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# INITIAL COEFFICIENT ESTIMATES FOR A CERTAIN FAMILIES OF BI-UNIVALENT FUNCTIONS RELATED TO BAZILEVIČ AND $\lambda$-PSEUDO FUNCTIONS

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Abstract. In this article, we define new families of normalized holormorphic and bi-univalent functions $\mathcal{R}_\Sigma(\mu,\gamma,\lambda;\vartheta)$ and $\mathcal{F}_\Sigma(\mu,\gamma,\lambda;\vartheta)$ which involve the Bazilevič functions and the $\lambda$-pseudo functions defined in the unit disk $U$. We determine the coefficient estimates for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ and resolve the Fekete-Szegö type inequalities for these families. In addition, we point out several special cases and consequences of our results.

## 1. Introduction

Denote by $\mathcal{A}$ the family of all holomorphic functions of the form

$$f(z)=z+\sum_{n=2}^{+\infty}a_nz^n \tag{1.1}$$

in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. We also denote by $\mathcal{S}$ the subfamily of $\mathcal{A}$ consisting of functions which are also univalent in $U$.

The famous Koebe one-quarter theorem [11] ensure that the image of $U$ under each univalent function $f\in\mathcal{A}$ contain a disk of radius $\frac{1}{4}$. Furthermore, each function $f\in\mathcal{S}$ has an inverse $f^{-1}$ defined by $f^{-1}(f(z))=z$ and

$$f^{-1}(f(w))=w,\quad |w|<r_0(f),r_0(f)\geq\frac{1}{4},$$

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where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is named bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. The family of all bi-univalent functions in $U$ denoted by $\Sigma$.

In fact, Srivastava et al. [31] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Frasin and Aouf [13], Ali et al. [1], Bulut et al. [7] and others (see, for example, [2, 8, 9, 14, 15, 25–28, 32, 35]). From the work of Srivastava et al. [31], we choose to recall the following examples of functions in the family $\Sigma$:

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

We notice that the family $\Sigma$ is not empty. However, the Koebe function is not a member of $\Sigma$.

The problem to obtain the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n|, \quad n \in \mathbb{N}, n \geq 3,$$

for functions $f \in \Sigma$ is still not completely addressed for many of the subfamilies of $\Sigma$ (see, for example, [32]). The Fekete-Szegö functional $|a_3 - \eta a_2^2|$ for $f \in \mathcal{S}$ is well known for its rich history in the field of Geometric Function Theory. Its origin was in the disproof by Fekete and Szegö [12] of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by unity. The functional has since received great attention, particularly in the study of many subclasses of the family of univalent functions. This topic has become of considerable interest among researchers in Geometric Function Theory (see, for example, [5, 17, 22, 29, 30]).

With a view to recalling the principle of subordination between holomorphic functions, let the functions $f$ and $g$ be holomorphic in $U$. We name the function $f$ is subordinate to $g$, if there exists a Schwarz function $\omega$, which is analytic in $U$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1, \quad z \in U,$$

such that

$$f(z) = g(\omega(z)).$$

This subordination is denoted by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad z \in U.$$

It is well known that (see [19]), if the function $g$ is univalent in $U$, then

$$f \prec g \quad (z \in U) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subseteq g(U).$$

A function $f \in \mathcal{A}$ is called Bazilevič function in $U$ if (see [24])

$$\operatorname{Re}\left\{\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right\} > 0, \quad z \in U, \gamma \geq 0.$$

On the other hand, a function $f \in \mathcal{A}$ is called a $\lambda$-pseudo-starlike function in $U$ if (see [3])

$$\operatorname{Re}\left\{\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right\} > 0, \quad z \in U, \lambda \geq 1.$$

Recently, several authors introduced and studied different subfamilies associated with Bazilevič and $\lambda$-pseudo functions (see, for example, [6, 10, 16, 21, 33, 34, 36–38]).

We shall need the following lemma in our investigation.

