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LOCAL EXISTENCE AND BLOW UP FOR A NONLINEAR VISCOELASTIC KIRCHHOFF-TYPE EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. The aim of this paper is to consider the initial boundary value problem of nonlinear viscoelastic Kirchhoff-type equation with logarithmic source term. Firstly, we prove the local existence of weak solution by applying Banach fixed theorem. Later, we derive the blow-up results by the combination of the perturbation energy method, concavity method and differential-integral inequality technique.

1. Introduction

In this article, we study the following viscolelastic Kirchhoff type problem
(1.1)
$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int\limits_0^t g\left(t-s\right)\triangle u\left(s\right)ds = u\ln\left|u\right|, & (x,t)\in\Omega\times\mathbb{R}^+, \\ u\left(x,0\right)=u_0\left(x\right), \quad u_t\left(x,0\right)=u_1\left(x\right), & x\in\Omega, \\ u\left(x,t\right)=0, \quad x\in\partial\Omega\times\mathbb{R}^+, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, $M\left(s\right)=\beta_1+\beta_2 s^\gamma$, $\gamma, s\geq 0$. Specially, we take $\beta_1=\beta_2=1$. We impose some conditions to be specified on the kernel function $g\left(t\right)$.

The equation with the logarithmic source term is related with many branches of physics. Cause of this is interest in it occures naturally in inflation cosmology, nuclear physics, supersymmetric field theories and quantum mechanics (see [3, 5, 10]). Later,

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by the motivation of this work, some authors gave necessary and sufficient conditions for the hyperbolic equation with logarithmic source term (see [6, 12, 15, 16]).

The Kirchhoff-type problem without the viscoelastic term has been extensively studied and many results for the existence, blow up and asymptotic behaviour of solutions have been established. For example, the following equation

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + |u_t|^{p-1} u_t - \Delta u_t = u^{k-1}\ln|u|,$$

has been considered by Yang et al. [19], where $M(s) = \alpha + \beta s^\gamma$, $\gamma > 0$, $\alpha \geq 1$, $\beta > 0$. They studied the local existence, asymptotic behavior and finite time blow up of solutions in cases subcritical energy and critical energy. And also, they proved the finite time blow up solutions in case arbitrary high energy.

In 2019, Pişkin and Irkıl [9] considered the global existence for the following equation

$$u_{tt} + M\left(\|\Delta u\|^2\right)\triangle^2 u + g(u_t)\, u_t = |u|^{p-1}\ln|u|^k.$$

In recent years, when by $g \neq 0$ and M is a constant function, problem have been offered by many authors. Al-Gharabli et al. [2] considered the following equation

$$|u_t|^\rho u_{tt} + \Delta^2 u_{tt} + \Delta^2 u - \int_0^t g(t-s)\,\Delta^2 u ds + u = u\ln|u|^k. \tag{1.2}$$

They investigated the local existence, global existence and stability for the problem (1.2). Later, they [11] proved the existence and decay results of problem (1.2) for $\rho = 0$ and absence $\Delta^2 u_{tt}$ term. Pişkin and Irkıl [18] studied the exponential growth of solutions of problem (1.2) for $\rho = 0$ and higher order viscoelastic term. In [17], the same authors studied the following equation

$$u_{tt} + [Pu_{tt} + Pu_t] + Pu + u - \int_0^t g(t-s)\, Puds + u_t = u\ln|u|^k,$$

where $P = (-\triangle)^m$, $m \geq 1$, and $m \in \mathbb{N}$. They obtained local existence by using Faedo-Galerkin method and a logaritmic Sobolev inequality. Later, they proved general decay results of solutions.

In [13], Peyravi considered

$$u_{tt} - \Delta u + u + \int_0^t g(t-s)\,\Delta u\, ds + h(u_t)\, u_t + |u|^2 u = u\ln|u|^k, \tag{1.3}$$

in $\Omega \subset \mathbb{R}^3$ with $h(s) = k_0 + k_1 |s|^{m-1}$. He studied the decay estimate and exponential growth of solutions for the problem (1.3).

In [20], Ye studied the logarithmic viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\triangle u(s)\, ds = u\ln|u|,$$

in three-dimensional space. The local and global existence for this problem are proved and the blow up of solutions is obtained.

In 2019, Boulaaras et al. [4] studied viscoleastic Kirchhoff equation with Balakrishnan-Taylor damping and logarithmic nonlinearity. They obtained an arbitrary rate of decay, which is not necessarily of polynomial or exponential decay.

In wiev of the articles mentioned above, much less effort has been devoted to initial boundary value problem for viscoelastic Kirchhoff type equation with logarithmic nonlinearity to our knowledge. Our purposes of this paper are to prove the local existence and blow up result by combining of Banach fixed point theorem, potential well theory and Logarithmic Sobolev inequality.

The structure of the work is as follows. To facilitate the description, firstly we give some definitions, notations, energy functional and some lemmas which will be used in our proof in Section 1. In Section 2 and in Section 3, respectively, we pove the local existence and blow up results for the solution of problem (1.1).

