta_{(j+1,\alpha,\mu)}(a_m, b_r)|a_j| z^j} \right| < \tau, \quad z \in U^*.
\end{aligned}$$

So, we have $h \in \mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$ (by the maximum modulus theorem).

Conversely, let $h \in \mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$ where h given by (1.6), then we obtain from inequality (1.5),

$$\left|\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'-\mathfrak{J}_{\alpha,\mu}h(z)}{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'+d\mathfrak{J}_{\alpha,\mu}h(z)}\right|$$

$$= \left|\frac{\sum_{j=1}^{\infty}[j^2-1]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\,|a_j|\,z^j}{(1+d)+\sum_{j=1}^{\infty}[j^2+d]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\,|a_j|\,z^j}\right| < \tau, \tag{2.2}$$

since the last inequality is real for all $z \in U^*$, choose values of z on the real axis. Following explanation, the denominator in (2.2) and letting $z \to 1^-$ through real values, we obtain

$$\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\,|a_j| \le \tau(1+d).$$

Therefore, we get the required inequality (2.1) of Theorem 2.1. □

Corollary 2.1. *If the function h given by* (1.6) *is in the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$, then*

$$|a_j| \le \frac{\tau(1+d)}{[j^2(1-\tau)-(1+\tau d)]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}, \quad j \ge 1, \tag{2.3}$$

the result is sharp of the function

$$h(z) = z^{-1} + \frac{\tau(1+d)}{[j^2(1-\tau)-(1+\tau d)]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}z^j, \quad j \ge 1.$$

Theorem 2.2. *Let $h_o(z) = z^{-1}$ and*

$$h_j(z) = z^{-1} + \frac{\tau(1+d)}{[j^2(1-\tau)-(1+\tau d)]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}z^j.$$

Then, $h \in \mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$ if and only if it can be expressed form

$$h(z) = z^{-1} + \sum_{j=0}^{\infty} v_j h_j(z), \tag{2.4}$$

where

$$v_j \ge 0 \quad \text{and} \quad \sum_{j=0}^{\infty} v_j = 1.$$

Proof. Using the function h which is defined in (2.4), then

$$h(z) = z^{-1} + \sum_{j=0}^{\infty} v_j \frac{\tau(1+d)}{[j^2(1-\tau)-(1+\tau d)]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}z^j,$$

and for last function, we get

$$\begin{aligned}&\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\\&\times v_j\frac{\tau(1+d)}{\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}\\=&\sum_{j=1}^{\infty}v_j\tau(1+d)=\tau(1+d)(1-v_o)=\tau(1+d),\end{aligned}$$

that is, condition (2.1) is met. Therefore, $h\in\mathfrak{T}_{\alpha,\mu}^{\tau,*}(a_m,b_r,d)$.

Conversely, we assume that $h\in\mathfrak{T}_{\alpha,\mu}^{\tau,*}(a_m,b_r,d)$, from equation (2.3), we have:

$$|a_j|\leq\frac{\tau(1+d)}{\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)},\quad j\geq1,$$

we set

$$v_i=\frac{\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\tau(1+d)}\;|a_j|,\quad j\geq1,$$

and

$$v_0=1-\sum_{j=1}^{\infty}v_j.$$

That is the result

$$h(z)=\sum_{j=0}^{\infty}v_jf_j.$$

The declaration of Theorem 2.2, is thus complete. □

Theorem 2.3. *If a function h defined by (1.6), is in the class $\mathfrak{T}_{\alpha,\mu}^{\tau,*}(a_m,b_r,d)$, then for $|z|=r$, we have*

$$\begin{aligned}&\frac{1}{r}-\frac{\tau(1+d)}{\left[(1-\tau)-(1+\tau d)\right]\Delta_{(2,\alpha,\mu)}(a_m,b_r)}r\\&\leq|h(z)|\leq\frac{1}{r}+\frac{\tau(1+d)}{\left[(1-\tau)-(1+\tau d)\right]\Delta_{(2,\alpha,\mu)}(a_m,b_r)}r\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{r^2}-\frac{\tau(1+d)}{\left[(1-\tau)-(1+\tau d)\right]\Delta_{(2,\alpha,\mu)}(a_m,b_r)}\\&\leq|h'(z)|\leq\frac{1}{r^2}+\frac{\tau(1+d)}{\left[(1-\tau)-(1+\tau d)\right]\Delta_{(2,\alpha,\mu)}(a_m,b_r)}.\end{aligned}$$