**Lemma 1.1** ([20])**.** *Let the function $\mathfrak{p} \in \mathfrak{P}$ be given by the following series:*

$$\mathfrak{p}(z) = 1 + \mathfrak{p}_1 z + \mathfrak{p}_2 z^2 + \cdots, \quad z \in \mathfrak{U}.$$

*The sharp estimate given by $|\mathfrak{p}_n| \leq 2$, $n \in \mathbb{N}$, holds true.*

## 2. Main Results

Denote by $\vartheta(z)$ the holomorphic function with positive real part in $U$ such that

$$\vartheta(0) = 1, \quad \vartheta'(0) > 0,$$

and $\vartheta(z)$ is symmetric with respect to real axis, which is of the type:

$$\vartheta(z) = 1 + \mathfrak{B}_1 z + \mathfrak{B}_2 z^2 + \mathfrak{B}_3 z^3 + \cdots, \tag{2.1}$$

where $\mathfrak{B}_1 > 0$.

Using the subordinations, we now provide the following subfamilies of holomorphic and bi-univalent functions.

**Definition 2.1.** For $0 \leq \mu \leq 1$, $\gamma \geq 0$ and $\lambda \geq 1$, a function $f \in \Sigma$ is said to be in the family $\mathcal{R}_{\Sigma}(\mu, \gamma, \lambda; \vartheta)$ if it fulfills the subordinations:

$$(1-\mu)\frac{z^{1-\gamma} f'(z)}{\Big(f(z)\Big)^{1-\gamma}} + \mu\, \frac{z\Big(f'(z)\Big)^{\lambda}}{f(z)} \prec \vartheta(z)$$

and

$$(1-\mu)\frac{w^{1-\gamma} g'(w)}{\Big(g(w)\Big)^{1-\gamma}} + \mu\, \frac{w\Big(g'(w)\Big)^{\lambda}}{g(w)} \prec \vartheta(w),$$

where $g(w) = f^{-1}(w)$.

**Definition 2.2.** For $0 \leq \mu \leq 1$, $\gamma \geq 0$ and $\lambda \geq 1$, a function $f \in \Sigma$ is said to be in the family $\mathcal{F}_{\Sigma}(\mu, \gamma, \lambda; \vartheta)$ if it fulfills the subordinations:

$$(1-\mu)\left(1 + \frac{z^{2-\gamma} f''(z)}{\left(zf'(z)\right)^{1-\gamma}}\right) + \mu \frac{\Big(\left(zf'(z)\right)'\Big)^{\lambda}}{f'(z)} \prec \vartheta(z)$$

and

$$(1-\mu)\left(1 + \frac{w^{2-\gamma} g''(w)}{\left(wg'(w)\right)^{1-\gamma}}\right) + \mu \frac{\Big(\left(wg'(w)\right)'\Big)^{\lambda}}{g'(w)} \prec \vartheta(w),$$

where $g(w) = f^{-1}(w)$.

*Remark* 2.1. The families $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ and $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ are a generalization of several known families studied in earlier investigations which are being recalled below.

(a) For $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $0 < \alpha \leq 1$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduce to the family $\mathcal{T}_\Sigma(\mu, \gamma, \lambda; \alpha)$ which was considered by Srivastava et al. [34].

(b) For $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{T}^*_\Sigma(\mu, \gamma, \lambda; \beta)$ which was studied by Srivastava et al. [34].

(c) For $\vartheta(z) = \frac{a+(b-ap)rz}{1-prz-qz^2} + 1 - a$, $r \in \mathbb{R}$, $a, b, p$ and $q$ are real constant, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{T}_\Sigma(\mu, \gamma, \lambda, r)$ which was investigated by Wanas et al. [39].

(d) For $\vartheta(z) = \frac{2-M(x)z}{1-M(x)z-N(x)z^2}$, $M(x)$ and $N(x)$ are polynomials with real coefficients, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{L}_{MN}(\mu, \gamma, \lambda; x)$ which was defined by Wanas et al. [38].

(e) For $\mu = 0$ and $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $0 < \alpha \leq 1$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $P_\Sigma(\alpha, \gamma)$ which was studied by Prema and Keerthi [21].

(f) For $\mu = 0$ and $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $P_\Sigma(\beta, \gamma)$ which was investigated by Prema and Keerthi [21].

(g) For $\mu = 1$ and $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $0 < \alpha \leq 1$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{L}B^\lambda_\Sigma(\alpha)$ which was considered by Joshi et al. [16].

