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# ENERGY AND SOMBOR ENERGY OF HYPERGRAPHS VIA A MATRIX

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ABSTRACT. Due to the limited availability of tools for analyzing the spectral properties of hypermatrices associated with hypergraphs, recent research has focused on studying these properties through related matrices derived from the hypergraph structure. To extend the concept of the degree-based extended adjacency matrix of graphs, we introduce a new definition of the (pairwise) Sombor index for hypergraphs. In this work, we characterize the extremal hypergraph that achieves the maximum Sombor spectral radius among all hypertrees. Additionally, we present preliminary results on computing the Sombor index and Sombor spectrum for specific classes of hypergraphs. Finally, we establish sharp bounds for energy and the Sombor energy of hypergraphs.

## 1. INTRODUCTION

The energy of a graph, defined [13] as the sum of the absolute values of the eigenvalues of its adjacency matrix, originated in theoretical chemistry to approximate the total $\pi$-electron energy in conjugated hydrocarbons.

Over time, this concept has found broader applications, notably in network analysis [1, 11, 12], where it aids in assessing structural properties like robustness and connectivity. Also, the application of energy in the context of macromolecular theory [9, 23], analysis of protein sequences [28, 31], pattern identification [32], and face recognition [27] is noteworthy. Additionally, areas like computer science [7], science and technology [24], biology [11], medicine [8], and many more could find application of energy in their respective fields. Thus, graph energy serves as a bridge between mathematical

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theory and practical applications across various scientific domains. A recent survey on the energy of graphs can be found in [15].

A hypergraph $\mathcal{H}$ is determined by the ordered pair $(\mathcal{V}, \mathcal{E})$. Here, the vertex set $\mathcal{V}$ is non-empty, and the edge set $\mathcal{E}$ is a subset of the power set $P^*(\mathcal{V})$ (set of all non-empty subsets) of the vertex set. The order of the hypergraph is the number of vertices ($|\mathcal{V}|$), and the size of the hypergraph is the number of hyperedges ($|\mathcal{E}|$) in the hypergraph. The rank (resp. co-rank) of a hypergraph $\mathcal{H}$ is defined as the maximum (resp. minimum) number of vertices in a hyperedge of $\mathcal{H}$. A hypergraph is said to be a $k$-uniform hypergraph if every hyperedge contains exactly $k$ vertices. The degree of a vertex $u$ in a hypergraph $\mathcal{H}$ is the number of hyperedges containing $u$ in $\mathcal{H}$. The maximum degree and minimum degree of the hypergraph $\mathcal{H}$ are denoted by $\Delta(\mathcal{H})$ and $\delta(\mathcal{H})$ (or simply $\Delta$ and $\delta$), respectively. A vertex of degree one is called a pendant vertex. The hyperedge containing only one non-pendant vertex is called a pendant hyperedge, and if it has more than one, then it is called a non-pendant hyperedge.

Codegree of two vertices $u$ and $v$ is the number of hyperedges containing both $u$ and $v$, denoted by $d_{uv}$. A walk of length $t$ is an alternating sequence of vertices and hyperedges $v_1e_1v_2e_2v_3\dots v_{t-1}e_tv_t$ such that $v_{i-1} \neq v_i$ and $v_i, v_{i+1} \in e_i$ for all $i = 1, \dots, t$. A path is a walk where all the vertices and hyperedges are distinct. The cycle is a closed path with initial and terminal vertices that are the same. The hypergraph $\mathcal{H}$ is said to be connected if there exists a path between two vertices. A hypertree $\mathcal{T}$ is a connected hypergraph with no cycle.

Several attempts have been made to encode hypergraphs using matrices [2, 20] and hypermatrices [6] (or tensors) to study their spectral properties. Recently, many authors have associated various (Laplacian, Seidel, and incidence) matrices with hypergraphs in order to analyze their spectral properties.

In this article, we consider the following definition of the adjacency matrix of the hypergraph that is defined by A. Banerjee [2].

**Definition 1.1** ([2])**.** Let $\mathcal{H}$ be a general hypergraph and $\mathcal{E}_u = \{e \in \mathcal{E} : u \in e\}$. The adjacency matrix of the hypergraph $\mathcal{H}$ of order $n$, denoted by $A(\mathcal{H}) = (a_{uv})$ is an $n \times n$ matrix (whose rows and columns correspond to vertices of the hypergraph) and the $uv$-th entry of which is given by

$$a_{uv} = \begin{cases} \sum\limits_{e \in (\mathcal{E}_u \cap \mathcal{E}_v)} \frac{1}{|e|-1}, & \text{if } u \sim v \text{ (i.e., } \mathcal{E}_u \cap \mathcal{E}_v \neq \emptyset), \\ 0, & \text{otherwise.} \end{cases}$$

For a $k$-uniform hypergraph, if $u$ and $v$ are adjacent, then $uv$-th entry of the adjacency matrix reduces to $\frac{d_{uv}}{k-1}$.

Similar to the case of graphs, the energy of a hypergraph is defined as the sum of the absolute eigenvalues of the adjacency matrix of the hypergraph. Developments in the study of energy of hypergraphs [4, 5, 18, 33] in the literature is based on the (non-normalized) adjacency matrix defined in [20].

In 2021, Gutman proposed [14] the definition of the Sombor index, a topological index based on vertex degrees. It has become well-known in molecular graph theory because of its capacity to predict the physicochemical properties of chemical compounds just based on its structural information (modeling via graphs). A review on the extremal results of the Sombor index of graphs can be found in [21].

An introductory work on the Sombor index of hypergraphs and the definitions of the vertex-degree-based (VDB) topological indices for hypergraphs have been listed in [25]. A very few studies addressing the VDB topological indices of hypergraphs can be found [10, 19, 26, 29]. A further generalization of the definition of the Sombor index of hypergraphs is studied in [30]. In this study, we consider the pairwise Sombor index of the hypergraph $\mathcal{H}$, that is defined as

$$PSO(\mathcal{H}) = \sum_{\substack{u,v\in\mathcal{V}(\mathcal{H}):\\ \{u,v\}\subseteq e\in\mathcal{E}(\mathcal{H})}} \sqrt{d_u^2 + d_v^2}.$$

The real motivation behind defining the pairwise Sombor index is to generalize the extended adjacency matrix to hypergraphs. Now, from the definition of the pairwise Sombor index and the adjacency matrix of the hypergraph, it is straightforward to define the pairwise Sombor matrix (or simply the Sombor matrix) of the hypergraph.

**Definition 1.2.** The Sombor matrix of the hypergraph $\mathcal{H}$ of order $n$, denoted by $A_{PSO}(\mathcal{H}) = (b_{uv})$ is an $n \times n$ matrix (whose rows and columns correspond to vertices of the hypergraph) and the $uv$-th entry of which is given by

$$b_{uv} = \begin{cases} \sqrt{d_u^2 + d_v^2} \sum\limits_{e\in(\mathcal{E}_u\cap\mathcal{E}_v)} \frac{1}{|e|-1}, & \text{if } u \sim v,\\ 0, & \text{otherwise.} \end{cases}$$

For all other terminologies, the reader can refer to [3] for the theory of hypergraphs and [16] for the theory of matrices.

Section 2 discusses the computation of the pairwise Sombor index and Sombor spectrum of specific classes of hypergraphs. The problem of characterizing the extremal hypertree attaining the maximum Sombor spectral radius, along with some bounds for the same, is considered in Section 3. Section 4 is dedicated to some sharp bounds for the Sombor energy of hypergraphs.

## 2. Preliminary Results

The expressions for the Sombor index and Sombor spectrum of some classes of hypergraphs that follow from direct computations are stated in this section.

**Definition 2.1.** • A *sunflower* hypergraph $\mathcal{S}(m, c, k)$ is a $k$-uniform hypergraph with $m$ hyperedges, where each hyperedge contain exactly $c$ vertices of degree $m$ and $k - c$ vertices of degree one.

• A *complete hypergraph* $\mathcal{K}_n$ is a hypergraph on the vertex set $\mathcal{V}$ ($|\mathcal{V}| = n$) with edge set as the set of all possible non-empty subsets of $\mathcal{V}$.