2. Preliminaries

In this part, we will present some notations and lemmas which will be used throughout this paper. We will write $\|\cdot\|_2$ and $\|\cdot\|_p$ for the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. We will use the Standart Lebesque Space $L^2(\Omega)$ with the inner product and the norm. The inner product can take as

$$\langle u, v\rangle = \int u(x)v(x)dx,$$

and the norm is defined as

$$\|u\|_2 = \langle u, u\rangle^{\frac{1}{2}}.$$

Let us begin with defining the following total energy functional

$$\begin{aligned} E(t) =& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2 + \frac{1}{4}\|u\|^2 \\ &+ \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{2}\int_\Omega u^2 \ln|u|\,dx. \end{aligned} \tag{2.1}$$

The potential energy functional

$$\begin{aligned} J(u) =& \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2 + \frac{1}{4}\|u\|^2 \\ &+ \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{2}\int_\Omega u^2 \ln|u|\,dx, \end{aligned}$$

and the Nehari functional

$$I(u) = \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g \circ \nabla u)(t) - \int_\Omega u^2 \ln|u|\,dx, \tag{2.2}$$

for $u \in H_0^1(\Omega)$, where

$$(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds.$$

Then, it is easy to show that for $u \in H_0^1(\Omega)$,

$$J(u) = \frac{1}{2}I(u) + \frac{1}{4}\|u\|^2 - \frac{\gamma}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}, \tag{2.3}$$

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(u). \tag{2.4}$$

The potential well depth is defined as

$$W = \left\{u \in H_0^1(\Omega) \mid J(u) < d, I(u) > 0\right\} \cup \{0\},$$

and the outer space of the potential well

$$V = \left\{u \in H_0^1(\Omega) \mid J(u) < d, I(u) < 0\right\}.$$

The depth of potential well is defined as

$$d = \inf_{u \in \mathcal{N}} J(u). \tag{2.5}$$

Now, we present following assumptions and some useful lemmas.

$(A1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$g(0) \geq 0, 1 - \int_0^\infty g(s)\, ds = l_0 > 0,$$

where

$$\int_0^\infty g(s)\, ds > \frac{\|\nabla u\|^2 + (g \circ \nabla u)(t) - \int_\Omega u^2 \ln|u|\, dx}{\|\nabla u\|^2}.$$

$(A2)$ There exists positive constant ϑ such that

$$g'(t) \leq \vartheta g(t), \quad t \geq 0.$$

Lemma 2.1 ([7,8] Logarithmic Sobolev Inequality). *Let u be any function $u \in H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain and $a > 0$ be any number. Then*

$$\int_\Omega \ln|u|\, u^2 dx < \frac{\alpha^2}{2\pi}\|\nabla u\|^2 + \ln\|u\|\,\|u\|^2 - \frac{3}{2}(1 + \ln\alpha)\|u\|_2^2.$$

Lemma 2.2 ([1, 14]). *Let $n = 3$. Then $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and there exists a constant c_p, the smallest positive number, satisfying*

$$\|u\|_6 \leq c_p \|\nabla u\|_2, \quad \text{for all } u \in H_0^1(\Omega).$$

Lemma 2.3. *Suppose that* $(A1)$ *and* $(A2)$ *hold. Then the energy functional* $E(t)$ *is decresing with respect to* t *and*

$$E'(t) = \frac{1}{2}\left[(g' \circ \nabla u)(t) - g(t)\|\nabla u(t)\|^2\right] \leq 0,$$

where

$$(g' \circ \nabla u)(t) = \int_0^t g'(t-s)\int_\Omega |\nabla u(s) - \nabla u(t)|^2 \, dxdt. \tag{2.6}$$

Proof. Multiplyingboth sides of (1.1) by u_t and then integrating from 0 to t, we have

$$E(t) = \int_0^t \frac{1}{2}\left[(g' \circ \nabla u)(t) - g(t)\|\nabla u(t)\|^2\right] + E(0),$$

which yields (2.6) by a simple calculation. □

Lemma 2.4. *For any* $u \in H_0^1(\Omega)$, $\|u\| \neq 0$, *we have*

i) $\lim_{\lambda\to 0} J(\lambda u) = 0$, $\lim_{\lambda\to\infty} J(\lambda u) = -\infty$*;*

ii) for $0 < \lambda < \infty$ *there exists a unique* λ_1 *such that*

$$\frac{d}{d\lambda} J(\lambda u)\mid_{\lambda=\lambda_1} = 0,$$

where λ_1 *is the unique root of equation*

$$l_0\|\nabla u\|^2 + (g \circ \nabla u)(t) - \int_\Omega u^2 \ln|u|\, dx = \ln\lambda \int_\Omega u^2 dx - \lambda^{2\gamma}\|\nabla u\|^{2\gamma+2};$$

iii) $J(\lambda u)$ *is strictly decreasing on* $\lambda_1 < \lambda < \infty$, *strictly increasing on* $0 < \lambda < \lambda_1$ *and attains the maximum at* $\lambda = \lambda_1$*;*

iv) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda_1$, $I(\lambda u) > 0$ *for* $\lambda_1 < \lambda < \infty$, *and* $I(\lambda_1 u) = 0$

$$I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) \begin{cases} > 0, & 0 \leq \lambda \leq \lambda_1, \\ = 0, & \lambda = \lambda_1, \\ < 0, & \lambda_1 \leq \lambda. \end{cases}$$