(h) For $\mu = 1$ and $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{L}B_\Sigma(\lambda, \beta)$ which was introduced by Joshi et al. [16].

(i) For $\mu = \gamma = 0$ and $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $0 < \alpha \leq 1$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $S^*_\Sigma(\alpha)$ which was considered by Brannan and Taha [4].

(j) For $\mu = \gamma = 0$ and $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $S^*_\Sigma(\beta)$ which was investigated by Brannan and Taha [4].

(k) For $\mu = \gamma = 0$ and $\vartheta(z) = \frac{a+(b-ap)rz}{1-prz-qz^2} + 1 - a$, $r \in \mathbb{R}$, $a, b, p$ and $q$ are real constants, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{W}_\Sigma(r)$ which was defined by Srivastava et al. [25].

(l) For $\mu = \gamma = 0$ and $\vartheta(z) = \frac{1}{1-2tz+z^2}$, $t \in (\frac{1}{2}, 1]$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $S^*_\Sigma(t)$ which was introduced by Bulut et al. [7].

(m) For $\mu = 0$, $\gamma = 1$ and $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $0 < \alpha \leq 1$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{H}_\Sigma(\alpha)$ which was investigated by Srivastava et al. [31].

(n) For $\mu = 0$, $\gamma = 1$ and $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$, the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{H}_\Sigma(\beta)$ which was defined by Srivastava et al. [31].

(o) For $\mu = 0$ and $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $0 < \alpha \leq 1$, the family $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{B}_\Sigma(\gamma; \alpha)$ which was investigated by Sakar and Wanas [23].

(p) For $\mu = 0$ and $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$, the family $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{B}^*_\Sigma(\gamma; \beta)$ which was defined by Sakar and Wanas [23].

(q) For $\mu = \gamma = 0$ and $\vartheta(z) = \frac{a+(b-ap)rz}{1-prz-qz^2} + 1 - a$, $r, a, b, p, q \in \mathbb{R}$, the family $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ reduces to the family $\mathcal{K}_\Sigma(r)$ which was introduced by Magesh et al. [18].

**Theorem 2.1.** *Let $f$, given by* (1.1), *be in the family $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$. Then,*

$$|a_2| \leq \min\left\{\frac{\mathfrak{B}_1}{(1-\mu)(\gamma+1)+\mu(2\lambda-1)}, \frac{\sqrt{2}\mathfrak{B}_1^{\frac{3}{2}}}{\sqrt{\left|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda\,(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}}\right\}$$

*and*

$$|a_3| \leq \min\left\{\frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)} + \frac{2\mathfrak{B}_2}{(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda\,(2\lambda-1)}, \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)} + \frac{\mathfrak{B}_1^2}{[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2}\right\},$$

*where the coefficients $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are defined as in* (2.1).

*Proof.* Let $f \in \mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ and $g = f^{-1}$. Then, there are holomorphic functions $\mathfrak{S}, \mathfrak{T} : U \to U$ with $\mathfrak{S}(0) = \mathfrak{T}(0) = 0$, fulfills the following conditions:

$$(1-\mu)\,\frac{z^{1-\gamma} f'(z)}{\big(f(z)\big)^{1-\gamma}} + \mu\,\frac{z\big(f'(z)\big)^{\lambda}}{f(z)} = \vartheta(\mathfrak{S}(z)), \quad z \in U, \tag{2.2}$$

and

$$(1-\mu)\,\frac{w^{1-\gamma} g'(w)}{\big(g(w)\big)^{1-\gamma}} + \mu\,\frac{w\big(g'(w)\big)^{\lambda}}{g(w)} = \vartheta(\mathfrak{T}(w)), \quad w \in U. \tag{2.3}$$

Define the functions $x$ and $y$ by

$$x(z) = \frac{1+\mathfrak{S}(z)}{1-\mathfrak{S}(z)} = 1 + x_1 z + x_2 z^2 + \cdots$$

and

$$y(z) = \frac{1+\mathfrak{T}(z)}{1-\mathfrak{T}(z)} = 1 + y_1 z + y_2 z^2 + \cdots.$$

Then, $x$ and $y$ are analytic in $U$ with $x(0) = y(0) = 1$. Since we have $\mathfrak{S}, \mathfrak{T} : U \to U$, each of the functions $x$ and $y$ has a positive real part in $U$.

