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HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR GEOMETRICALLY CONVEX FUNCTIONS II

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Abstract. In this paper, we prove some Hermite-Hadamard type inequalities for operator geometrically convex functions for non-commutative operators.

1. Introduction and Preliminaries

Let $B(H)$ stand for C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. An operator $A \in B(H)$ is strictly positive and write $A > 0$ if $\langle Ax, x \rangle > 0$ for all $x \in H$. Let $B(H)^{++}$ stand for all strictly positive operators on $B(H)$.

Let A be a self-adjoint operator in $B(H)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(\mathrm{Sp}(A))$ of all continuous functions defined on the spectrum of A, denoted $\mathrm{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows.

For any $f, g \in C(\mathrm{Sp}(A)))$ and any $\alpha, \beta \in \mathbb{C}$ we have:

- $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
- $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- $\|\Phi(f)\| = \|f\| := \sup_{t \in \mathrm{Sp}(A)} |f(t)|$;
- $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathrm{Sp}(A)$.

With this notation we define $f(A) = \Phi(f)$ for all $f \in C(\mathrm{Sp}(A))$, and we call it the continuous functional calculus for a self-adjoint operator A. If A is a self-adjoint operator and both f and g are real valued functions on $\mathrm{Sp}(A)$ then the following important property holds: $f(t) \geq g(t)$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq g(A)$,

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in the operator order of $B(H)$, see [12]. A real valued continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be convex (concave) if

$$f(\lambda a + (1-\lambda)b) \leq (\geq) \lambda f(a) + (1-\lambda) f(b),$$

for $a, b \in \mathbb{R}$ and $\lambda \in [0,1]$. The following Hermite-Hadamard inequality holds for any convex function f defined on $\mathbb{R}$

$$\begin{aligned}(b-a) f\left(\frac{a+b}{2}\right) &\leq \int_a^b f(x)dx \\ &\leq (b-a)\frac{f(a)+f(b)}{2}, \quad \text{for } a, b \in \mathbb{R}.\end{aligned}$$

The author of [8, Remark 1.9.3] gave the following refinement of Hermite-Hadamard inequalities for convex functions

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right) \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\left(f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right) \\ &\leq \frac{f(a)+f(b)}{2}.\end{aligned}$$

A real valued continuous function is operator convex if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B),$$

for self-adjoint operator $A, B \in B(H)$ and $\lambda \in [0,1]$. In [2] Dragomir investigated the operator version of the Hermite-Hadamard inequality for operator convex functions. Let $f : \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval I then, for any self-adjoint operators A and B with spectra in I, the following inequalities hold

$$\begin{aligned}f\left(\frac{A+B}{2}\right) &\leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} f(tA+(1-t)B)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \\ &\leq \int_0^1 f\left((1-t)A+tB\right)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] \\ &\leq \frac{f(A)+f(B)}{2}.\end{aligned}$$

For the first inequality in above, see [10].

A continuous function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ ($\mathbb{R}^+$ denoted positive real numbers) is said to be geometrically convex function (or multiplicatively convex function) if

$$f(a^\lambda b^{1-\lambda}) \leq f(a)^\lambda f(b)^{1-\lambda},$$

for $a, b \in I$ and $\lambda \in [0,1]$.

The author of [7, p. 158] showed that every polynomial $P(x)$ with non-negative coefficients is a geometrically convex function on $[0,\infty)$. More generally, every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients is geometrically convex function on $(0, R)$ where R denotes the radius of convergence. Also, see [9,11]. In [10], the following inequalities were obtained for a geometrically convex function

$$\begin{aligned} f(\sqrt{ab}) &\leq \sqrt{\left(f(a^{\frac{3}{4}}b^{\frac{1}{4}})f(a^{\frac{1}{4}}b^{\frac{3}{4}})\right)} \\ &\leq \exp\left(\frac{1}{\log b - \log a}\int_a^b \frac{\log f(t)}{t}dt\right) \\ &\leq \sqrt{f(\sqrt{ab})}.\sqrt[4]{f(a)}.\sqrt[4]{f(b)} \\ &\leq \sqrt{f(a)f(b)}. \end{aligned}$$

In this paper, we prove some Hermite-Hadamard inequalities for operator geometrically convex functions. Moreover, in the final section, we present some examples and remarks.