Proof. i) By the definition of $J(u)$, we get

$$\begin{aligned} J(\lambda u) =& \frac{\lambda^2}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2 + \frac{\lambda^2}{2}(g \circ \nabla u)(t) \\ &+ \frac{\lambda^{2\gamma+2}}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{\lambda^2}{4}\int_\Omega u^2 dx \\ &- \frac{\lambda^2}{2}\int_\Omega u^2 \ln|u|\, dx - \frac{\lambda^2 \ln\lambda}{2}\int_\Omega u^2 dx. \end{aligned} \tag{2.7}$$

Considering $\|u\| \neq 0$, so $\lim_{\lambda\to 0} J(\lambda u) = 0$ and $\lim_{\lambda\to\infty} J(\lambda u) = -\infty$ hold.

ii) Taking derivative of $J(\lambda u)$ with respect to λ, (2.7) yields

$$\begin{aligned}\frac{d}{d\lambda}J(\lambda u) =&\lambda\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\lambda(g\circ\nabla u)(t)\\&+\lambda^{2\gamma+1}\|\nabla u\|^{2(\gamma+1)}-\lambda\int_\Omega u^2\ln|u|\,dx-\lambda\ln\lambda\int_\Omega u^2dx\\=&\lambda\left(l_0\|\nabla u\|^2+(g\circ\nabla u)(t)+\lambda^{2\gamma}\|\nabla u\|^{2(\gamma+1)}-\int_\Omega u^2\ln|u|\,dx\right.\\&\left.-\ln\lambda\int_\Omega u^2dx\right),\end{aligned}$$

which means that there is a unique λ_1 such that $\frac{d}{d\lambda}J(\lambda u)\,|_{\lambda=\lambda_1}=0$, where λ_1 is the unique root of equation

$$l_0\|\nabla u\|^2+(g\circ\nabla u)(t)-\int_\Omega u^2\ln|u|\,dx=\ln\lambda\int_\Omega u^2dx-\lambda^{2\gamma}\|\nabla u\|^{2(\gamma+1)},$$

where $l_0\|\nabla u\|^2+(g\circ\nabla u)(t)-\int_\Omega u^2\ln|u|\,dx<0$.

iii) A simple corollary of the *ii*) we get

$$\frac{d}{d\lambda}J(\lambda u)>0,\qquad \text{for } 0<\lambda<\lambda_1,$$

and

$$\frac{d}{d\lambda}J(\lambda u)<0,\qquad \text{for } \lambda_1<\lambda<\infty.$$

iv) From (2.2), we get

$$\begin{aligned}I(\lambda u) =&\lambda^2\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+\lambda^2(g\circ\nabla u)(t)\\&-\int_\Omega(\lambda u)^2\ln|\lambda u|\,dx\\=&\lambda^2\left(l_0\|\nabla u\|^2+(g\circ\nabla u)(t)+\lambda^{2\gamma}\|\nabla u\|^{2(\gamma+1)}-\int_\Omega u^2\ln|u|\,dx-\ln\lambda\int_\Omega u^2dx\right)\\=&\lambda^2\frac{d}{d\lambda}J(\lambda u),\end{aligned}$$

which implies $I(\lambda_1 u)=0$, then $I(\lambda u)>0$ for $0<\lambda<\lambda_1$, $I(\lambda u)>0$ for $\lambda_1<\lambda<\infty$. □

Lemma 2.5. *Assume that $u\in H_0^1(\Omega)$. Then $d=\frac{1}{4}(2\pi l_0)^{\frac{3}{2}}e^3$.*

Proof. Combining Logarithmic Sobolev inequality and $(A1)$ yields that

$$\begin{aligned} I(u) &= \left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+(g\circ\nabla u)(t)-\int_\Omega u^2\ln|u|\,dx \\ &\geq \left(l_0-\frac{\alpha^2}{2\pi}\right)\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+(g\circ\nabla u)(t)+\left[\frac{3}{2}(1+\ln\alpha)-\ln\|u\|\right]\|u\|^2, \end{aligned} \tag{2.8}$$

for any $\alpha>0$. Taking $\alpha=\sqrt{2\pi l_0}$, by (2.8) and $(A1)$, we arrive that

$$I(u)>\left[\frac{3}{2}(1+\ln\alpha)-\ln\|u\|\right]\|u\|^2. \tag{2.9}$$

From Lemma 2.4 and (2.3), we conclude that

$$\begin{aligned} \sup_{\lambda\geq 0} J(\lambda u) &= J(\lambda_1 u)=\frac{1}{2}I(\lambda_1 u)+\frac{1}{4}\|\lambda_1 u\|^2-\frac{\gamma}{\gamma+1}\|\lambda_1\nabla u\|^{2(\gamma+1)} \\ &\geq \frac{1}{2}I(\lambda_1 u)+\frac{1}{4}\|\lambda_1 u\|^2. \end{aligned} \tag{2.10}$$

It follows from (2.9) and Lemma 2.4 that

$$0=I(\lambda_1 u)\geq\left[\frac{3}{2}(1+\ln\alpha)-\ln\|\lambda_1 u\|\right]\|\lambda_1 u\|^2,$$

which implies that

$$\|\lambda_1 u\|^2\geq(2\pi l_0)^{\frac{3}{2}}e^3. \tag{2.11}$$

We gain from (2.10) and (2.11) that

$$\sup_{\lambda\geq 0} J(\lambda u)\geq\frac{1}{4}(2\pi l_0)^{\frac{3}{2}}e^3. \tag{2.12}$$