Proof. By Theorem 2.1, we have

$$\begin{aligned}
&[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}\,(a_m,b_r)\sum_{j=1}^{\infty}|a_j| \\
\leq&\sum_{j=1}^{\infty}\left[j^2\,(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}\,(a_m,b_r)\,|a_j| \\
\leq&\tau\,(1+d)\,,
\end{aligned}$$

which results

$$\sum_{j=1}^{\infty}|a_j|\leq\frac{\tau\,(1+d)}{[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}\,(a_m,b_r)}.$$

Therefore,

$$|h\,(z)|\leq\frac{1}{|z|}+|z|\sum_{j=1}^{\infty}|a_j|\leq\frac{1}{|z|}+\frac{\tau\,(1+d)}{[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}\,(a_m,b_r)}\,|z|$$

and

$$|h\,(z)|\geq\frac{1}{|z|}-|z|\sum_{j=1}^{\infty}|a_j|\geq\frac{1}{|z|}-\frac{\tau\,(1+d)}{[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}\,(a_m,b_r)}\,|z|\,.$$

On the other hand, for (1.6), differentiating both sides with respect to z, we get:

$$|h'\,(z)|\leq\frac{1}{|z|^2}+\sum_{j=1}^{\infty}|a_j|\leq\frac{1}{|z|}+\frac{\tau\,(1+d)}{[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}\,(a_m,b_r)}$$

and

$$|h'\,(z)|\geq\frac{1}{|z|^2}-\sum_{j=1}^{\infty}|a_j|\geq\frac{1}{|z|}-\frac{\tau\,(1+d)}{[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}\,(a_m,b_r)}.$$

Define the functions h_i, $i=1,2$, by

$$h_i\,(z)=z^{-1}+\sum_{j=1}^{\infty}|a_{j,i}|\,z^j,\quad z\in\ U^*. \tag{2.5}$$

□

Theorem 2.4. *Let the functions h_i, $i=1,2$, which are defined in* (2.5)*, be in the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}\,(a_m,b_r,d)$. Then for $0\leq s\leq 1$, the function $h\,(z)=sh_1\,(z)+(1-s)\,h_2\,(z)$, in the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}\,(a_m,b_r,d)$.*

Proof. Using

$$h_i\,(z)=z^{-1}+\sum_{j=1}^{\infty}|a_{j,i}|\,z^j,\quad i=1,2,$$

we have:

$$h\,(z)=z^{-1}+\sum_{j=1}^{\infty}\{s\,|a_{j,1}|+(1-s)\,|a_{j,2}|\}\,z^j,\quad 0\leq s\leq 1.$$

Now, by Theorem 2.1, we obtain

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\left\{s\left|a_{j,1}\right|+(1-s)\left|a_{j,2}\right|\right\}\\
=&s\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\left|a_{j,1}\right|\\
&+(1-s)\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\left|a_{j,2}\right|\\
\leq& s\tau(1+d)+(1-s)\tau(1+d)=\tau(1+d),
\end{aligned}$$

that demonstrates $h(z)\in\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m,b_r,d)$. □

Theorem 2.5. *Let the function h_i, $i=1,2$, which are defined in (2.5), be in the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m,b_r,d)$. Then $h_1*h_2\in\mathcal{T}^{\delta,*}_{\alpha,\mu}(a_m,b_r,d)$, where*

$$\delta\leq\frac{(j^2-1)\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\tau(1+d)+(j^2+d)\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}.$$

Proof. It's enough to find the Littlest δ, such that

$$\sum_{j=1}^{\infty}\frac{[j^2(1-\delta)-(1+\delta d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\delta(1+d)}a_{j,1}a_{j,2}\leq 1.$$

Since $h_i\in\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m,b_r,d)$, $i=1,2$, then

$$\sum_{j=1}^{\infty}\frac{[j^2(1-\tau)-(1+\tau d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\tau(1+d)}a_{j,1}a_{j,2}\leq 1.$$

By Cauchy-Schwarz inequality, we get

$$\sum_{j=1}^{\infty}\frac{[j^2(1-\tau)-(1+\tau d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\tau(1+d)}\sqrt{a_{j,1}a_{j,2}}\leq 1. \tag{2.6}$$

We just want to demonstrate that

$$\begin{aligned}
&\sum_{j=1}^{\infty}\frac{[j^2(1-\delta)-(1+\delta d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\delta(1+d)}a_{j,1}a_{j,2}\\
\leq&\sum_{j=1}^{\infty}\frac{[j^2(1-\tau)-(1+\tau d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\tau(1+d)}\sqrt{a_{j,1}a_{j,2}},
\end{aligned}$$

or equivalent to

$$\sqrt{a_{j,1}a_{j,2}}\leq\frac{[j^2(1-\delta)-(1+\delta d)]\tau}{[j^2(1-\tau)-(1+\tau d)]\delta}.$$