• A $k$-uniform ($k \geq 2$) complete hypergraph $\mathcal{K}_n^{(k)}$ is a hypergraph on the vertex set $\mathcal{V}$ ($|\mathcal{V}| = n$) with edge set as the set of all possible $k$-subsets of $\mathcal{V}$.

• A hypergraph $\mathcal{H}$ is said to be *bipartite* if its vertex set can be partitioned into two non-empty subsets $V_1$ and $V_2$ such that every hyperedge of $\mathcal{H}$ has a non-empty intersection with both $V_1$ and $V_2$. A bipartite hypergraph ($\mathcal{V} = V_1 \cup V_2, \mathcal{E}$) with $|V_1| = p$ and $|V_2| = q$ is said to be a *complete bipartite hypergraph* $\mathcal{K}_{p,q}$, if $\mathcal{E}$ contain all possible subsets of $\mathcal{V}$ such that every hyperedge has a non-empty intersection with both $V_1$ and $V_2$.

• A $k$-uniform complete bipartite hypergraph is a bipartite hypergraph with edge set as the set of all possible $k$-subsets of the vertex set such that each of which has a non-empty intersection with both partite sets.

• Let $G = (V(G), E(G))$ be a simple graph. For an integer $r \geq 3$, the $r$ power hypergraph of $G$ is an $r$-uniform hypergraph, $G^r = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}(G^r) = V(G) \cup \left(\bigcup_{e \in E(G)} V_e\right)$, where $V_e = \{u_1^{(e)}, \dots, u_{r-2}^{(e)}\}$ is a newly added vertices corresponding to each edge of $G$, and $\mathcal{E}(G^r) = \{e \cup V_e : e \in E(G)\}$. Figure 1 presents an example of the graph $G$ and its 4-power hypergraph.

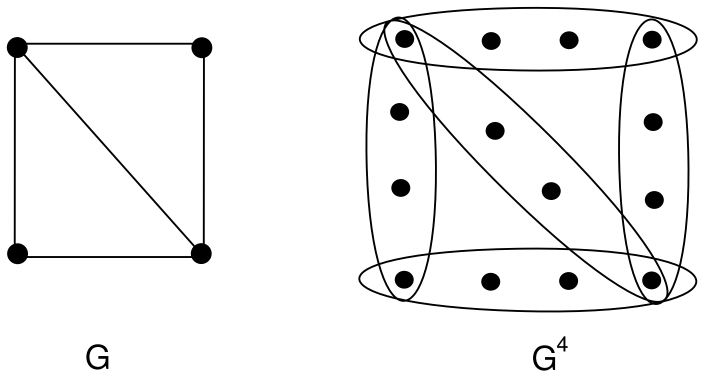


FIGURE 1. The graph $G$ and corresponding 4-power hypergraph.

• A $k$-uniform hyperstar, hypercycle, and hyperpath are respectively, the power hypergraph of star, cycle, and path.

Figures 2 to 5 represent pictorial depiction of hyperpath $\mathcal{P}_3^4$, hyperstar $\mathcal{S}_5^4$, sunflower hypergraph $\mathcal{S}(5, 3, 5)$ and 4-uniform hypercycle, respectively.

In the following proposition, the expressions for the pairwise Sombor index of the above mentioned class of hypergraphs are discussed.

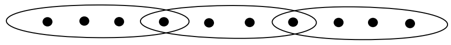

FIGURE 2. The hyperpath $\mathcal{P}_3^4$.

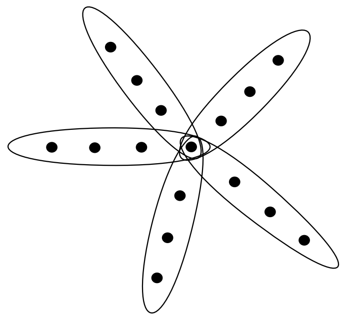

FIGURE 3. The hyperstar $\mathcal{S}_5^4$.

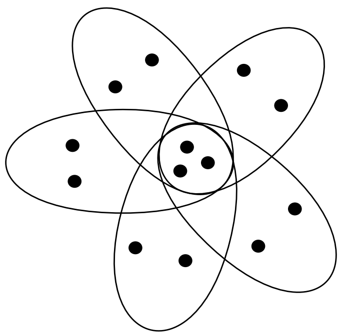

FIGURE 4. The sunflower hypergraph $\mathcal{S}(5,3,5)$.

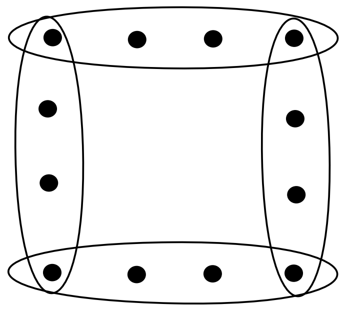

FIGURE 5. 4-uniform hypercycle.

**Proposition 2.1.** *Let $\mathcal{H}$ be a hypergraph of order $n$ and size $m$.*

- *The pairwise Sombor index of a single hyperedge (i.e., $\mathcal{E} = \{\mathcal{V}\}$) is given by $\binom{n}{2}\sqrt{2}$.*

*The pairwise Sombor index of the sunflower hypergraph $\mathcal{S}(m,c,k)$ is given by*

$$PSO(\mathcal{S}(m,c,k)) = \binom{c}{2}\sqrt{2m^2} + mc(k-c)\sqrt{m^2+1} + m\binom{k-c}{2}\sqrt{2}.$$

- *The pairwise Sombor index of the linear hyperpath $\mathcal{P}_n^k$ ($k$ power hypergraph of the path) is given by*

$$PSO(\mathcal{P}_n^k) = \left(2\binom{k-1}{2} + (m-2)\binom{k-2}{2}\right)\sqrt{2} \\ + 2\left(mk - k - 2m + 3\right)\sqrt{5} + (m-2)\sqrt{8},$$

*where $m = \frac{n-1}{k-1}$.*

- *The pairwise Sombor index of the hypercycle $\mathcal{C}_n^k$ ($k$ power hypergraph of the cycle) is given by*

$$PSO(\mathcal{C}_n^k) = \left(m\binom{k-2}{2}\right)\sqrt{2} + 2m\left(k-2\right)\sqrt{5} + m\sqrt{8},$$

*where* $m = \frac{n}{k-1}$.

• *The pairwise Sombor index of the complete bipartite hypergraph* $\mathcal{K}_{p,q}$, $p, q \geq 2$ *is given by*

$$PSO(\mathcal{K}_{p,q}) = \binom{p}{2} D_1\sqrt{2} + \binom{q}{2} D_2\sqrt{2} + pq\sqrt{D_1^2 + D_2^2},$$

*where* $D_1 = 2^{p-1}(2^q - 1)$ *and* $D_2 = 2^{q-1}(2^p - 1)$.

• *The pairwise Sombor index of the complete hypergraph* $\mathcal{K}_n$ *(hyperedge of size one is allowed) is given by*

$$PSO(\mathcal{K}_n) = \binom{n}{2}\sqrt{2^{2n-1}}.$$

• *The pairwise Sombor index of the* $k$*-uniform* $(k \geq 2)$ *complete hypergraph* $\mathcal{K}_n^{(k)}$ *is given by*

$$PSO(\mathcal{K}_n^{(k)}) = \binom{n}{2}\binom{n-1}{k-1}\sqrt{2}.$$

2.1. **Sombor Spectrum of Hypergraphs.** In the following results, we obtain the Sombor spectrum of some classes of hypergraphs.

**Theorem 2.1.** *Let* $\mathcal{K}_n^*$ *be the hypergraph obtained from* $\mathcal{K}_n$ *by removing all hyperedges of size* 1*, be a complete hypergraph on* $n$ *vertices. Then,*

(a) $Spec(A_{PSO}(\mathcal{K}_n^*)) = \begin{pmatrix} \sqrt{2}(2^{n-1}-1)D_n(n-1) & -\sqrt{2}(2^{n-1}-1)D_n \\ 1 & (n-1) \end{pmatrix}$,

(b) $\mathscr{E}(A_{PSO}(\mathcal{K}_n^*) = 2\sqrt{2}(2^{n-1}-1)D_n(n-1)$,

*where* $D_n = \sum_{t=1}^{n-1}\frac{1}{t}$.