By (2.5) and (2.12), $d=\frac{1}{4}(2\pi l_0)^{\frac{3}{2}}e^3>0$. □

3. Local Existence

In this part, we state and prove the local existence result for the problem (1.1). Firstly, we consider linear problem

$$\begin{cases} u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u+\int_0^t g(t-s)\,\triangle\, u(s)\;ds+u=v\ln|v|, & (x,t)\in\Omega\times(0,T), \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & x\in\Omega, \\ u(x,t)=0, \quad x\in\partial\Omega\times\mathbb{R}^+, \end{cases} \tag{3.1}$$

in which $T>0$.

Lemma 3.1. *Assume that* $(A1)$ *and* $(A2)$ *hold. Then for every* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* $v \in C([0,T]; H_0^1(\Omega))$, *problem* (3.1) *has a unique local solution for some* $T > 0$

$$u \in C\left([0,T); H_0^1(\Omega)\right), \quad u_t \in C\left([0,T); L^2(\Omega)\right).$$

Proof. Suppose that $\{w_j\}_{j=1}^{\infty}$ be the eigenfunctions of the Laplace operator with the Dirichlet boundary condition

$$-\Delta w_j = \lambda_j w_j, \quad w_j \mid_{\partial\Omega} = 0.$$

Then, we choose an orthogonal basis $\{w_j\}_{j=1}^{\infty}$ in $H_0^1(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let V_m be the subspace of $H_0^1(\Omega)$ generated by $\{w_1, w_2, \ldots, w_m\}$, $m \in \mathbb{N}$. We search for an approximate solution

$$u^m(x,t) = \sum_{j=1}^{m} h_j^m(t)\, w_j(x),$$

which satisfies the following Cauchy problem in V_m

$$\tag{3.2} \begin{cases} \left(u_{tt}^m(t), w_j\right) - M\left(\|\nabla^m u\|^2\right)\left(\Delta^m u(t), w_j\right) + \int\limits_0^t g(t-s)\left(\triangle^m u(s), w_j\right) ds \\ = \left(v \ln|v|, w_j\right), \quad j = 1,2,\ldots,m \in V_m, \\ u^m(0) = u_0^m = \sum\limits_{j=1}^{m} \left(u_0, w_j\right) w_j, \quad \text{in } H_0^1(\Omega), m \to \infty, \\ u_t^m(0) = u_1^m = \sum\limits_{j=1}^{m} \left(u_1, w_j\right) w_j, \quad \text{in } L^2(\Omega), m \to \infty. \end{cases}$$

This leads to the initial value problem for a system second-order differantial equations for unknown functions $h_j^m(t)$

$$\tag{3.3} \begin{cases} h_{jtt}^m(t) + M\left(\|\nabla^m u\|^2\right)\lambda_j h_j^m(t) = G_j\left(h_j^m(t)\right), \quad j = 1,2,\ldots,m, \\ h_j^m(0) = \int\limits_\Omega u_0 w_j dx, \quad h_{jt}^m(0) = \int\limits_\Omega u_1 w_j dx, \quad j = 1,2,\ldots,m, \end{cases}$$

where

$$G_j\left(h_j^m(t)\right) = \int_0^t g(t-s)\,\lambda_j h_j^m(s)\; ds + \int_\Omega v \ln|v|\, w_j, \quad j = 1,2,\ldots,m.$$

Multiplying (3.3) by $h_{jt}^m(t)$ and sum over j from 1 to m, and later integrating over $[0,t]$, we obtain

$$\begin{aligned} &\|u_t^m(t)\|^2 + \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u^m\|^2 + \frac{1}{\gamma+1}\|\nabla u^m\|^{2(\gamma+1)} + (g \circ \nabla u^m)(t) \\ = &\|u_1^m(t)\|^2 + \|\nabla u_0^m\|^2 + \frac{1}{\gamma+1}\|\nabla u_0^m\|^{2(\gamma+1)} \\ &+ 2\int_0^t\int_\Omega v(s)\ln|v(s)|\, u_t^m(s)\,dxds + \int_0^t \left[(g' \circ \nabla u)(s) - g(s)\|\nabla u(s)\|^2\right] ds \end{aligned}$$

$$\leq \|u_1^m(t)\|^2 + \|\nabla u_0^m\|^2 + \frac{1}{\gamma+1}\|\nabla u_0^m\|^{2(\gamma+1)} + 2\int_0^t \int_\Omega v(s)\ln|v(s)|\, u_t^m(s)\, dxds. \tag{3.4}$$

We estimate the last term in the right-hand side as follows. By Hölder's and Young's inequalities, we have

$$\begin{aligned} 2\int_0^t \int_\Omega v(s)\ln|v(s)|\, u_t^m(s)\, dxds &\leq 2\int_0^t \int_\Omega |v(s)\ln|v(s)||^2\, dxds \int_0^t \int_\Omega |u_t^m(s)|^2\, dxds \\ &\leq \int_0^t \int_\Omega |v(s)\ln|v(s)||^2\, dxds + \int_0^t \||u_t^m(s)|\|^2\, ds. \end{aligned} \tag{3.5}$$