2. Hermite-Hadamard Inequalities for Geometrically Convex Functions

In this section, we introduce the concept of operator geometrically convex function for positive operators and prove the Hermite-Hadamard type inequalities for this function.

Proposition 2.1. *Let $A, B \in B(H)^{++}$ such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$, and $t \in [0,1]$. Then $\mathrm{Sp}(A\sharp_t B) \subseteq I$, where $A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ is t-geometric mean.*

Proof. Let $I = [m, M]$ for some positive real numbers m, M with $m < M$. Since $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$ it is equivalent to $m1_H \leq A \leq M1_H$ and $m1_H \leq B \leq M1_H$. So, by virtue of the fact that if a, b be self-adjoint operators in C^*-algebra $\mathcal{A}$ which $a \leq b$ and $c \in \mathcal{A}$, then $c^*ac \leq c^*bc$, and also by using the operator monotonicity property of the function $f(x) = x^t$ on $(0,\infty)$ for $t \in [0,1]$, we get the result. $\square$

Now, by applying Proposition 2.1, we present the following definition.

Definition 2.1. A continuous function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ is said to be operator geometrically convex if

$$f(A\sharp_t B) \leq f(A)\sharp_t f(B),$$

for $A, B \in B(H)^{++}$ such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$ and $t \in [0,1]$.

We need the following lemmas for proving our theorems.

Lemma 2.1 ([4,5]). *Let $A, B \in B(H)^{++}$ and let $t, s, u \in \mathbb{R}$. Then*
$$(A\sharp_t B)\sharp_s(A\sharp_u B) = A\sharp_{(1-s)t+su}B.$$

Lemma 2.2 ([4]). *Let A, B, C and D be operators in $B(H)^{++}$ and let $t \in \mathbb{R}$. Then, we have*
$$A\sharp_t B \le C\sharp_t D,$$
for $A \le C$ and $B \le D$.

Lemma 2.3. *Let $A, B \in B(H)^{++}$. If $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, then*
$$\int_0^1 f\left(A\sharp_t B\right)\sharp f\left(A\sharp_{1-t}B\right)dt \le \left(\int_0^1 f\left(A\sharp_t B\right)dt\right)\sharp\left(\int_0^1 f\left(A\sharp_{1-t}B,\right)dt\right)$$
such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$.

Proof. Since the function $t^{\frac{1}{2}}$ is operator concave, we can write
$$\left(\left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}}\left(\int_0^1 f(A\sharp_u B)du\right)\left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
(by change of variable $v = 1-u$)
$$=\left(\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}\left(\int_0^1 f(A\sharp_u B)du\right)\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
$$=\left(\int_0^1\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{1}{2}} f(A\sharp_u B)\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{1}{2}}du\right)^{\frac{1}{2}}$$
$$=\left(\int_0^1\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}(f(A\sharp_{1-u}B))^{\frac{1}{2}}\left((f(A\sharp_{1-u}B))^{\frac{-1}{2}}f(A\sharp_u B)(f(A\sharp_{1-u}B))^{\frac{-1}{2}}\right)\right.$$
$$\left.\times(f(A\sharp_{1-u}B))^{\frac{1}{2}}\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}du\right)^{\frac{1}{2}}$$
(by the operator Jensen inequality)
$$\ge\int_0^1\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}(f(A\sharp_{1-u}B))^{\frac{1}{2}}\left(\left(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}f(A\sharp_u B)(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
$$\times(f(A\sharp_{1-u}B))^{\frac{1}{2}}\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}du$$
$$=\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}\int_0^1(f(A\sharp_{1-u}B))^{\frac{1}{2}}\left(\left(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}f(A\sharp_u B)(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
$$\times(f(A\sharp_{1-u}B))^{\frac{1}{2}}du\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}$$