*Proof.* The Sombor matrix of the complete hypergraph can be expressed as,

$$A_{PSO}(\mathcal{K}_n^*) = \sqrt{2}(2^{n-1}-1)D_n(J_n - I_n),$$

from which the proof follows. □

**Theorem 2.2.** *Let* $\mathcal{S}(m,c,k)$ *be a* $k$*-uniform sunflower hypergraph with* $m$ *hyperedges. Then,*

(a) $Spec(A_{PSO}(\mathcal{S}(m,c,k))) = \begin{pmatrix} \mu_1 & \mu_2 & \frac{-\sqrt{2}}{k-1} & \sqrt{2}(\frac{k-c-1}{k-1}) & \frac{-\sqrt{2}m^2}{k-1} \\ 1 & 1 & n-m-c & m-1 & c-1 \end{pmatrix}$,

(b) $\mathscr{E}(A_{PSO}(\mathcal{S}(m,c,k))) = \frac{\sqrt{2}}{k-1}((m(c-1)+k-c-2)m + n - k + 1 + \theta_1)$,

*where*

$$\mu_1, \mu_2 = \frac{(m^2(c-1)+(k-c-1)) \pm \sqrt{(m^2(c-1)+(k-c-1))^2 - 2(2m^2(c-1)(k-c-1) - mc(k-c)(m^2+1))}}{\sqrt{2}(k-1)},$$

*and* $\theta_1 = \sqrt{(m^2(c-1)+(k-c-1))^2 - 2(2m^2(c-1)(k-c-1) - mc(k-c)(m^2+1))}$.

*Proof.* Let the Sombor matrix of the $k$-uniform sunflower hypergraph be expressed as a block matrix as follows:

$$A_{PSO}(\mathcal{S}(m,c,k)) = \begin{pmatrix} \frac{\sqrt{2}m^2}{k-1}A(\mathcal{K}_c) & \frac{\sqrt{m^2+1}}{k-1}J_{c,m(k-c)} \\ \frac{\sqrt{m^2+1}}{k-1}J_{m(k-c),c} & I_m \otimes \frac{\sqrt{2}}{k-1}A(\mathcal{K}_{k-c}) \end{pmatrix}.$$

By performing row operations, we get

$$\begin{aligned} |\mu I - A_{PSO}(\mathcal{S}(m,c,k))| = {} & \left(\mu + \frac{\sqrt{2}m^2}{k-1}\right)^{c-1}\left(\mu - \sqrt{2}\left(\frac{k-c-1}{k-1}\right)\right)^{m-1} \\ & \times \left(\mu + \frac{\sqrt{2}}{k-1}\right)^{n-m-c}(\mu-\mu_1)(\mu-\mu_2), \end{aligned}$$

where $\mu_1$ and $\mu_2$ are the remaining two eigenvalues of the matrix that can be obtained from the quotient matrix

$$Q(A_{PSO}(\mathcal{S}(m,c,k))) = \begin{pmatrix} \frac{m^2(c-1)\sqrt{2}}{k-1} & \frac{m(k-c)\sqrt{m^2+1}}{k-1} \\ \frac{c\sqrt{m^2+1}}{k-1} & \frac{(k-c-1)\sqrt{2}}{k-1} \end{pmatrix}.$$

□

It is direct from the definition of the hyperstar that the Sombor spectrum of which can be obtained by putting $c = 1$ in the expression for the Sombor spectrum of the sunflower hypergraph.

**Corollary 2.1.** *Let $\mathcal{S}_m^{(k)}$ be a $k$-uniform hyperstar with $m$ hyperedges. Then,*

(a) $Spec(A_{PSO}(\mathcal{S}_m^{(k)})) = \begin{pmatrix} \mu_1 & \mu_2 & \frac{-\sqrt{2}}{k-1} & \sqrt{2}(\frac{k-2}{k-1}) \\ 1 & 1 & n-m-1 & m-1 \end{pmatrix}$,

(b) $\mathscr{E}(A_{PSO}(\mathcal{S}_m^{(k)})) = \frac{\sqrt{2}}{k-1}((m-1)k + n - 3m + 1 + \theta_2)$,

*where* $\mu_1, \mu_2 = \frac{(k-2) \pm \sqrt{(k-2)^2 + 2m(k-1)(m^2+1)}}{\sqrt{2}(k-1)}$ *and* $\theta_2 = \sqrt{(k-2)^2 + 2m(k-1)(m^2+1)}$.

**Theorem 2.3.** *Given $k \geq 3$ and $p, q \geq 2$, let $\mathcal{K}_{p,q}^{(k)}$ be a $k$-uniform complete bipartite hypergraph with $m$ hyperedges. Then,*

(a) $Spec(A_{PSO}(\mathcal{K}_{p,q}^{(k)})) = \begin{pmatrix} -\sqrt{2}t_1D_1 & -\sqrt{2}t_2D_2 & \mu_1 & \mu_2 \\ p-1 & q-1 & 1 & 1 \end{pmatrix}$,

(b) $\mathscr{E}(A_{PSO}(\mathcal{K}_{p,q}^{(k)})) = \sqrt{2}(t_1D_1(p-1) + t_2D_2(q-1) + \eta_1)$,

*where* $D_1 = \sum_{i=0}^{k-2}\binom{p-1}{i}\binom{q}{k-i-1}$, $D_2 = \sum_{j=0}^{k-2}\binom{p}{k-j-1}\binom{q-1}{j}$, $t_1 = \frac{1}{k-1}\sum_{i=0}^{k-3}\binom{p-2}{i}\binom{q}{k-i-2}$, $t_2 = \frac{1}{k-1}\sum_{i=0}^{k-3}\binom{p}{k-i-2}\binom{q-2}{i}$,

$$\mu_1, \mu_2 = \frac{((p-1)t_1D_1 + (q-1)t_2D_2) \pm \sqrt{((p-1)t_1D_1 + (q-1)t_2D_2)^2 + 2pqt_3^2(D_1^2 + D_2^2)}}{\sqrt{2}},$$

$\eta_1 = \sqrt{((p-1)t_1D_1 + (q-1)t_2D_2)^2 + 2pqt_3^2(D_1^2 + D_2^2)}$, $t_3 = \frac{1}{k-1}\sum_{i=0}^{k-2}\binom{p-1}{i}\binom{q-1}{k-i-2}$, *provided* $\binom{r}{s} = 0$ *if* $r < s$.

*Proof.* Let the Sombor matrix of a $k$-uniform complete bipartite hypergraph be expressed as a block matrix as follows:

$$A_{PSO}(\mathcal{K}_{p,q}^{(k)}) = \begin{pmatrix} \sqrt{2}t_1 D_1 A(\mathcal{K}_p) & t_3\sqrt{D_1^2+D_2^2}J_{p,q} \\ t_3\sqrt{D_1^2+D_2^2}J_{q,p} & \sqrt{2}t_2 D_2 A(\mathcal{K}_q) \end{pmatrix}.$$

By performing row operations, we get

$$|\mu I - A_{PSO}(\mathcal{K}_{p,q}^{(k)})| = \left(\mu + \sqrt{2}t_1 D_1\right)^{p-1}\left(\mu + \sqrt{2}t_2 D_2\right)^{q-1}(\mu-\mu_1)(\mu-\mu_2),$$

where $\mu_1$ and $\mu_2$ are the remaining two eigenvalues that can be obtained from the quotient matrix,

$$Q(A_{PSO}(\mathcal{K}_{p,q}^{(k)})) = \begin{pmatrix} \sqrt{2}(p-1)D_1 t_1 & \sqrt{D_1^2+D_2^2}pt_3 \\ \sqrt{D_1^2+D_2^2}qt_3 & \sqrt{2}(q-1)D_2 t_2 \end{pmatrix}.$$

□

**Theorem 2.4.** *Given $p, q \geq 2$, let $\mathcal{K}_{p,q}$ be the complete bipartite hypergraph with $m$ hyperedges. Then,*