For $v \in H_0^1(\Omega)$, by direct calculation and using of Lemma 2.2, we obtain

$$\begin{aligned} \int_\Omega |v\ln|v||^2\, dx &= \int_{\{x\in\Omega;|v(x)|\leq 1\}} v^2(\ln|v|)^2\, dx + \int_{\{x\in\Omega;|v(x)|>1\}} v^2(\ln|v|)^2\, dx \\ &\leq e^{-2}|\Omega| + \frac{1}{4}\int_{\{x\in\Omega;|v(x)|>1\}} |v|^6\, dx \leq e^{-2}|\Omega| + \frac{1}{4}\|v\|_6^6 \\ &\leq e^{-2}|\Omega| + \frac{1}{4}c_p\|\nabla v\|^6 = C, \end{aligned} \tag{3.6}$$

since

$$\begin{cases} \ln|u| < \frac{u^2}{2}, & |u(x)| > 1, \\ u\ln|u| < e^{-1}, & |u(x)| \leq 1. \end{cases}$$

It follows from $(A1)$, (3.4), (3.5) and (3.6) that

$$\begin{aligned} &\|u_t^m(t)\|^2 + l_0\|\nabla u^m\|^2 + \frac{1}{\gamma+1}\|\nabla u_0^m\|^{2(\gamma+1)} \\ &\leq \|u_1^m(t)\|^2 + \|\nabla u_0^m\|^2 + \frac{1}{\gamma+1}\|\nabla u_0^m\|^{2(\gamma+1)} + CT + \int_0^t \||u_t^m(s)|\|^2\, ds \\ &\leq C_* + \int_0^t \left[\|u_t^m(s)\|^2 + l_0\|\nabla u^m\|^2 + \frac{1}{\gamma+1}\|\nabla u^m\|^{2(\gamma+1)}\right] ds, \end{aligned} \tag{3.7}$$

where $C_* = \|u_1^m(t)\|^2 + l_0\|\nabla u_0^m\|^2 + \frac{1}{\gamma+1}\|\nabla u_0^m\|^{2\gamma+2} + CT$. By using of Gronwall inequality and (3.7), we get

$$\|u_t^m(t)\|^2 + l_0\|\nabla u^m\|^2 + \frac{1}{\gamma+1}\|\nabla u^m\|^{2(\gamma+1)} \leq C_2 e^T. \tag{3.8}$$

We obtain from (3.8) that

$$\begin{cases} u^m \text{ is a bounded sequence in } L^\infty([0,T]; H_0^1(\Omega)), \\ u_t^m \text{ is a bounded sequence in } L^\infty([0,T]; L^2(\Omega)). \end{cases}$$

Hence, there exists a subsequence of $\{u^m\}$, still denoted by $\{u^m\}$, such that

$$\begin{cases} u_m \to u, \text{ weakly star in } L^\infty\left(0,T;H_0^1(\Omega)\right), \\ u_{mt} \to u_t, \text{ weakly star in } L^\infty\left(0,T;L^2(\Omega)\right), \\ u_{mtt} \to u_{tt}, \text{ weakly in } L^2\left(0,T;H_0^{-1}(\Omega)\right). \end{cases} \tag{3.9}$$

Setting up $m \to \infty$ and passing to the limit in (3.2), and combining by (3.9), we obtain

$$(u_{tt}(t), w_j) - M\left(\|\nabla u\|^2\right)(\Delta u(t), w_j) + \int_0^t g(t-s)(\triangle u(s), w_j)\,ds = (v \ln|v|, w_j),$$

for $j = 1, 2, \ldots$ Since $\{w_j\}_{j=1}^\infty$ is a base in the corresponding space, we deduce that u satisfies the equation in (3.1). We finished this section by proving a local existence result of the problem (1.1). □

Theorem 3.1. *Suppose that* $(A1)$ *holds. Assume further that* $u_0 \in H_0^1(\Omega)$ *and* $u_1 \in L^2(\Omega)$. *Then problem* (1.1) *has a unique local solution*

$$u \in C\left([0,T]; H_0^1(\Omega)\right), \quad u_t \in C\left([0,T]; L^2(\Omega)\right).$$

Proof. We define the following set

$$X_{r_0,T} = \left\{u \in \Pi \mid \|u(t)\|_\Pi \le r_0^2, t \in [0,T]\right\},$$

here the space

$$\Pi = \left\{u \mid u \in C\left([0,T]; H_0^1(\Omega)\right), u_t \in C\left([0,T]; L^2(\Omega)\right)\right\},$$

equipped with the norm

$$\|u(t)\|_\Pi = \sup_{0 \le t \le T}\left(\|u_t^m(t)\|^2 + l_0\|\nabla u^m\|^2 + \frac{1}{\gamma+1}\|\nabla u^m\|^{2(\gamma+1)}\right).$$

Then $X_{r_0,T}$ is a complete metric space with the distance

$$d(u_1, u_2) = \|u_1 - u_2\|_\Pi.$$

By Lemma 3.1, we define the nonlinear mapping $\Psi : v \to u = \Psi v$ in the following way. For $v \in X_{r_0,T}$, $u = \Psi v$ is the unique solution of problem (3.1). We claim that Ψ is a contraction mapping from $X_{r_0,T}$ into itself for $r_0 > 0$ and $T > 0$.