Solving for $\mathfrak{S}(z)$ and $\mathfrak{T}(z)$, we have

$$\mathfrak{S}(z) = \frac{x(z)-1}{x(z)+1} = \frac{1}{2}\left[x_1 z + \left(x_2 - \frac{x_1^2}{2}\right) z^2\right] + \cdots, \quad z \in U, \tag{2.4}$$

and

$$\mathfrak{T}(z) = \frac{y(z)-1}{y(z)+1} = \frac{1}{2}\left[y_1 z + \left(y_2 - \frac{y_1^2}{2}\right) z^2\right] + \cdots, \quad z \in U. \tag{2.5}$$

By substituting (2.4) and (2.5) into (2.2) and (2.3) and applying (2.1), we get

$$\begin{aligned} &(1-\mu)\,\frac{z^{1-\gamma} f'(z)}{\left(f(z)\right)^{1-\gamma}} + \mu\,\frac{z\left(f'(z)\right)^{\lambda}}{f(z)} \\ =&\vartheta\left(\frac{x(z)-1}{x(z)+1}\right) = 1 + \frac{1}{2}\mathfrak{B}_1 x_1 z + \left[\frac{1}{2}\mathfrak{B}_1\left(x_2 - \frac{x_1^2}{2}\right) + \frac{1}{4}\mathfrak{B}_2 x_1^2\right] z^2 + \cdots \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &(1-\mu)\,\frac{w^{1-\gamma} g'(w)}{\left(g(w)\right)^{1-\gamma}} + \mu\,\frac{w\left(g'(w)\right)^{\lambda}}{g(w)} \\ =&\vartheta\left(\frac{y(w)-1}{y(w)+1}\right) = 1 + \frac{1}{2}\mathfrak{B}_1 y_1 w + \left[\frac{1}{2}\mathfrak{B}_1\left(y_2 - \frac{y_1^2}{2}\right) + \frac{1}{4}\mathfrak{B}_2 y_1^2\right] w^2 + \cdots. \end{aligned} \tag{2.7}$$

Equating the coefficients in (2.6) and (2.7), yields

$$\left[(1-\mu)(\gamma+1) + \mu(2\lambda-1)\right] a_2 = \frac{1}{2}\mathfrak{B}_1 x_1, \tag{2.8}$$

$$\begin{aligned} &\left[(1-\mu)(\gamma+2) + \mu(3\lambda-1)\right] a_3 \\ &+ \left[\frac{1}{2}\,(1-\mu)(\gamma+2)(\gamma-1) + \mu\left(2\lambda(\lambda-2)+1\right)\right] a_2^2 \\ =&\frac{1}{2}\mathfrak{B}_1\left(x_2 - \frac{x_1^2}{2}\right) + \frac{1}{4}\mathfrak{B}_2 x_1^2, \end{aligned} \tag{2.9}$$

$$-\left[(1-\mu)(\gamma+1) + \mu(2\lambda-1)\right] a_2 = \frac{1}{2}\mathfrak{B}_1 y_1 \tag{2.10}$$

and

$$\begin{aligned} &\left[(1-\mu)(\gamma+2) + \mu(3\lambda-1)\right](2a_2^2 - a_3) \\ &+ \left[\frac{1}{2}\,(1-\mu)(\gamma+2)(\gamma-1) + \mu\left(2\lambda(\lambda-2)+1\right)\right] a_2^2 \\ =&\frac{1}{2}\mathfrak{B}_1\left(y_2 - \frac{y_1^2}{2}\right) + \frac{1}{4}\mathfrak{B}_2 y_1^2. \end{aligned} \tag{2.11}$$

From (2.8) and (2.10), we have

$$x_1 = -y_1 \tag{2.12}$$

and

$$2\left[(1-\mu)(\gamma+1) + \mu(2\lambda-1)\right]^2 a_2^2 = \frac{1}{4}\mathfrak{B}_1^2(x_1^2 + y_1^2). \tag{2.13}$$