(by change of variable $u = 1 - v$)

$$= \left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}} \int_0^1 (f(A\sharp_{1-u}B))^{\frac{1}{2}} \left(\left(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}} f(A\sharp_u B)(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$

$$\times (f(A\sharp_{1-u}B))^{\frac{1}{2}} du \left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}}.$$

So, we obtain

$$\left(\left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}} \left(\int_0^1 f(A\sharp_u B)du\right) \left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$

$$\geq \left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}} \int_0^1 (f(A\sharp_{1-u}B))^{\frac{-1}{2}} \left(\left(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}} f(A\sharp_u B)(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$

$$\times (f(A\sharp_{1-u}B))^{\frac{1}{2}} du \left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}}.$$

Multiplying both side of the above inequality by $\left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{1}{2}}$ we obtain

$$\left(\int_0^1 f\left(A\sharp_u B\right) du\right) \sharp \left(\int_0^1 f\left(A\sharp_{1-u}B\right) du\right) \geq \int_0^1 f\left(A\sharp_u B\right) \sharp f\left(A\sharp_{1-u}B\right) du. \qquad \square$$

Before giving our theorems in this section, we mention the following remark.

Remark 2.1. Let $p(x) = x^t$ and $q(x) = x^s$ on $[1, \infty)$, where $0 \leq t \leq s$. If $f(A) \leq f(B)$ then $\mathrm{Sp}\left(f(A)^{\frac{-1}{2}} (f(B)) f(A)^{\frac{-1}{2}}\right) \subseteq [1, \infty)$. By functional calculus, we have

$$p\left(f(A)^{\frac{-1}{2}} f(B) f(A)^{\frac{-1}{2}}\right) \leq q\left(f(A)^{\frac{-1}{2}} f(B) f(A)^{\frac{-1}{2}}\right).$$

So,

$$\left(f(A)^{\frac{-1}{2}} f(B) f(A)^{\frac{-1}{2}}\right)^t \leq \left(f(A)^{\frac{-1}{2}} f(B) f(A)^{\frac{-1}{2}}\right)^s.$$

Now, we are ready to prove Hermite-Hadamard type inequality for operator geometrically convex functions.

Theorem 2.1. *Let f be an operator geometrically convex function. Then, we have*

$$f(A\sharp B) \leq \int_0^1 f(A\sharp_t B) dt \leq \int_0^1 f(A)\sharp_t f(B) dt. \tag{2.1}$$

Moreover, if $f(A) \leq f(B)$, then we have

$$\int_0^1 f(A\sharp_t B)dt \leq \int_0^1 f(A)\sharp_t f(B)dt \leq \frac{1}{2}((f(A)\sharp f(B)) + f(B)), \tag{2.2}$$

for $A, B \in B(H)^{++}$.

Proof. Let f be a geometrically convex function. Then we have

$$\begin{aligned} f\left(A\sharp B\right) &= f\left(\left(A\sharp_t B\right)\sharp\left(A\sharp_{1-t}B\right)\right) \qquad \text{(by Lemma 2.1)} \\ &\leq f\left(A\sharp_t B\right)\sharp f\left(A\sharp_{1-t}B\right) \qquad (f \text{ is operator geometrically convex}). \end{aligned}$$

Taking integral of the both sides of the above inequalities on $[0,1]$, we obtain

$$\begin{aligned} f\left(A\sharp B\right) &\leq \int_0^1 f\left(A\sharp_t B\right)\sharp f\left(A\sharp_{1-t}B\right)dt \\ &\leq \left(\int_0^1 f\left(A\sharp_t B\right)dt\right)\sharp\left(\int_0^1 f\left(A\sharp_{1-t}B\right)dt\right) \quad \text{(by Lemma 2.3)} \\ &= \int_0^1 f\left(A\sharp_t B\right)dt \\ &\leq \int_0^1 f\left(A\right)\sharp_t f\left(B\right)dt. \end{aligned}$$