(a) $Spec(A_{PSO}(\mathcal{K}_{p,q})) = \begin{pmatrix} -\sqrt{2}f_1R_1 & -\sqrt{2}f_2R_2 & \mu_1 & \mu_2 \\ p-1 & q-1 & 1 & 1 \end{pmatrix}$,

(b) $\mathscr{E}(A_{PSO}(\mathcal{K}_{p,q})) = \sqrt{2}(f_1R_1(p-1) + f_2R_2(q-1) + \eta_2)$,

*where* $R_1 = 2^{(p-1)}(2^q-1)$, $R_2 = 2^{(q-1)}(2^p-1)$,

$$f_1 = \sum_{k=3}^{p+q}\frac{1}{k-1}\sum_{i=0}^{k-3}\binom{p-2}{i}\binom{q}{k-i-2},$$

$$f_2 = \sum_{k=3}^{p+q}\frac{1}{k-1}\sum_{i=0}^{k-3}\binom{p}{k-i-2}\binom{q-2}{i},$$

$$\mu_1, \mu_2 = \frac{((p-1)f_1R_1 + (q-1)f_2R_2) \pm \sqrt{((p-1)f_1R_1 + (q-1)f_2R_2)^2 + 2pqf_3^2(R_1^2+R_2^2)}}{\sqrt{2}},$$

$$\eta_2 = \sqrt{((p-1)f_1R_1 + (q-1)f_2R_2)^2 + 2pqf_3^2(R_1^2+R_2^2)},$$

$$f_3 = \sum_{k=2}^{p+q}\frac{1}{k-1}\sum_{i=0}^{k-2}\binom{p-1}{i}\binom{q-1}{k-i-2},$$

*provided* $\binom{r}{s} = 0$ *if* $r < s$.

*Proof.* Let the Sombor matrix of a complete bipartite hypergraph be expressed as the block matrix as follows:

$$A_{PSO}(\mathcal{K}_{p,q}) = \begin{pmatrix} \sqrt{2}f_1R_1A(\mathcal{K}_p) & f_3\sqrt{R_1^2+R_2^2}J_{p,q} \\ f_3\sqrt{R_1^2+R_2^2}J_{q,p} & \sqrt{2}f_2R_2A(\mathcal{K}_q) \end{pmatrix}.$$

Suppose that $|\mu I - A_{PSO}(\mathcal{K}_{p,q})|$ is the characteristic polynomial of the matrix

$A_{PSO}(\mathcal{K}_{p,q})$, and by performing row operations, we get

$$|\mu I - A_{PSO}(\mathcal{K}_{p,q})| = \left(\mu + \sqrt{2} f_1 R_1\right)^{p-1} \left(\mu + \sqrt{2} f_2 R_2\right)^{q-1} (\mu - \mu_1)(\mu - \mu_2),$$

where $\mu_1$ and $\mu_2$ are the remaining two eigenvalues that are obtained from the quotient matrix,

$$Q(A_{PSO}(\mathcal{K}_{p,q})) = \begin{pmatrix} \sqrt{2}(p-1) f_1 R_1 & \sqrt{R_1^2 + R_2^2} p f_3 \\ \sqrt{R_1^2 + R_2^2} q f_3 & \sqrt{2}(q-1) f_2 R_2 \end{pmatrix}.$$

□

## 3. Sombor Spectral Radius

This section deals with the Sombor spectral radius of hypergraphs, and in later parts, we attempt to characterize the extremal hypertree that attains the maximum Sombor spectral radius.

Throughout this article $\mu_1(\mathcal{H})$ denotes the largest eigenvalue of $A_{PSO}(\mathcal{H})$, and $\lambda_1(\mathcal{H})$ denotes the largest (adjacency) eigenvalue of $\mathcal{H}$. We have that if $A_1 \leq A_2$, then $\rho(A_1) \leq \rho(A_2)$. By using this, we get the following simple result.

**Lemma 3.1.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph and $e \notin \mathcal{E}$. Then, $\mu_1(\mathcal{H} + e) > \mu_1(\mathcal{H})$.*

By using the above lemma, we have the following upper bound for the spectral radius of a hypergraph.

**Theorem 3.1.** *For a general hypergraph $\mathcal{H}$ with* co-rank$(\mathcal{H}) \geq 2$*, we have*

$$\mu_1(A_{PSO}(\mathcal{H})) \leq D_n(n-1)(2^{n-1}-1)\sqrt{2}, \qquad \text{where } D_n = \sum_{t=1}^{n-1} \frac{1}{t},$$

*and the equality is attained by the hypergraph $\mathcal{K}_n^*$, that is obtained from $\mathcal{K}_n$ by removing all hyperedges of size* 1*.*

**Theorem 3.2.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a non-trivial connected hypergraph of* rank$(\mathcal{H}) = p$ *and* co-rank$(\mathcal{H}) = q$*. Then,*

$$\frac{\sqrt{2}\delta^2(q-1)}{p-1} \leq \mu_1 \leq \frac{\sqrt{2}\Delta^2(p-1)}{q-1}.$$

*Proof.* Let $\mathbf{x}$ be the Perron vector of $A_{PSO}(\mathcal{H})$, and $x_u$ and $x_v$, respectively be the component of $\mathbf{x}$ corresponding to the vertices $u$ and $v$ of $\mathcal{H}$, such that $x_v = \max_{i\in\mathcal{V}} x_i$ and $x_u = \min_{i\in\mathcal{V}} x_i$.

We know that, $(A_{PSO}(\mathcal{H})\mathbf{x})_v = \sum_{u\sim v} \sum_{e\in(\mathcal{E}_u\cap\mathcal{E}_v)} \frac{\sqrt{d_u^2+d_v^2}}{|e|-1} x_u = \mu_1 x_v$. Therefore,

$$\mu_1 \leq \frac{\sqrt{2}\Delta^2(p-1)x_v}{(q-1)x_v}. \tag{3.1}$$

Equality in (3.1) holds for uniform and regular hypergraphs. Since $\mathcal{H}$ is connected, we have $\mathbf{x} > 0$, and hence $x_u = \min_{i \in \mathcal{V}} x_i > 0$ and

$$\mu_1 \geq \frac{\sqrt{2}\delta^2(q-1)}{(p-1)}. \tag{3.2}$$

□

**Corollary 3.1.** *Let $\mathcal{H}$ be a $k$-uniform linear hypergraph on $n$ vertices with $m$ hyperedges. Then,* $\sqrt{2}\delta^2 \leq \mu_1 \leq \sqrt{2}\Delta^2$.

**Theorem 3.3.** *Let $\mathcal{H}$ be a linear hypergraph of* $\operatorname{rank}(\mathcal{H}) = p$ *and* $co - \operatorname{rank}(\mathcal{H}) = q$. *Then,*

$$\mu_1 \geq \frac{\sqrt{2}q(q-1)\delta m}{n(p-1)}.$$

*Proof.* Let $\mathbf{1} := (1, \ldots, 1)$ be a all-one vector of length $n$ and $\mu_1$ be the spectral radius of the matrix $A_{PSO}(\mathcal{H})$. Then,

$$\begin{aligned}\mu_1 &\geq \frac{\mathbf{1}A_{PSO}\mathbf{1}^\intercal}{\mathbf{1}\mathbf{1}^\intercal} = \frac{1}{n(k-1)}\sum_{i=1}^{n}\sum_{i\sim j}\sqrt{d_i^2 + d_j^2} \\ &\geq \frac{\sqrt{2}\delta}{n(p-1)}\sum_{i=1}^{n} d_i(q-1) \geq \frac{\sqrt{2}q(q-1)\delta m}{n(p-1)}.\end{aligned}$$

□

**Corollary 3.2.** *Let $\mathcal{H}$ be a $k$-uniform linear hypergraph on $n$ vertices and $m$ hyperedges. If $\lambda_1$ and $\mu_1$ are the adjacency and Sombor spectral radius of $\mathcal{H}$, respectively, then*

$$\mu_1 \geq \frac{\sqrt{2}km\delta}{n} \quad \text{and} \quad \lambda_1 \geq \frac{km}{n}.$$

In the following, we state a couple of transformation of a hypergraph that are used to maximize the spectral radius.