Let $v \in X_{r_0,T}$, for $t \in [0,T]$, we get from $(A1)$ and (3.4) that

$$\begin{aligned} &\|u_t\|^2 + l_0\|\nabla u\|^2 + \frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)} \\ &\le \|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)} + 2\int_0^t\int_\Omega v(s)\ln|v(s)|\,u_t(s)\,dx\,ds \end{aligned}$$

(3.10)

$$\leq \|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)} + \int_0^t \|v(s)\ln|v(s)|\|^2 ds + \int_0^t \||u_t(s)|\|^2 ds.$$

Next we estimate the $\int_0^t \|v(s)\ln|v(s)|\|^2 ds$ term in (3.10), by using of Hölder ineqality, Lemma 2.2, the definition of $\|u(t)\|_\Pi$ and the inequality $\ln x < x$ as $x > 1$ such that we obtain

$$\begin{aligned} \|v(s)\ln|v(s)|\|^2 &= \int_{\{x\in\Omega;|v(x)|\leq 1\}} v^2(\ln|v|)^2 dx + \int_{\{x\in\Omega;|v(x)|>1\}} v^2(\ln|v|)^2 dx \\ &\leq \int_{\{x\in\Omega;|v(x)|>1\}} |v|^4 dx \\ &\leq \sqrt[3]{\Omega}\|v\|_6^4 \leq \sqrt[3]{\Omega}c_p^4\|\nabla v\|^4 \leq \frac{\sqrt[3]{\Omega}c_p^4 r_0^4}{l_0^2}. \end{aligned} \tag{3.11}$$

By combining of (3.10) and (3.11) and using of the definition of $\|u(t)\|_\Pi$, we have

$$\begin{aligned} \|u_t\|^2 + l_0\|\nabla u\|^2 + \frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)} &\leq \Xi(u_0,u_1,r_0,T) + \int_0^t \||u_t(s)|\|^2 ds \\ &\leq \Xi(u_0,u_1,r_0,T) + \int_0^t \|u(s)\|_\Pi ds, \end{aligned} \tag{3.12}$$

where $\Xi(u_0,u_1,r_0,T) = \|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)} + \frac{\sqrt[3]{\Omega}c_p^4 r_0^4}{l_0^2}T.$

We get from (3.12) and Gronwall's inequality that

$$\|u\|_\Pi \leq \Xi(u_0,u_1,r_0,T)e^T. \tag{3.13}$$

Choosing

$$r_0 > \sqrt{\|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}}$$

and

$$T < \left[\frac{r_0^2 - \left(\|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}\right)l_0^2}{\sqrt[3]{\Omega}c_p^4 r_0^4}\right],$$

such that $\Xi(u_0,u_1,r_0,T) \leq r_0^2$, we see that $u \in X_{r_0,T}$ by (3.13). This shows that Ψ maps $X_{r_0,T}$ into itself.

Next, we shall show that Ψ is a contraction mapping. Let $v_1, v_2 \in X_{r_0,T}$ and $u_1 = \Psi v_1,\ u_2 = \Psi v_2,$ be the corresponding solution for problem (3.1). Taking $U = u_1 - u_2$,

$V = v_1 - v_2$, then U satisfies the following problem

$$\begin{cases} U_{tt} - M\left(\|\nabla U\|^2\right)\Delta U + \int_0^t g(t-s) \triangle U(s)\, ds \\ = v_1 \ln|v_1| - v_2 \ln|v_2|, \quad (x,t) \in \Omega \times (0,T), \\ U(x,0) = U_t(x,0) = 0, \quad x \in \Omega, \\ \frac{\partial^j U(x,t)}{\partial v^j} = 0, \quad j = 0,1,2,\ldots,m-1, (x,t) \in \partial\Omega \times (0,T). \end{cases} \tag{3.14}$$

Multiplying (3.14) by U_t and then integrate it over $\Omega \times (0,T)$, we obtain

$$\begin{aligned} &\|U_t\|^2 + \left(1 - \int_0^t g(s)\, ds\right) \|\nabla U(t)\|^2 + \frac{1}{\gamma+1} \|\nabla U(t)\|^{2(\gamma+1)} \\ &+ (g \circ \nabla U)(t) - \int_0^t \left[(g' \circ \nabla U)(s) - g(s) \|\nabla U(s)\|^2\right] ds \\ =& 2\int_0^t \int_\Omega (v_1 \ln|v_1| - v_2 \ln|v_2|)\, U_t(x,s)\, dxds. \end{aligned} \tag{3.15}$$