For the case $f(A) \leq f(B)$, by applying Remark 2.1 for $s = \frac{1}{2}$, we have

$$\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t \leq \left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^{\frac{1}{2}}.$$

By integrating the above inequality over $t \in [0, \frac{1}{2}]$, we obtain

$$\int_0^{\frac{1}{2}} \left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t dt \leq \frac{1}{2}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Multiplying both sides of the above inequality by $f(A)^{\frac{1}{2}}$, we have

$$\begin{aligned} &\int_0^{\frac{1}{2}} f(A)^{\frac{1}{2}}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t f(A)^{\frac{1}{2}}dt \\ \leq&\frac{1}{2}\left(f(A)^{\frac{1}{2}}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^{\frac{1}{2}} f(A)^{\frac{1}{2}}\right). \end{aligned}$$

It follows that

$$\int_0^{\frac{1}{2}} f(A)\sharp_t f(B) \leq \frac{f(A)\sharp f(B)}{2}. \tag{2.3}$$

On the other hand, by considering Remark 2.1 for $s = 1$, we have

$$\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t \leq f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}.$$

Integrating the above inequality over $t \in [\frac{1}{2}, 1]$, we get

$$\int_{\frac{1}{2}}^1 \left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t dt \leq \frac{1}{2}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right).$$

By multiplying both side of the above inequality by $f(A)^{\frac{1}{2}}$, we have

$$\int_{\frac{1}{2}}^1 f(A)^{\frac{1}{2}}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t f(A)^{\frac{1}{2}}dt \leq \frac{f(B)}{2}.$$

It follows that

$$\int_{\frac{1}{2}}^{1} f(A)\sharp_t f(B) \leq \frac{f(B)}{2}. \tag{2.4}$$

From inequalities (2.3) and (2.4) we obtain

$$\begin{aligned}\int_0^{\frac{1}{2}} f(A\sharp_t B)dt + \int_{\frac{1}{2}}^{1} f(A\sharp_t B)dt &\leq \int_0^{\frac{1}{2}} f(A)\sharp_t f(B)dt + \int_{\frac{1}{2}}^{1} f(A)\sharp_t f(B)dt \\ &\leq \frac{f(A)\sharp f(B)}{2} + \frac{f(B)}{2}.\end{aligned}$$

It follows that

$$\int_0^1 f(A\sharp_t B)dt \leq \int_0^1 f(A)\sharp_t f(B)dt \leq \frac{1}{2}((f(A)\sharp f(B)) + f(B)). \qquad \square$$

By making use of inequalities (2.1) and (2.2), we have the following result.

Corollary 2.1. *Let f be an operator geometrically convex function. Then, if $f(A) \leq f(B)$ we have*

$$f(A\sharp B) \leq \int_0^1 f\left(A\sharp_t B\right) dt \leq \frac{1}{2}\left((f(A)\sharp f(B)) + f(B)\right),$$

for $A, B \in B(H)^{++}$.

Theorem 2.2. *Let f be an operator geometrically convex function. Then, we have*

$$f\left(A\sharp B\right) \leq \int_0^1 f\left(A\sharp_t B\right)\sharp f\left(A\sharp_{1-t} B\right) dt \leq f(A)\sharp f(B),$$

for $A, B \in B(H)^{++}$.

Proof. We can write

$$\begin{aligned} f(A\sharp B) &= f\left((A\sharp_t B)\sharp(A\sharp_{1-t} B)\right) \quad \text{(by Lemma 2.1)} \\ &\leq f(A\sharp_t B)\sharp f(A\sharp_{1-t} B) \quad (f \text{ is operator geometrically convex}) \\ &\leq \left(f(A)\sharp_t f(B)\right)\sharp\left(f(A)\sharp_{1-t} f(B)\right) \quad \text{(by Lemma 2.2)} \\ &= f(A)\sharp f(B). \end{aligned}$$

So, we obtain

$$f(A\sharp B) \leq f(A\sharp_t B)\sharp f(A\sharp_{1-t} B) \leq f(A)\sharp f(B).$$

Integrating the above inequality over $t \in [0, 1]$ we obtain the desired result. $\square$

We divide the interval $[0, 1]$ to the interval $[\nu, 1-\nu]$ when $\nu \in [0, \frac{1}{2})$ and to the interval $[1-\nu, \nu]$ when $\nu \in (\frac{1}{2}, 1]$. The we have the following inequalities.