If we add (2.9) to (2.11), we obtain
(2.14)

$$[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]\,a_2^2=\frac{1}{2}\mathfrak{B}_1\left[x_2+y_2-\frac{x_1^2+y_1^2}{2}\right]+\frac{1}{4}\mathfrak{B}_2[x_1^2+y_1^2].$$

Substituting the value of $x_1^2+y_1^2$ from (2.13) in the right hand side of (2.14), we deduce that

$$a_2^2=\frac{\mathfrak{B}_1^3(x_2+y_2)}{2\left(\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)\right)}. \tag{2.15}$$

Applying Lemma 1.1 for the coefficients $x_1, x_2, y_1, y_2$ in (2.13) and (2.15), we get

$$|a_2|\le\frac{\sqrt{2}\mathfrak{B}_1^{\frac{3}{2}}}{\sqrt{\left|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}},$$

i.e.,

$$|a_2|\le\frac{\mathfrak{B}_1}{(1-\mu)(\gamma+1)+\mu(2\lambda-1)},$$

which gives the estimates of $|a_2|$.

Furthermore, in order to find the bound on $|a_3|$, we subtract (2.11) from (2.9) and also applying (2.12), we obtain $x_1^2=y_1^2$, hence

$$2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)](a_3-a_2^2)=\frac{1}{2}\mathfrak{B}_1(x_2-y_2). \tag{2.16}$$

Then, by substituting of the value of $a_2^2$ from (2.13) into (2.16), gives

$$a_3=\frac{\mathfrak{B}_1(x_2-y_2)}{4[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}+\frac{\mathfrak{B}_1^2(x_1^2+y_1^2)}{8[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2}.$$

So, we have

$$|a_3|\le\frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}+\frac{\mathfrak{B}_1^2}{[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2}.$$

Also, substituting the value of $a_2^2$ from (2.14) into (2.16), we get

$$a_3=\frac{\mathfrak{B}_1(x_2-y_2)}{4[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}+\frac{\mathfrak{B}_1(x_2+y_2)+\frac{1}{2}(x_1^2+y_1^2)(\mathfrak{B}_2-\mathfrak{B}_1)}{2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]},$$

and we have

$$|a_3|\le\frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}+\frac{2\mathfrak{B}_2}{(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)},$$

which gives us the desired estimates on the coefficient $|a_3|$. □

Taking $\vartheta(z)=\left(\frac{1+z}{1-z}\right)^\alpha=1+2\alpha z+2\alpha^2z^2+\cdots$, $0<\alpha\le1$, in Theorem 2.1, we obtain the next corollary.

**Corollary 2.1.** *Let $f$, given by* (1.1)*, be in the family* $\mathcal{T}_{\Sigma}(\mu,\gamma,\lambda;\alpha)$*, where* $0<\alpha\leq 1$*. Then,*

$$|a_2|\leq \min\left\{\frac{2\alpha}{(1-\mu)(\gamma+1)+\mu(2\lambda-1)},\right.$$

$$\left.\frac{2\alpha\sqrt{\alpha}}{\sqrt{\left|\alpha^2\left[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda\left(2\lambda-1\right)\right]+\alpha(1-\alpha)\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2\right|}}\right\}$$

*and*

$$|a_3|\leq \min\left\{\frac{2\alpha}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}+\frac{4\alpha^2}{(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda\left(2\lambda-1\right)},\right.$$

$$\left.\frac{2\alpha}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}+\frac{4\alpha^2}{\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2}\right\}.$$

Taking $\vartheta(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots$, $0\leq\beta<1$, in Theorem 2.1, we obtain the next corollary.

**Corollary 2.2.** *Let $f$, given by* (1.1)*, be in the family* $\mathcal{T}^*_{\Sigma}(\mu,\gamma,\lambda;\beta)$*, where* $0\leq\beta<1$*. Then,*

$$|a_2|\leq \min\left\{\frac{2(1-\beta)}{(1-\mu)(\gamma+1)+\mu(2\lambda-1)},\frac{2\sqrt{1-\beta}}{\sqrt{\left|(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda\left(2\lambda-1\right)\right|}}\right\}$$

*and*

$$|a_3|\leq \min\left\{\frac{2(1-\beta)}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}+\frac{4(1-\beta)}{(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda\left(2\lambda-1\right)},\right.$$

$$\left.\frac{2(1-\beta)}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}+\frac{4\left(1-\beta\right)^2}{\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2}\right\}.$$

The families $\mathcal{T}_{\Sigma}(\mu,\gamma,\lambda;\alpha)$ and $\mathcal{T}^*_{\Sigma}(\mu,\gamma,\lambda;\beta)$ were given by Srivastava et al. [34] and defined as follows.