From (2.6), we get

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{\tau(1+d)}{[j^2(1-\tau)-(1+\tau d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}.$$

Therefore, it is sufficient to show that

$$\frac{\tau\left(1+d\right)}{\left[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\right]\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)}\leq\frac{\left[j^{2}\left(1-\delta\right)-\left(1+\delta d\right)\right]\tau}{\left[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\right]\delta}.$$

Finally, we have

$$\delta\leq\frac{\left(j^{2}-1\right)\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)}{\tau\left(1+d\right)+\left(j^{2}+d\right)\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)}.$$ □

Theorem 2.6. *If the function h_i, $i=1,2$, given by equation* (2.5) *is in the class $\mathfrak{T}^{\tau,*}_{\alpha,\mu}\left(a_m,b_r,d\right)$, then $h_1 * h_2 \in \mathfrak{T}^{\tau,*}_{\alpha,\mu}\left(a_m,b_r,d\right)$.*

Proof. Because $h_1 \in \mathfrak{T}^{\tau,*}_{\alpha,\mu}\left(a_m,b_r,d\right)$, by Theorem 2.1, we obtain

$$\sum_{j=1}^{\infty}\left[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\right]\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)\left|a_{j}\right|\leq\tau\left(1+d\right).$$

Since

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\right]\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)\left|a_{j,1}a_{j,2}\right|\\
&=\sum_{j=1}^{\infty}\left[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\right]\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)\left|a_{j,1}\right|\left|a_{j,2}\right|\\
&\leq\sum_{j=1}^{\infty}\left[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\right]\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)\left|a_{j,1}\right|\\
&\leq 1,
\end{aligned}$$

we have $h_1 * h_2 \in \mathfrak{T}^{\tau,*}_{\alpha,\mu}\left(a_m,b_r,d\right)$. □

Acknowledgements. The authors would like to thank the referee(s) for their helpful comments and suggestions.

References

[1] H. Aldweby and M. Darus, *Integral operator defined by q-analogue of Liu-Srivastava operator*, Stud. Univ. Babeş-Bolyai Math. **58**(4) (2013), 529–537.

[2] A. Apelblat *Differentiation of the Mittag-Leffler functions with respect to parameters in the Laplace transform approach*, Mathematics **8**(5) (2020), Article ID 657, 22 pages. `https://doi.org/10.3390/math8050657`

[3] R. Asadi, A. Ebadian, S. Shams and J. Sokół, *On a certain classes of meromorphic functions with positive coefficients*, Appl. Math. J. Chinese Univ. **34**(3) (2019), 253–260. `https://doi.org/10.1007/s11766-019-3432-8`

[4] A. Attiya, *Some applications of Mittag-Leffler function in the unit disk*, Filomat **30**(7) (2016), 2075–2081. `https://doi.org/10.2298/FIL1607075A`

[5] D. Bansal and J. K. Prajapat, *Certain geometric properties of the Mittag-Leffler functions*, Complex Var. Elliptic Equ. **61**(3) (2016), 338–350. `https://doi.org/10.1080/17476933.2015.1079628`

[6] A. Ebadian, N. E. Cho, E. A. Adegani and S. Yalçin, *New criteria for meromorphic starlikeness and close-to-convexity*, Mathematics **8**(5) (2020). https://doi.org/10.3390/math8050847

[7] S. Elhaddad and M. Darus, *On meromorphic functions defined by a new operator containing the Mittag-Leffler function*, Symmetry **11**(2) (2019), Article ID 210, 11 pages. https://doi.org/10.3390/sym11020210

[8] S. Elhaddad and M. Darus, *Certain properties on analytic p-valent functions*, Int. J. Math. Comput. Sci. **15**(1) (2020), 433–442.

[9] H. Exton, *q-Hypergeometric Functions and Applications*, Ellis Horwood Series, Mathematics and Its Applications, Ellis Horwood, Chichester, UK, 1983.

[10] M. R. Ganigi and B. A. Uralegaddi, *New criteria for meromorphic univalent functions*, Bulletin Mathématique dela Société des Sciences Mathématiques de la République Socialiste de Roumanie Nouvelle Séerie **33** (1989), 9–13.

[11] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, UK, 1990.