**Definition 3.1** (Edge moving operation)**.** Suppose that $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a $k$-uniform hypergraph with $u \in \mathcal{V}$ and $e_1, \ldots, e_t \in \mathcal{E}$, such that $u \notin e_i$ for $i = 1, \ldots, t$. Furthermore, suppose that $v_i \in e_i$ be the non-pendant vertex. Construct $e_i' = (e_i \backslash \{v_i\}) \cup u$. Let $\mathcal{H}^\dagger = (\mathcal{V}, \mathcal{E}^\dagger)$ be the hypergraph with $\mathcal{E}^\dagger = (\mathcal{E} \backslash \{e_1, \ldots, e_t\}) \cup \{e_1', \ldots, e_t'\}$. Then we say that $\mathcal{H}^\dagger$is obtained from $\mathcal{H}$ by moving the edges $\{e_1, \ldots, e_t\}$ from $(v_1, \ldots, v_t)$ to $u$ (all $e_i$'s are distinct but $v_i, 1 \leq i \leq t$ need not be distinct).

An example of moving edges $(e_1, e_2, e_3)$ from $(v_1, v_2, v_2)$ to $u$ has been pictorially depicted in Figure 6.

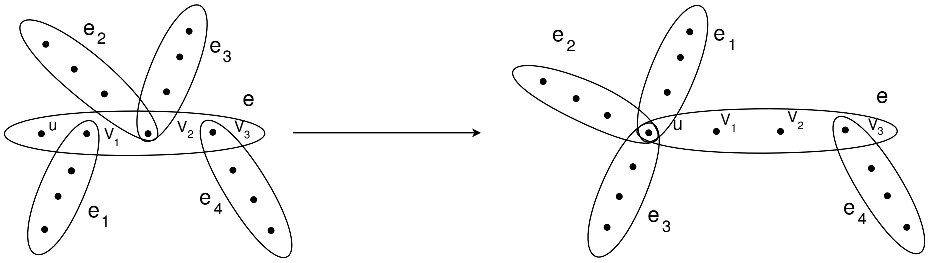


FIGURE 6. Moving of edges $(e_1, e_2, e_3)$ from $(v_1, v_2, v_2)$ to $u$.

**Definition 3.2** (Edge-releasing operation)**.** Suppose that $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a $k$-uniform hypergraph and $e$ be a non-pendant edge with $u \in e$. Let $\{e_1, \dots, e_t\}$ be all the edges of $\mathcal{H}$ adjacent to the edge $e$. Also let $e \cap e_i = v_i$ for $i = 1, \dots, t$. Let $\mathcal{H}^*$ be the hypergraph obtained from $\mathcal{H}$ by moving edges $\{e_1, \dots, e_t\}$ from $(v_1, \dots, v_t)$ to $u$. Then, $\mathcal{H}^*$ is said to be obtained from $\mathcal{H}$ by edge-releasing operation on $e$ at $u$.

Figure 7 depicts the edge-releasing of $e$ at $v_1$ and $v_2$.

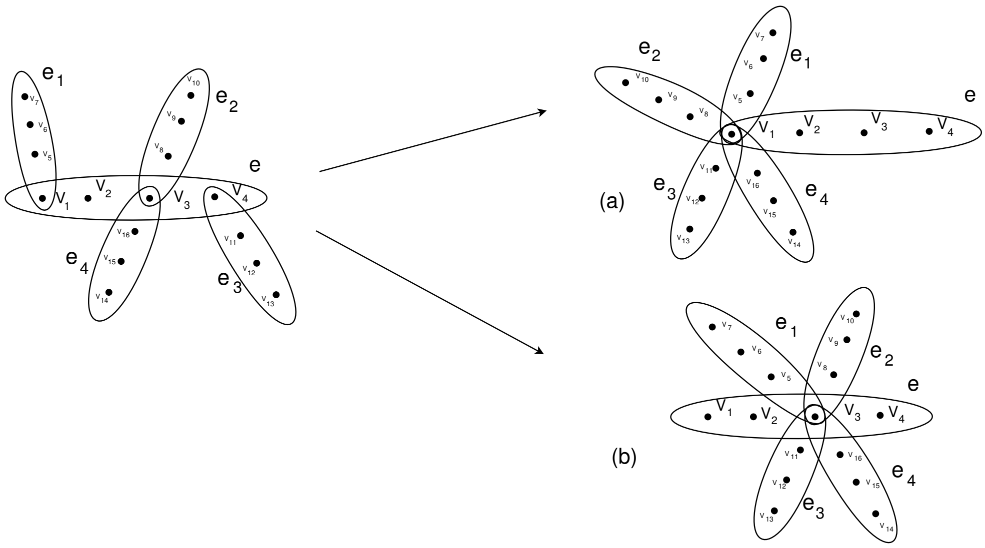


FIGURE 7. Edge-releasing of $e$ at $v_1$ (in $(a)$) and $v_2$ (in $(b)$).

**Lemma 3.2.** *Suppose that $e$ is a non-pendant edge of $\mathcal{T}$ and $u \in e$. Then, $\mu_1(\mathcal{T}') > \mu_1(\mathcal{T})$.*

*Proof.* (Incomplete) Let us consider $u \in e$ such that $x_u = \max\{x_v : v \in e\}$ for the principal eigenvector $\mathbf{x}$ of $\mathcal{T}$, and $e$ be a non-pendant edge with $u \in e$, also $\{e_1, \dots, e_t\}$ be all the edges of $\mathcal{T}$ adjacent to the edge $e$, and $e \cup e_i = \{v_i\}$ for $i = 1, \dots, t$.

Suppose $\mathcal{T}'$ is the hypergraph obtained from $\mathcal{T}$ by moving edges $\{e_1, \dots, e_t\}$ from $(v_1, \dots, v_t)$ to $u$. Consider,

$$\begin{aligned}
&\mu_1(\mathcal{T}') - \mu_1(\mathcal{T}) \geq \mathbf{x}^\intercal A_{PSO}(\mathcal{T}')\mathbf{x} - \mu_1(\mathcal{T})\\
=&\mathbf{x}^\intercal A_{PSO}(\mathcal{T}')\mathbf{x} - \mathbf{x}^\intercal A_{PSO}(\mathcal{T})\mathbf{x}\\
=&\frac{1}{m-1}\left(\sum_{i\sim j\ in\ \mathcal{T}'} 2\sqrt{d_i'^2 + d_j'^2}x_i x_j - \sum_{i\sim j\ in\ \mathcal{T}} 2\sqrt{d_i^2 + d_j^2}x_i x_j\right)\\
=&\frac{2}{m-1}\left(\sum_{i=1}^{t}\left(\sum_{j\in e_i; j\neq v_i}\sqrt{d_u'^2 + d_j^2}x_u x_j - \sum_{j\in e_i; j\neq v_i}\sqrt{d_{v_i}^2 + d_j^2}x_{v_i} x_j\right)\right.\\
&+\left(\sum_{u\sim j; j\notin e, e_i}\sqrt{d_u'^2 + d_j^2}x_u x_j - \sum_{u\sim j; j\notin e, e_i}\sqrt{d_u^2 + d_j^2}x_u x_j\right)\\
&\left.+\left(\sum_{i\sim j; i,j\in e}\sqrt{d_i'^2 + d_j'^2}x_i x_j - \sum_{i\sim j; i,j\in e}\sqrt{d_i^2 + d_j^2}x_i x_j\right)\right)
\end{aligned}$$

$$=\frac{2}{m-1}\left(\sum_{i=1}^{t}\left(\sum_{j\in e_i; j\neq v_i}\left(\sqrt{d_u'^2 + d_j^2}x_u - \sqrt{d_{v_i}^2 + d_j^2}x_{v_i}\right)x_j\right)\right. \tag{3.3}$$

$$+\left(\sum_{u\sim j; j\notin e, e_i}\left(\sqrt{d_u'^2 + d_j^2} - \sqrt{d_u^2 + d_j^2}\right)x_u x_j\right) \tag{3.4}$$

$$\left.+\left(\sum_{i\sim j; i,j\in e}\left(\sqrt{d_i'^2 + d_j'^2} - \sqrt{d_i^2 + d_j^2}\right)x_i x_j\right)\right). \tag{3.5}$$