**Definition 2.3** ([34])**.** For $0<\alpha\leq 1$, $0\leq\mu\leq 1$, $\gamma\geq 0$ and $\lambda\geq 1$, a function $f\in\Sigma$ is said to be in the family $\mathcal{T}_{\Sigma}(\mu,\gamma,\lambda;\alpha)$ if it fulfills the subordinations:

$$\left|\arg\left((1-\mu)\,\frac{z^{1-\gamma}f'(z)}{\left(f(z)\right)^{1-\gamma}}+\mu\,\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right)\right|<\frac{\alpha\pi}{2}$$

and

$$\left|\arg\left((1-\mu)\,\frac{w^{1-\gamma}g'(w)}{\left(g(w)\right)^{1-\gamma}}+\mu\,\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}\right)\right|<\frac{\alpha\pi}{2},$$

where $g(w)=f^{-1}(w)$.

**Definition 2.4** ([34])**.** For $0 \le \beta < 1$, $0 \le \mu \le 1$, $\gamma \ge 0$ and $\lambda \ge 1$, a function $f \in \Sigma$ is said to be in the family $\mathcal{T}^*_\Sigma(\mu, \gamma, \lambda; \beta)$ if it fulfills the subordinations:

$$\mathrm{Re}\left((1-\mu)\,\frac{z^{1-\gamma} f'(z)}{\big(f(z)\big)^{1-\gamma}} + \mu\,\frac{z\big(f'(z)\big)^{\lambda}}{f(z)}\right) > \beta$$

and

$$\mathrm{Re}\left((1-\mu)\,\frac{w^{1-\gamma} g'(w)}{\big(g(w)\big)^{1-\gamma}} + \mu\,\frac{w\big(g'(w)\big)^{\lambda}}{g(w)}\right) > \beta,$$

where $g(w) = f^{-1}(w)$.

**Theorem 2.2.** *Let $f$, given by* (1.1)*, be in the family $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$. Then,*

$$|a_2| \le \min\left\{\frac{\mathfrak{B}_1}{2\left(2\mu(\lambda-1)+1\right)},\right.$$
$$\left.\frac{\mathfrak{B}_1^{\frac{3}{2}}}{\sqrt{\left|\mathfrak{B}_1^2\left(2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)\right)+4\left(2\mu(\lambda-1)+1\right)^2\left(\mathfrak{B}_1-\mathfrak{B}_2\right)\right|}}\right\}$$

*and*

$$|a_3| \le \min\left\{\frac{\mathfrak{B}_1}{3\left(3\mu(\lambda-1)+2\right)} + \frac{\mathfrak{B}_2}{2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)},\right.$$
$$\left.\frac{\mathfrak{B}_1}{3\left(3\mu(\lambda-1)+2\right)} + \frac{\mathfrak{B}_1^2}{4\left(2\mu(\lambda-1)+1\right)^2}\right\},$$

*where the coefficients $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are defined as in* (2.1)*.*

*Proof.* Let $f \in \mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ and $g = f^{-1}$. Then, there are holomorphic functions $\mathfrak{S}, \mathfrak{T} : U \to U$ such that

$$(1-\mu)\left(1 + \frac{z^{2-\gamma} f''(z)}{\left(zf'(z)\right)^{1-\gamma}}\right) + \mu\frac{\left(\left(zf'(z)\right)'\right)^{\lambda}}{f'(z)} = \vartheta(\mathfrak{S}(z)), \quad z \in U, \tag{2.17}$$