[12] J. L. Liu, *New applications of the Srivastava-Tomovski generalization of the Mittag-Leffler function*, Iran. J. Sci. Technol. Trans. A Sci. **43**(2) (2019), 519–524. https://doi.org/10.1007/s40995-017-0409-4

[13] J. L. Liu and H. M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl. **259** (2001), 566–581. https://doi.org/10.1006/jmaa.2000.7430

[14] J. L. Liu and H. M. Srivastava, *Classes of meromorphically multivalent functions associated with the generalized hypergeometric function*, Math. Comput. Model. **39** (2004), 21–34. https://doi.org/10.1016/S0895-7177(03)00391-1

[15] T. Mahmood, M. Naeem, S. Hussain, S. Kjan, and Ş. Altınkaya, *A subclass of analytic functions defined by using Mittag-Leffler function*, Honam Math. J. **42**(3) (2020), 577–590. https://doi.org/10.5831/HMJ.2020.42.3.577

[16] G. M. Mittag-Leffler, *Sur la representation analytique d'une branche uniforme d'une function monogene*, Acta Math. **29** (1905), 101–181.

[17] G. M. Mittag-Leffler, *Sur la nouvelle function* $E\alpha(x)$, Comptes Rendus de l'Académie des Sciences **137**(2) (1903), 554–558.

[18] G. Murugusundaramoorthy and T. Janani, *Meromorphic parabolic starlike functions associated with-hypergeometric series*, International Scholarly Research Notices **2014** (2014), Article ID 923607, 9 pages. https://doi.org/10.1155/2014/923607

[19] J. K. Prajapat, S. Maharana and D. Bansal, *Radius of starlikeness and Hardy space of Mittag-Leffler functions*, Filomat **32**(18) (2018), 6475–6486. https://doi.org/10.2298/FIL1818475P

[20] R. S. Qahtan, H. Shamsan and S. Latha, *Some subclasses of meromorphic with p-Valent q-Spirallike functions*, International Journal of Mathematical Combinatorics **4** (2019), 19–28.

[21] D. Răducanu, *On partial sums of normalized Mittag-Leffler functions*, Analele Universitatii Ovidius Constanta-Seria Matematica **25**(2) (2017), 123–133. https://doi.org/10.1515/auom-2017-0024

[22] D. Răducanu, *Third-order differential subordinations for analytic functions associated with generalized Mittag-Leffler functions*, Mediterr. J. Math. **14**(4) (2017), Article ID 167, 18 pages. https://doi.org/10.1007/s00009-017-0969-8

[23] H. Rehman, M. Darus and J. Salah, *Coefficient properties involving the generalized k-Mittag-Leffler functions*, Transylvanian Journal of Mathematics and Mechanics **9** (2017), 155–164.

[24] H. M. Srivastava, N. Khan, M. Darus, S. Khan, Q. Z. Ahmad and S. Hussain, *Fekete-Szegö type problems and their applications for a subclass of q-starlike functions with respect to symmetrical points*, Mathematics **8**(5) (2020), Article ID 842, 18 pages. https://doi.org/10.3390/math8050842

[25] H. M. Srivastava, *Some generalizations and basic* (*or* q-)*extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inform. Sci. **5**(3) (2011), 390–444.

[26] A. Wiman, *Über den fundamentalsatz in der teorie der funktionen* $ea(x)$, Acta Math. **29** (1905), 191–201.

[27] Y. H. Xu and J. L. Liu. *Convolution and partial sums of certain multivalent analytic functions involving Srivastava-Tomovski generalization of the Mittag-Leffler function*, Symmetry **10**(11) (2018), Article ID 597, 8 pages. https://doi.org/10.3390/sym10110597

[28] C. Yan and J. Liu, *A family of meromorphic functions involving generalized Mittag-Leffler function*, J. Math. Inequal. **12**(4) (2018), 943–951. https://doi.org/10.7153/jmi-2018-12-71

[29] D. Yang, *On a class of meromorphic starlike multivalent functions*, Bull. Inst. Math. Acad. Sin. **24** (1996), 151–157.

[30] A. Zireh and S. Salehian, *Initial coefficient bounds for certain class of meromorphic bi-univalent functions*, Acta Univ. Sapientiae Math. **11**(1) (2019), 234–245. https://doi.org/10.2478/ausm-2019-0018

[1]Department of Mathematics,
College of Education for Pure Sciences, The University of Babylon,
51002 Babylon, Iraq
Email address: aqeel.ketab@uobabylon.edu.iq

[2]Department of Mathematics,
College of Science, University of Al-Qadisiyah,
Al Diwaniyah, Al-Qadisiyah, Iraq
Email address: abbas.kareem.w@qu.edu.iq