Theorem 2.3. *Let $A, B \in B(H)^{++}$ such that $f(A) \leq f(B)$. Then, we have*

(a) *for $\nu \in [0, \frac{1}{2})$*

$$f(A)\sharp_\nu f(B) \leq \frac{1}{1-2\nu}\int_\nu^{1-\nu} f(A)\sharp_t f(B)dt \leq f(A)\sharp_{1-\nu} f(B); \tag{2.5}$$

(b) *for* $\nu \in (\frac{1}{2}, 1]$

$$f(A)\sharp_{1-\nu}f(B) \leq \frac{1}{2\nu-1}\int_{1-\nu}^{\nu} f(A)\sharp_t f(B)dt \leq f(A)\sharp_\nu f(B). \tag{2.6}$$

Proof. Let $\nu \in [0, \frac{1}{2})$, then by Remark 2.1 we have

$$\begin{aligned}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{\nu} &\leq \left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{t}\\ &\leq \left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{1-\nu},\end{aligned}$$

for $\nu \leq t \leq 1-\nu$ and $A, B \in B(H)^{++}$ such that $\mathrm{Sp}\,(A), \mathrm{Sp}\,(B) \subseteq I$.
By integrating the above inequality over $t \in [\nu, 1-\nu]$ we obtain

$$\begin{aligned}\int_{\nu}^{1-\nu}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{\nu}dt &\leq \int_{\nu}^{1-\nu}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{t}dt\\ &\leq \int_{\nu}^{1-\nu}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{1-\nu}dt.\end{aligned}$$

It follows that

$$\begin{aligned}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{\nu} &\leq \frac{1}{1-2\nu}\int_{\nu}^{1-\nu}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{t}dt\\ &\leq \left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{1-\nu}.\end{aligned}$$

Multiplying the both sides of the above inequality by $f(A)^{\frac{1}{2}}$ gives us

$$f(A)\sharp_\nu f(B) \leq \frac{1}{1-2\nu}\int_{\nu}^{1-\nu} f(A)\sharp_t f(B)dt \leq f(A)\sharp_{1-\nu}f(B).$$

Also, we know that

$$\begin{aligned}\lim_{\nu\to\frac{1}{2}} f(A)\sharp_\nu f(B) &= \lim_{\nu\to\frac{1}{2}}\frac{1}{1-2\nu}\int_{\nu}^{1-\nu} f(A)\sharp_t f(B)dt\\ &= \lim_{\nu\to\frac{1}{2}} f(A)\sharp_{1-\nu}f(B)\\ &= f(A)\sharp f(B).\end{aligned}$$

Similarily, for $\nu \in (\frac{1}{2}, 1]$, by the same proof as above, we get

$$f(A)\sharp_{1-\nu}f(B) \leq \frac{1}{2\nu-1}\int_{1-\nu}^{\nu} f(A)\sharp_t f(B)dt \leq f(A)\sharp_\nu f(B).$$

□

By definition of geometrically convex function and (2.5) we have

$$\begin{aligned}f\left(A\sharp_\nu B\right) &\leq \frac{1}{1-2\nu}\int_{\nu}^{1-\nu} f\left(A\sharp_t B\right)dt\\ &\leq \frac{1}{1-2\nu}\int_{\nu}^{1-\nu} f(A)\sharp_t f(B)dt\\ &\leq f(A)\sharp_{1-\nu}f(B),\end{aligned}$$

for $\nu \in [0, \frac{1}{2})$. We should mention here that

$$\lim_{\nu \to \frac{1}{2}} \frac{1}{1-2\nu} \int_{\nu}^{1-\nu} f\left(A \sharp_t B\right) dt = \lim_{\nu \to \frac{1}{2}} f(A \sharp_\nu B) = f(A \sharp B).$$