and

$$(1-\mu)\left(1 + \frac{w^{2-\gamma} g''(w)}{\left(wg'(w)\right)^{1-\gamma}}\right) + \mu\frac{\left(\left(wg'(w)\right)'\right)^{\lambda}}{g'(w)} = \vartheta(\mathfrak{T}(w)), \quad w \in U, \tag{2.18}$$

where $\mathfrak{S}(z)$ and $\mathfrak{T}(z)$ have the forms (2.4) and (2.5). From (2.17), (2.18) and (2.1), we deduce that

$$(1-\mu)\left(1+\frac{z^{2-\gamma}f''(z)}{(zf'(z))^{1-\gamma}}\right)+\mu\frac{\left((zf'(z))'\right)^{\lambda}}{f'(z)} \\ =\vartheta\left(\frac{x(z)-1}{x(z)+1}\right)=1+\frac{1}{2}\mathfrak{B}_1x_1z+\left[\frac{1}{2}\mathfrak{B}_1\left(x_2-\frac{x_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2x_1^2\right]z^2+\cdots \tag{2.19}$$

and

$$(1-\mu)\left(1+\frac{w^{2-\gamma}g''(w)}{(wg'(w))^{1-\gamma}}\right)+\mu\frac{\left((wg'(w))'\right)^{\lambda}}{g'(w)} \\ =\vartheta\left(\frac{y(w)-1}{y(w)+1}\right)=1+\frac{1}{2}\mathfrak{B}_1y_1w+\left[\frac{1}{2}\mathfrak{B}_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2y_1^2\right]w^2+\cdots. \tag{2.20}$$

Equating the coefficients in (2.19) and (2.20), yields

$$2\left(2\mu(\lambda-1)+1\right)a_2=\frac{1}{2}\mathfrak{B}_1x_1, \tag{2.21}$$

$$3\left(3\mu(\lambda-1)+2\right)a_3+4\left[2\lambda\mu(\lambda-2)+\mu(2-\gamma)+\gamma-1\right]a_2^2 \\ =\frac{1}{2}\mathfrak{B}_1\left(x_2-\frac{x_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2x_1^2, \tag{2.22}$$

$$-2\left(2\mu(\lambda-1)+1\right)a_2=\frac{1}{2}\mathfrak{B}_1y_1 \tag{2.23}$$

and

$$3\left(3\mu(\lambda-1)+2\right)\left(2a_2^2-a_3\right)+4\left[2\lambda\mu(\lambda-2)+\mu(2-\gamma)+\gamma-1\right]a_2^2 \\ =\frac{1}{2}\mathfrak{B}_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{1}{4}\mathfrak{B}_2y_1^2. \tag{2.24}$$

From (2.21) and (2.23), we have

$$x_1=-y_1 \tag{2.25}$$

and

$$8\left(2\mu(\lambda-1)+1\right)^2a_2^2=\frac{1}{4}\mathfrak{B}_1^2(x_1^2+y_1^2). \tag{2.26}$$

If we add (2.22) to (2.24), we obtain

$$2\left[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)\right]a_2^2 \\ =\frac{1}{2}\mathfrak{B}_1\left[x_2+y_2-\left(\frac{x_1^2+y_1^2}{2}\right)\right]+\frac{1}{4}\mathfrak{B}_2[x_1^2+y_1^2]. \tag{2.27}$$

Substituting the value of $x_1^2 + y_1^2$ from (2.26) in the right hand side of (2.27), we deduce that

$$a_2^2 = \frac{\mathfrak{B}_1^3(x_2+y_2)}{4\left[\mathfrak{B}_1^2\left[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)\right]+4\left(2\mu(\lambda-1)+1\right)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right]}. \tag{2.28}$$

Applying Lemma 1.1 for the coefficients $x_1, x_2, y_1, y_2$ in (2.26) and (2.28), we get

$$|a_2| \leq \frac{\mathfrak{B}_1^{\frac{3}{2}}}{\sqrt{\left|\mathfrak{B}_1^2\left[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)\right]+4\left(2\mu(\lambda-1)+1\right)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}},$$

i.e.,

$$|a_2| \leq \frac{\mathfrak{B}_1}{2\left(2\mu(\lambda-1)+1\right)},$$

which gives the estimates of $|a_2|$.