Our aim is to prove that, $\mu_1(\mathcal{T}') - \mu_1(\mathcal{T}) \geq 0$. Since $x_u \geq x_{v_i}$, we have $x_u\sqrt{d_u'^2 + d_j^2} - x_{v_i}\sqrt{d_{v_i}^2 + d_j^2} = x_u\sqrt{(d_u+t)^2 + d_j^2} - x_{v_i}\sqrt{d_u^2 + d_j^2} > 0$ for each $j \in e_i, j \neq v_i$. Hence, the summand in (3.3) is strictly positive. Also, the summand (3.4) $\sqrt{d_u'^2 + d_j^2} - \sqrt{d_u^2 + d_j^2} = \sqrt{(d_u+t)^2 + d_j^2} - \sqrt{d_u^2 + d_j^2}$, $u \sim j$, $j \notin e, e_i$ strictly positive. The last summand in (3.5), $\sum_{i\sim j; i,j\in e}\left(\sqrt{d_i'^2 + d_j'^2} - \sqrt{d_i^2 + d_j^2}\right)x_i x_j$ is correspond to the hyperedge that is edge released. Unfortunately, we couldn't prove that the summand (3.5) is greater than or equal to zero. □

**Theorem 3.4.** *Let $\mathcal{T}$ be a $k$-uniform hypertree with $m$ hyperedges. Also, let $\mu_1(\mathcal{T})$ be the Sombor spectral radius of $\mathcal{T}$. If Lemma* 3.2 *holds good, then $\mu_1(\mathcal{T}) \leq \mu_1(\mathcal{S}_m^{(k)})$. Equality holds if and only if $\mathcal{T} \cong \mathcal{S}_m^{(k)}$.*

*Proof.* Let $\mathbf{x}$ be the Perron vector of $A_{PSO}(\mathcal{T})$ and $\mu_1$ be its corresponding eigenvalue. Suppose $\mathcal{T}$ contains $t$ non-pendant hyperedges ($t \geq 2$). Let us assume that $\mathcal{T}$ has a maximum Sombor spectral radius. Let $u \in e$ be such that $x_u = \max\{x_v : v \in e\}$. Suppose $\mathcal{T}_1$ denotes the hypertree obtained from $\mathcal{T}$ by edge releasing $e$ at $u$. By using Lemma 3.2, we have $\mu_1(\mathcal{T}_1) > \mu_1(\mathcal{T})$, a contradiction to the assumption that $\mathcal{T}$ has maximum Sombor spectral radius. Therefore, by repeatedly performing the edge-releasing operation on the non-pendant hyperedges of the hypertree, we finally arrive at a hypergraph that does not contain non-pendant hyperedges. But we know that hyperstar is the unique hypertree without any non-pendant hyperedges.

Suppose there exist a hypertree $\mathcal{T}_2 \ncong \mathcal{S}_m^{(k)}$ with $\mu_1(\mathcal{T}_2) = \mu_1(\mathcal{S}_m^{(k)})$. Since $\mathcal{T}_2 \ncong \mathcal{S}_m^{(k)}$, the existence of the non-pendant hyperedge in $\mathcal{T}_2$ is guaranteed, and hence by a similar argument as above, we arrive at the contradiction. □

## 4. Bounds for Energy of Hypergraphs

In this section, we have obtained sharp bounds for the energy of linear hypergraphs and the Sombor energy of (general) hypergraphs in terms of order, size, rank, co-rank, maximum and minimum degree of the vertices of the hypergraph.

**Lemma 4.1.** *Let $\mathcal{H}$ be a $k$-uniform linear hypergraph with $m$ hyperedges. Then, $\sum\limits_{u\in\mathcal{V}(\mathcal{H})} d_u = km$. Also, $Tr(A(\mathcal{H})^2) = \frac{km}{(k-1)}$.*

*Proof.* Note that, if $\mathcal{H}$ is a linear hypergraph, then co-degree of any pair of adjacent vertices $u$ and $v$ in $\mathcal{H}$ is $d_{uv} = 1$. □

For the case of simple graphs, McClelland [22] proved that the energy is upper bounded by $\sqrt{2mn}$.

**Theorem 4.1** (McClelland-type bound for linear uniform hypergraphs)**.** *Let $\mathcal{H}$ be a $k$-uniform linear hypergraph on $n$ vertices with $m$ hyperedges. If $E(\mathcal{H})$ denotes the (adjacency) energy of $\mathcal{H}$, then*

$$E(\mathcal{H}) \leq \sqrt{\frac{kmn}{k-1}}.$$

*Proof.* If $\lambda_1, \ldots, \lambda_n$ denotes the eigenvalues of the adjacency matrix of $\mathcal{H}$, then

$$\begin{aligned}0 \leq \sum_{r=1}^{n}\sum_{s=1}^{n}(|\lambda_r| - |\lambda_s|)^2 &= n\sum_{r=1}^{n}|\lambda_r|^2 + n\sum_{s=1}^{n}|\lambda_s|^2 - 2\sum_{r=1}^{n}\sum_{s=1}^{n}|\lambda_r|\cdot|\lambda_s|\\ &= 2n\sum_{s=1}^{n}|\lambda_s|^2 - 2E(\mathcal{H})^2.\end{aligned}$$

Now, using Lemma 4.1, we have $\sum_{s=1}^{n}|\lambda_s|^2 = \frac{km}{k-1}$, from which the result follows. □

Initially, the following upper bound for simple graphs was proved [17] by Koole and Moulton.

**Theorem 4.2** (Koolen-Moulton type bound for linear uniform hypergraphs)**.** *Let $\mathcal{H}$ be a $k$-uniform ($k \geq 2$) linear hypergraph on $n$ vertices and $m$ hyperedges, with*

*$km \geq n$. If the spectral radius of $\mathcal{H}$ is upper bounded (strictly) by $\sqrt{km/(k-1)}$, then*

$$E(\mathcal{H}) \leq \frac{km}{n} + \sqrt{(n-1)\left(\frac{km}{k-1} - \left(\frac{km}{n}\right)^2\right)}.$$

*For $k = 2$, equality case is same as that of simple graphs.*

*Proof.* Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of the $k$-uniform linear hypergraph $\mathcal{H}$. If $s_1, \ldots, s_n$ denotes the row sum of the adjacency matrix of $\mathcal{H}$, then we know that

$$\lambda_1 \geq \frac{\sum_{i=1}^n s_i}{n} = \frac{\sum_{i=1}^n ((k-1)d_i)/(k-1)}{n} = \frac{km}{n}.$$

Using Lemma 4.1, we have

$$\sum_{i=2}^n \lambda_i^2 = \frac{km}{k-1} - \lambda_1^2.$$

Now, using the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=2}^n |\lambda_i|\right)^2 \leq (n-1)\left(\frac{km}{k-1} - \lambda_1^2\right).$$

Hence,

$$E(\mathcal{H}) \leq \lambda_1 + \sqrt{(n-1)\left(\frac{km}{k-1} - \lambda_1^2\right)}.$$

Let $g(y) := y + \sqrt{(n-1)((km/(k-1)) - y^2)}$. Then, $g(y)$ decreases in the interval $\left[\sqrt{(km)/n}, \sqrt{(km)/(k-1)}\right)$. Also, for $km \geq n$, $\sqrt{km/n} \leq km/n \leq \lambda_1$. Therefore, for $\lambda < \sqrt{km/(k-1)}$, we have

$$g(\lambda_1) \leq g(km/n) = \frac{km}{n} + \sqrt{(n-1)\left(\frac{km}{k-1} - \left(\frac{km}{n}\right)^2\right)}.$$

□

### 4.1. **Sombor energy of hypergraphs.**

**Definition 4.1.** For a hypergraph $\mathcal{H}$, the *Zagreb index* of $\mathcal{H}$ is denoted by $M_1(\mathcal{H})$ and is defined as the sum of the squares of the degrees of its vertices. Also, the *forgotten index* of the hypergraph $\mathcal{H}$, denoted by $F(\mathcal{H})$, is defined as the sum of the cubes of the degree of all the vertices of the hypergraph.