Thanks to Lagrange mean value Theorem, we get $v_1 \ln|v_1| - v_2 \ln|v_2| = V(1 + \ln|\beta|)$, where $|\beta| = |v_1 + \theta(v_2 - v_1)| = |(1-\theta)v_1 + \theta v_2|$, $0 < \theta < 1$. Thus, by applying the same process as (3.11), we estimate the last term in (3.15) as follows

$$\begin{aligned} &\int_0^t \int_\Omega (v_1 \ln|v_1| - v_2 \ln|v_2|)\, U_t(x,s)\, dxds \\ \le& \int_0^t \int_\Omega V U_t(x,s)\, dxds + \int_0^t \int_\Omega V(|v_1| + |v_2|)\, U_t(x,s)\, dxds \\ \le& \int_0^t \|V\| \|U_t\|\, ds + \int_0^t \|V\|_6 \||v_1| + |v_2|\|_3 \|U_t\|\, ds \\ \le& c_p \int_0^t \|\nabla V\| \|U_t\|\, ds + c_p^2 \int_0^t \|\nabla V\| (|\nabla v_1| + |\nabla v_2|) \|U_t\|\, ds \\ \le& \int_0^t c_p\left(1 + 2l_0^{-\frac{1}{2}} c_p r_0\right) \|\nabla V\| \|U_t\|\, ds \\ \le& \frac{1}{2}\left[c_p\left(1 + 2l_0^{-\frac{1}{2}} c_p r_0\right)\right]^2 \int_0^t \|\nabla V\|^2 + \frac{1}{2}\int_0^t \|U_t(s)\|^2\, ds. \end{aligned} \tag{3.16}$$

We have from $(A1)$, (3.15) and (3.16) that

$$\|U_t\|^2 + l_0 \|\nabla U(t)\|^2 + \frac{1}{\gamma+1} \|\nabla U(t)\|^{2(\gamma+1)}$$

$$\leq \left[c_p\left(1+2l_0^{-\frac{1}{2}}c_pr_0\right)\right]^2\int_0^t \|\nabla V\|^2+\int_0^t \|U_t(s)\|^2\,ds,$$

which implies that

$$\|U\|_\Pi \leq l_0^{-1}\left[c_p\left(1+2l_0^{-\frac{1}{2}}c_pr_0\right)\right]^2 T\|V\|_\Pi+\int_0^t \|U\|_\Pi\,ds. \tag{3.17}$$

By the Gronwall inequality and (3.17), we have

$$\|U\|_\Pi \leq l_0^{-1}\left[c_p\left(1+2l_0^{-\frac{1}{2}}c_pr_0\right)\right]^2 T\|V\|_\Pi e^T.$$

By choosing

$$T<l_0\left[c_p\left(1+2l_0^{-\frac{1}{2}}c_pr_0\right)\right]^{-2}e^{-T},$$

such that

$$l_0^{-1}\left[c_p\left(1+2l_0^{-\frac{1}{2}}c_pr_0\right)\right]^2 T\|V\|_\Pi e^T<1,$$

then Ψ is a contraction mapping.

In summary, when we choose

$$r_0>\sqrt{\|u_1\|^2+\|\nabla u_0\|^2+\frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}},$$

and

$$T<\min\left\{\frac{r_0^2-\left(\|u_1\|^2+\|\nabla u_0\|^2+\frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}\right)l_0^2}{\sqrt[3]{\Omega}c_p^4r_0^4},\right.$$
$$\left.l_0\left[c_p\left(1+2l_0^{-\frac{1}{2}}c_pr_0\right)\right]^{-2}e^{-T}\right\},$$

Ψ is a contraction mapping from $X_{r_0,T}$ to itself. According to Banach fixed point theorem, we have the local existence result. The proof is completed. □

4. Blow Up

In this part, we prove the blow up result of solution for the problem (1.1). We give some lemmas which will e used in our proof.

Lemma 4.1. *If a solution u of the problem* (1.1) *meets $u\in V$, then*

$$I(u(t))<2(J(u)-d).$$

Proof. By $u\in V$ and Lemma 2.4, there exists a λ_1 such that $0<\lambda_1<1$ and $I(\lambda_1 u)=0$. By taking of $I(\lambda_1 u)=0$, definition of d in (2.5) and (2.3), we get

$$d<J(\lambda_1 u)=\frac{1}{2}I(\lambda_1 u)+\frac{1}{4}\|\lambda_1 u\|^2-\frac{\gamma}{\gamma+1}\|\lambda_1\nabla u\|^{2(\gamma+1)}$$

$$< \lambda_1^2 \left(\frac{1}{4} \|u\|^2 - \frac{\gamma}{\gamma+1} \|\nabla u\|^{2(\gamma+1)} \right)$$

$$< \frac{1}{4} \|u\|^2 - \frac{\gamma}{\gamma+1} \|\nabla u\|^{2(\gamma+1)}. \tag{4.1}$$

Combining (4.1) and (2.3) yields that

$$d < J(u) - \frac{1}{2} I(u),$$

which implies that

$$I(u) < 2 \left(J(u) - d \right). \tag{4.2}$$

□

Lemma 4.2. *Assume that* $u(t)$ *is a solution of the problem* (1.1). *If* $u_0 \in V$ *and* $E(0) < d$, *then* $E(t) < d$ *for all* $t \geq 0$.