On the other hand, by the definition of geometrically convex function and (2.6) we have

$$\begin{aligned} f\left(A \sharp_{1-\nu} B\right) &\leq \frac{1}{2\nu-1} \int_{1-\nu}^{\nu} f\left(A \sharp_t B\right) dt \\ &\leq \frac{1}{2\nu-1} \int_{1-\nu}^{\nu} f(A) \sharp_t f(B) dt \\ &\leq f(A) \sharp_\nu f(B), \end{aligned}$$

for $\nu \in (\frac{1}{2}, 1]$.

3. Examples and Remarks

In this section we give some examples of the results that obtained in the previous section.

Remark 3.1. For positive $A, B \in B(H)$, Ando proved in [1] that if Ψ is a positive linear map, then we have

$$\Psi(A \sharp B) \leq \Psi(A) \sharp \Psi(B).$$

The above inequality shows that we can find some examples for Definition 2.1 when f is linear.

Example 3.1. It is easy to check that the function $f(t) = t^{-1}$ is operator geometrically convex for operators in $B(H)^{++}$.

Definition 3.1. Let ϕ be a map on C^*-algebra $B(H)$. We say that ϕ is 2-positive if the 2×2 operator matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then we have $\begin{bmatrix} \phi(A) & \phi(B) \\ \phi(B^*) & \phi(C) \end{bmatrix} \geq 0$.

In [6], M. Lin gave an example of a 2-positive map over contraction operators (i.e., $\|A\| < 1$). He proved that

$$\phi(t) = (1-t)^{-1} \tag{3.1}$$

is 2-positive.

Example 3.2. Let A and B be two contraction operators in $B(H)^{++}$. Then it is easy to check $A \sharp B$ is also a contraction and positive. Also, we know the 2×2 operator matrix

$$\begin{bmatrix} A & A \sharp B \\ A \sharp B & B \end{bmatrix}$$

is semidefinite positive. Hence, by (3.1) we obtain

$$\begin{bmatrix} (I-A)^{-1} & (I-(A \sharp B))^{-1} \\ (I-(A \sharp B))^{-1} & (I-B)^{-1} \end{bmatrix}$$

is semidefinite positive.

On the other hand, by Ando's characterization of the geometric mean if X is a Hermitian matrix and

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0,$$

then $X \leq A\sharp B$. So we conclude that $(I-(A\sharp B))^{-1} \leq (I-A)^{-1}\sharp(I-B)^{-1}$. Therefore, the function $\phi(t) = (1-t)^{-1}$ is operator geometrically convex.

Also, Lin proved that the function

$$\phi(t) = \frac{1+t}{1-t}$$

is 2-positive over contractions. By the same argument as Example 3.2 we can say the above function is operator geometrically convex too.

Example 3.3. In the proof of [3, Theorem 4.12], by applying Hölder-McCarthy inequality the authors showed the following inequalities

$$\begin{aligned} \langle A\sharp_\alpha Bx, x\rangle &= \left\langle \left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^\alpha A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right\rangle \\ &\leq \left\langle \left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right) A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right\rangle^\alpha \left\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right\rangle^{1-\alpha} \\ &= \langle Ax, x\rangle^{1-\alpha}\langle Bx, x\rangle^\alpha \\ &= \langle Ax, x\rangle \sharp_\alpha \langle Bx, x\rangle, \end{aligned}$$

for $x \in H$ and $\alpha \in [0,1]$. By taking the supremum over unit vector x, we obtain that $f(x) = \|x\|$ is geometrically convex function for usual operator norms.

By the above example and Corollary 2.1, when $\|A\| \leq \|B\|$ we have

$$\|A\sharp B\| \leq \int_0^1 \|A\sharp_t B\| dt \leq \frac{1}{2}(\sqrt{\|A\|\|B\|} + \|B\|),$$

for $A, B \in B(H)^{++}$.

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