$$M_1(\mathcal{H}) = \sum_{v \in \mathcal{V}(\mathcal{H})} d_v^2 \quad \text{and} \quad F(\mathcal{H}) = \sum_{v \in V(\mathcal{H})} d_v^3.$$

**Theorem 4.3.** *Let $\mathcal{H}$ be a hypergraph on $n$ vertices and $m$ hyperedges. If* $\operatorname{rank}(\mathcal{H})$ *is* $p$ *and* $\operatorname{co-rank}(\mathcal{H})$ *is* $q$, *then*

$$\mathscr{E}_{PSO}(\mathcal{H}) \leq \sqrt{2n(p-1)F(\mathcal{H})}\frac{\Delta}{q-1} \leq \sqrt{2nmp(p-1)}\frac{\Delta^2}{q-1}.$$

*The equality holds if and only if $\mathcal{H}$ is a 2-graph with isolated edges and no isolated vertices, or if $\mathcal{H}$ is a trivial hypergraph (edgeless).*

*Proof.* Suppose $\mu_1, \dots, \mu_n$ are the eigenvalues of $A_{PSO}(\mathcal{H})$. Then,

$$\sum_{i=1}^{n} \mu_i^2 = Tr(A_{PSO}^2(\mathcal{H})) = \sum_{v\in\mathcal{V}(\mathcal{H})} \sum_{u\in N(v)} (d_u^2 + d_v^2) \left( \sum_{e\in(\mathcal{E}_u\cap\mathcal{E}_v)} \frac{1}{|e|-1} \right)^2$$

$$\leq \frac{1}{(q-1)^2} \sum_{v\in\mathcal{V}(\mathcal{H})} \sum_{u\in N(v)} (d_u^2 + d_v^2) d_{uv}^2 \tag{4.1}$$

$$\leq \frac{1}{(q-1)^2} \sum_{v\in\mathcal{V}(\mathcal{H})} d_v^2 \sum_{u\in N(v)} (d_u^2 + d_v^2) \tag{4.2}$$

$$\leq \frac{2\Delta^2(p-1)}{(q-1)^2} \sum_{v\in\mathcal{V}(\mathcal{H})} d_v^3 \tag{4.3}$$

$$\leq \frac{2\Delta^2(p-1)}{(q-1)^2} F(\mathcal{H}). \tag{4.4}$$

The equality in (4.1) holds for a uniform hypergraph, (4.2) holds if for any two vertices $u$ and $v$, $N(u) = N(v)$ holds. But this can happen only in a hypergraph made of isolated hyperedges. Therefore,

$$\mathscr{E}_{PSO}(\mathcal{H})^2 \leq \left( \sum_{i=1}^{n} |\mu_i| \right)^2 \leq n \left( \sum_{i=1}^{n} \mu_i^2 \right) \leq \frac{2n\Delta^2(p-1)}{(q-1)^2} F(\mathcal{H}).$$

The above equality holds if all the eigenvalues have the same absolute value. For uniform hypergraphs made of isolated edges, this only happens when there is no isolated vertex and the edges have size 2. We have,

$$F(\mathcal{H}) = \sum_{v\in\mathcal{V}(\mathcal{H})} d_v^3 \leq \sum_{v\in\mathcal{V}(\mathcal{H})} \Delta d_v^2 \leq \Delta^2 pm,$$

and the equality holds if the hypergraph is both uniform and regular. Hence,

$$\mathscr{E}_{PSO}(\mathcal{H}) \leq \sqrt{2n(p-1)F(\mathcal{H})}\frac{\Delta}{q-1} \leq \sqrt{2nmp(p-1)}\frac{\Delta^2}{q-1}.$$

All the equalities happen only if $\mathcal{H}$ is a 2-graph with isolated edges and no isolated vertices or if $\mathcal{H}$ is trivial. □

**Theorem 4.4.** *Let $\mathcal{H}$ be a hypergraph with $n$ vertices and $m$ hyperedges. If* rank($\mathcal{H}$) *is $p$ and* co-rank($\mathcal{H}$) *is $q$, then*

$$\mathscr{E}_{PSO}(\mathcal{H}) \leq \mu_1 + \sqrt{(n-1)\left(\frac{2\Delta^2(p-1)}{(q-1)^2}F(\mathcal{H}) - \mu_1^2\right)}.$$

*Equality holds if and only if $\mathcal{H}$ is a 2-graph with isolated edges and no isolated vertices or if $\mathcal{H}$ is an edgeless hypergraph.*

*Proof.* From Theorem 4.3, we have

$$\mu_1^2 + \sum_{i=2}^{n} \mu_i^2 = \sum_{i=1}^{n} \mu_i^2 \leq \frac{2\Delta^2(p-1)}{(q-1)^2}F(\mathcal{H}),$$

equality holding only on uniform hypergraphs made of isolated edges and possibly some isolated vertices. Therefore,

$$\begin{aligned}(\mathscr{E}_{PSO}(\mathcal{H}) - \mu_1)^2 &= \left(\sum_{i=2}^{n} |\mu_i|\right)^2 \\ &\leq (n-1)\sum_{i=2}^{n} \mu_i^2 \leq (n-1)\left(\frac{2\Delta^2(p-1)}{(q-1)^2}F(\mathcal{H}) - \mu_1^2\right).\end{aligned}$$

Equality holds when $|\mu_2| = |\mu_3| = \cdots = |\mu_n|$. Uniform hypergraphs made of isolated edges can only occur when all the edges are size two and there are no isolated vertices. □

**Corollary 4.1.** *Let $\mathcal{H}$ be a $k$-uniform and $r$-regular hypergraph. Then,*

$$\mathscr{E}_{PSO}(\mathcal{H}) \leq r^2\sqrt{2}\left(1 + \sqrt{(n-1)\left(\frac{rn}{k-1} - 1\right)}\ \right).$$

**Lemma 4.2.** *Let $\mathcal{H}$ be a hypergraph with $n$ vertices and $m$ hyperedges. Then,*

$$\mathscr{E}_{PSO}(\mathcal{H})^2 \geq 2\sum_{i=1}^{n} \mu_i^2.$$

*Proof.* First, notice that $0 = \left(\sum_{i=1}^{n} \mu_i\right)^2 = \sum_{i=1}^{n} \mu_i^2 + 2\sum_{i<j} \mu_i\mu_j$. Therefore,

$$\sum_{i=1}^{n} \mu_i^2 = -2\sum_{i<j} \mu_i\mu_j, \tag{4.5}$$

$$\sum_{i<j} |\mu_i\mu_j| \geq \left|\sum_{i<j} \mu_i\mu_j\right| = \frac{1}{2}\sum_{i=1}^{n} \mu_i^2. \tag{4.6}$$

Then,

$$\mathscr{E}_{PSO}^2(\mathcal{H}) = \left(\sum_{i=1}^{n} |\mu_i|\right)^2 = \sum_{i=1}^{n} |\mu_i|^2 + 2\sum_{i<j} |\mu_i\mu_j| \geq 2\sum_{i=1}^{n} \mu_i^2. \tag{4.7}$$

Notice that equality in (4.6) and consequently in (4.7) occurs only if every non-zero eigenvalue has the same sign. If $\mathcal{H}$ has at least three non-zero eigenvalues (not necessarily distinct), it is impossible for this to happen. Therefore, $\mathcal{H}$ must have at most one 2-uniform complete bipartite connected component and possibly isolated vertices. □

**Theorem 4.5.** *Let $\mathcal{H}$ be a hypergraph with $n$ vertices and $m$ hyperedges. If* $\operatorname{rank}(\mathcal{H})$ *is $p$,* $\operatorname{co-rank}(\mathcal{H})$ *is $q$ and average degree of $\mathcal{H}$ is $d(\mathcal{H})$, then*

$$\mathscr{E}_{PSO}(\mathcal{H}) \geq \frac{2\delta}{p-1}\sqrt{n(q-1)d(\mathcal{H})}.$$