Proof. By Lemma 2.3 and (2.1), we get

$$J(u) \leq E(t) \leq E(0) < d, \quad \text{for all } t \geq 0.$$

Suppose that there exists $t^* \in [0, \infty)$ such that $u(t^*) \notin V$, then by continuity of $I(u(t))$, we obtain $I(u(t^*)) = 0$. This means that $u(t^*) \in \mathcal{N}$. Thus, from definition of d, we get that $J(u(t^*)) \geq d$, which is a contradiction with (4.2). Consequently, Lemma 4.1 is valid. □

Theorem 4.1. *Assume that* $u_0 \in V$, $u_1 \in L^2(\Omega)$, $\int\limits_{\Omega} u_0 u_1 dx > 0$ *and* $E(0) < d$. *Then the solution* $u(t)$ *in Theorem* 3.1 *of the problem* (1.1) *blows up as time* t *goes to infinity.*

Proof. We set

$$G(t) = \int\limits_{\Omega} u^2 dx, \tag{4.3}$$

for all $t \in [0, \infty)$. It is obvious that $G(t) > 0$. Moreover, by using of (4.3) and (1.1), we get

$$G'(t) = 2 \int\limits_{\Omega} u_t u dx \tag{4.4}$$

and

$$\begin{aligned} G''(t) =& 2 \|u_t\|^2 + 2 \int\limits_{\Omega} u_{tt} u dx \\ =& 2 \|u_t\|^2 - 2 \int\limits_{\Omega} M\left(\|\nabla u\|^2\right) \|\nabla u\|^2 dx \\ & + 2 \int\limits_0^t g(t-s) \nabla u(s) \nabla u(t) ds dx + 2 \int\limits_{\Omega} u^2 \ln|u| \end{aligned}$$

$$=2\left\|u_t\right\|^2-2\left\|\nabla u\right\|^2-2\left\|\nabla u\right\|^{2(\gamma+1)}+2\int_0^t g\left(t-s\right)ds\left\|\nabla u\right\|^2$$

$$+2\int_0^t g\left(t-s\right)\int_\Omega \nabla u\left(t\right)\left(\nabla u\left(s\right)-\nabla u\left(t\right)\right)dxds+2\int_\Omega u^2\ln\left|u\right|. \tag{4.5}$$

By using Young inequality, we have

$$\int_0^t g\left(t-s\right)\int_\Omega \left|\nabla u\left(t\right)\right|\left|\nabla u\left(s\right)-\nabla u\left(t\right)\right|dxds\le\int_0^t g\left(s\right)ds\left\|\nabla u\right\|^2+\frac{1}{4}\left(g\circ\nabla u\right)\left(t\right). \tag{4.6}$$

Combining (4.5) and (4.6) yields that

$$\begin{aligned}G''\left(t\right)\ge&2\left\|u_t\right\|^2-2\left\|\nabla u\right\|^2-2\left\|\nabla u\right\|^{2(\gamma+1)}\\&-2\int_0^t g\left(s\right)ds\left\|\nabla u\right\|^2+2\int_\Omega u^2\ln\left|u\right|-\frac{1}{2}\left(g\circ\nabla u\right)\left(t\right)\\\ge&2\left\|u_t\right\|^2-2I(u).\end{aligned} \tag{4.7}$$

From (4.4) and (4.3) and using of the Cauchy inequality, we have

$$\left|G'\left(t\right)\right|^2\le 4\int_\Omega \left|u_t\right|^2dx\int_\Omega \left|u\right|^2dx=4G\left(t\right)\left\|u_t\right\|^2. \tag{4.8}$$

Combining (4.7), (4.8) and (2.4), we arrive at

$$\begin{aligned}G''\left(t\right)G\left(t\right)-\left(G'\left(t\right)\right)^2&\ge G\left(t\right)\Big(2\left\|u_t\right\|^2-2I(u)\Big)-4G\left(t\right)\left\|u_t\right\|^2\\&=-2G\left(t\right)\Big(\left\|u_t\right\|^2+I(u\left(t\right))\Big)\\&\ge-2G\left(t\right)\left(2E\left(t\right)-2J\left(u\left(t\right)\right)+I(u\left(t\right))\right).\end{aligned} \tag{4.9}$$

Combining $u_0\in V$, $E\left(0\right)<d$ with Lemma 4.2 obtain $u\in V$, $E\left(t\right)<d$. By Lemma 4.1, we have

$$2E\left(t\right)-2J\left(u\left(t\right)\right)+I(u)\le 2d-2J\left(u\left(t\right)\right)+2\left(J(u\left(t\right))-d\right)=0. \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$G''\left(t\right)G\left(t\right)-\left(G'\left(t\right)\right)^2>0.$$

By directly calculation, we have

$$\left(\ln\left|G\left(t\right)\right|\right)'=\frac{G'\left(t\right)}{G\left(t\right)}$$

and

$$\left(\ln\left|G\left(t\right)\right|\right)''=\frac{G''\left(t\right)G\left(t\right)-\left(G'\left(t\right)\right)^2}{\left(G\left(t\right)\right)^2}>0. \tag{4.11}$$

By (4.11), we know that $(\ln|G(t)|)'$ is increasing with respect to t. Integrating both sides of (4.11) over $[0,t]$, we get

$$\ln|G(t)| - \ln|G(0)| = \int_0^t (\ln|G(\tau)|)'\,d\tau = \int_0^t \frac{G'(\tau)}{G(\tau)}d\tau \geq \frac{G'(0)}{G(0)}t,$$

which implies that

$$G(t) \geq G(0)\exp\left(\frac{G'(0)}{G(0)}t\right).$$

$G(t)$ tends to infinity as time goes to infinity. This completed our proof. □

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