*Equality holds if and only if $\mathcal{H}$ is the 2-uniform, regular hypergraph with at most one complete bipartite connected component.*

*Proof.* By the above lemma, we have

$$\begin{aligned}
\mathscr{E}_{PSO}(\mathcal{H})^2 &\geq 2\sum_{i=1}^{n}\mu_i^2 = 2\sum_{v\in\mathcal{V}(\mathcal{H})}\sum_{u\in N(v)}(d_u^2+d_v^2)\left(\sum_{e\in(\mathcal{E}_u\cap\mathcal{E}_v)}\frac{1}{|e|-1}\right)^2\\
&\geq \frac{2}{(p-1)^2}\sum_{v\in\mathcal{V}(\mathcal{H})}\sum_{u\in N(v)}(d_u^2+d_v^2)d_{uv}^2\\
&\geq \frac{2}{(p-1)^2}\sum_{v\in\mathcal{V}(\mathcal{H})}\sum_{u\in N(v)}(d_u^2+d_v^2)d_{uv}\\
&\geq \frac{4n\delta^2(q-1)d(\mathcal{H})}{(p-1)^2}.
\end{aligned}$$

□

**Theorem 4.6.** *Let $\mathcal{H}$ be a hypergraph with $n$ vertices and $m$ hyperedges. If* $\operatorname{rank}(\mathcal{H})$ *is $p$ and* $\operatorname{co-rank}(\mathcal{H})$ *is $q$, then*

$$\mathscr{E}_{PSO}(\mathcal{H}) \geq \frac{2\delta(q-1)}{p-1}\sqrt{\frac{M_1(\mathcal{H})}{n}}.$$

*Equality holds if and only if $\mathcal{H}$ is the 2-uniform, regular hypergraph with only one edge and no isolated vertices, or if $\mathcal{H}$ is trivial.*

*Proof.* Note that,

$$\begin{aligned}
\mathscr{E}_{PSO}(\mathcal{H})^2 &\geq \frac{2}{(p-1)^2}\sum_{v\in\mathcal{V}(\mathcal{H})}\sum_{u\in N(v)}(d_u^2+d_v^2)d_{uv}^2\\
&\geq \frac{2}{(p-1)^2}\sum_{v\in\mathcal{V}(\mathcal{H})}\frac{1}{n}\left(\sum_{u\in N(v)}(d_u^2+d_v^2)d_{uv}\right)^2 \qquad (4.8)\\
&\geq \frac{4\delta^2(q-1)^2M_1(\mathcal{H})}{n(p-1)^2}.
\end{aligned}$$

Equality in (4.8) holds only $d_{uv}$ is constant for each $u$. □

**Theorem 4.7.** *Let $\mathcal{H}$ be a hypergraph with $n$ vertices and $m$ hyperedges. Then,*

$$\mathscr{E}_{PSO}(\mathcal{H}) \geq \mu_1 \sqrt{\frac{2n}{n-1}}.$$

*Equality holds if and only if $\mathcal{H}$ is the 2-uniform hypergraph with one edge and two vertices, or if $\mathcal{H}$ is trivial.*

*Proof.* We have

$$\sum_{i=1}^{n} \mu_i^2 = \mu_1^2 + \sum_{i=2}^{n} \mu_i^2 \geq \mu_1^2 + \frac{1}{n-1}\left(\sum_{i=2}^{n} \mu_i\right)^2 = \frac{n\mu_1^2}{n-1}.$$

Equality holds in above inequality, whenever $|\mu_2| = |\mu_3| = \cdots = |\mu_n|$. For complete bipartite graph, this only holds when $\mathcal{H} \cong \mathcal{K}_2^{(2)}$. Therefore,

$$\mathscr{E}_{PSO}(\mathcal{H})^2 \geq 2\sum_{i=1}^{n} \mu_i^2 \geq \frac{2n\mu_1^2}{n-1}.$$

□

**Lemma 4.3.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with $n$ vertices and $m$ hyperedges. If* rank$(\mathcal{H})$ *is $p$ and* co-rank$(\mathcal{H})$ *is $q$. Then,*

1) $\sum_{i=1}^{n} \mu_i^2 \geq \frac{2n\delta^2(q-1)d(\mathcal{H})}{(p-1)^2}$;
2) $\sum_{i=1}^{n} \mu_i^2 \geq \frac{2\delta^2(q-1)^2 M_1(\mathcal{H})}{n(p-1)^2}$;
3) $\sum_{i=1}^{n} \mu_i^2 \geq \frac{n\mu_1^2}{n-1}$.

**Lemma 4.4.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with $n$ vertices and $m$ hyperedges. If* rank$(\mathcal{H})$ *is $p$ and* co-rank$(\mathcal{H})$ *is $q$, then*

$$\mathscr{E}_{PSO}(\mathcal{H})^2 \geq \sum_{i=1}^{n} \mu_i^2 + n(n-1)|\det(A_{PSO}(\mathcal{H}))|^{\frac{2}{n}}.$$

*Equality holds if $|\mu_i| = |\mu_j|$ for every $i \neq j$, that is, if $\mathcal{H}$ is the graph with one edge and two vertices.*

*Proof.* Suppose,

$$\frac{1}{n(n-1)} \sum_{i\neq j} |\mu_i\mu_j| \geq \prod_{i\neq j} |\mu_i\mu_j|^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n} |\mu_i|^{\frac{2}{n}} = |\det(A_{PSO}(\mathcal{H}))|^{\frac{2}{n}},$$

where the above inequality is the arithmetic mean of non-negative numbers is greater than or equal to the geometric mean, and equality holds if and only if $|\mu_i| = |\mu_j|$ for every $i \neq j$. Therefore,

$$\begin{aligned}\mathscr{E}_{PSO}(\mathcal{H})^2 &= \left(\sum_{i=1}^{n} |\mu_i|\right)^2 = \sum_{i=1}^{n} \mu_i^2 + \sum_{i\neq j} |\mu_i \mu_j| \\ &\geq \sum_{i=1}^{n} \mu_i^2 + n(n-1)|\det(A_{PSO}(\mathcal{H}))|^{\frac{2}{n}}.\end{aligned}$$

□

By using Lemma 4.3 and Lemma 4.4, we found some lower bounds for the Sombor energy of a hypergraph, depending on the determinant of its Sombor matrix.

**Theorem 4.8.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with $n$ vertices and $m$ hyperedges. If* rank$(\mathcal{H})$ *is $p$ and* co-rank$(\mathcal{H})$ *is $q$, then*

$$\mathscr{E}_{PSO}(\mathcal{H}) \geq \sqrt{\frac{2n\delta^2(q-1)}{(p-1)^2} d(\mathcal{H}) + n(n-1)|\det(A_{PSO}(\mathcal{H}))|^{\frac{2}{n}}},$$

$$\mathscr{E}_{PSO}(\mathcal{H}) \geq \sqrt{\frac{2\delta^2(q-1)^2}{n(p-1)^2} M_1(\mathcal{H}) + n(n-1)|\det(A_{PSO}(\mathcal{H}))|^{\frac{2}{n}}},$$

$$\mathscr{E}_{PSO}(\mathcal{H}) \geq \sqrt{\frac{n}{n-1}\mu_1^2 + n(n-1)|\det(A_{PSO}(\mathcal{H}))|^{\frac{2}{n}}}.$$

*Proof.* Combining Lemma 4.4 and the inequalities in Lemma 4.3, we get the above results. □

## 5. Conclusion

In this article, the Sombor index of the graph has been extended to the pairwise Sombor index of hypergraphs with the motive of studying the spectra of the Sombor matrix of the hypergraph. Similar to this work, one can study the spectra of degree-based extended adjacency matrices corresponding to many VDB topological indices, as well as their application in predicting the physico-chemical properties of the compounds (which can be modeled through hypergraphs). We have obtained some bounds for the Sombor spectral radius and characterized the hypertree attaining the maximum Sombor spectral radius. We present some lower and upper bounds for energy and the Sombor energy of a hypergraph in terms of maximum (or minimum) degree, size, order, rank, co-rank and uniformity of the hypergraph.

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