Furthermore, in order to find the bound on $|b_3|$, we subtract (2.24) from (2.22) and also applying (2.25), we obtain $x_1^2 = y_1^2$, hence

$$6\left(3\mu(\lambda-1)+2\right)(a_3-a_2^2) = \frac{1}{2}\mathfrak{B}_1(x_2-y_2). \tag{2.29}$$

Then, by substituting of the value of $a_2^2$ from (2.26) into (2.29), gives

$$a_3 = \frac{\mathfrak{B}_1(x_2-y_2)}{12\left(3\mu(\lambda-1)+2\right)} + \frac{\mathfrak{B}_1^2(x_1^2+y_1^2)}{32\left(2\mu(\lambda-1)+1\right)^2}.$$

So, we have

$$|a_3| \leq \frac{\mathfrak{B}_1}{3\left(3\mu(\lambda-1)+2\right)} + \frac{\mathfrak{B}_1^2}{4\left(2\mu(\lambda-1)+1\right)^2}.$$

Also, substituting the value of $a_2^2$ from (2.27) into (2.29), we get

$$a_3 = \frac{\mathfrak{B}_1(x_2-y_2)}{12\left(3\mu(\lambda-1)+2\right)} + \frac{\mathfrak{B}_1(x_2+y_2)+\frac{1}{2}(x_1^2+y_1^2)(\mathfrak{B}_2-\mathfrak{B}_1)}{4\left[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)\right]},$$

and we have

$$|a_3| \leq \frac{\mathfrak{B}_1}{3\left(3\mu(\lambda-1)+2\right)} + \frac{\mathfrak{B}_2}{2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)},$$

which gives us the desired estimates on the coefficient $|a_3|$. □

Taking $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots$, $0 < \alpha \leq 1$, in Theorem 2.2, we obtain the next corollary.

**Corollary 2.3.** *Let $f$ given by* (1.1) *be in the family $\mathcal{M}_\Sigma(\mu, \gamma, \lambda; \alpha)$, where $0 < \alpha \leq 1$. Then,*

$$|a_2| \leq \min\left\{\frac{\alpha}{2\mu(\lambda-1)+1},\right.$$

$$\left.\frac{\alpha\sqrt{2\alpha}}{\sqrt{\left|\alpha^2\left[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)\right]+2\alpha(1-\alpha)\left(2\mu(\lambda-1)+1\right)^2\right|}}\right\}$$

*and*

$$|a_3| \le \min\left\{\frac{2\alpha}{3\left(3\mu(\lambda-1)+2\right)}+\frac{2\alpha^2}{2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)},\right.$$
$$\left.\frac{2\alpha}{3\left(3\mu(\lambda-1)+2\right)}+\frac{\alpha^2}{\left(2\mu(\lambda-1)+1\right)^2}\right\}.$$

Taking $\vartheta(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots$, $0\le\beta<1$, in Theorem 2.2, we obtain the next corollary.

**Corollary 2.4.** *Let $f$, given by* (1.1), *be in the family $\mathcal{M}^*_\Sigma(\mu,\gamma,\lambda;\beta)$, where $0\le\beta<1$. Then,*

$$|a_2| \le \min\left\{\frac{1-\beta}{\left(2\mu(\lambda-1)+1\right)},\sqrt{\frac{2(1-\beta)}{|2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)|}}\right\}$$

*and*

$$|a_3| \le \min\left\{\frac{2(1-\beta)}{3\left(3\mu(\lambda-1)+2\right)}+\frac{2(1-\beta)}{2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)},\right.$$
$$\left.\frac{2(1-\beta)}{3\left(3\mu(\lambda-1)+2\right)}+\frac{(1-\beta)^2}{\left(2\mu(\lambda-1)+1\right)^2}\right\}.$$

The families $\mathcal{M}_\Sigma(\mu,\gamma,\lambda;\alpha)$ and $\mathcal{M}^*_\Sigma(\mu,\gamma,\lambda;\beta)$ are defined as follows:

**Definition 2.5.** For $0<\alpha\le1$, $0\le\mu\le1$, $\gamma\ge0$ and $\lambda\ge1$, a function $f\in\Sigma$ is said to be in the family $\mathcal{T}_\Sigma(\mu,\gamma,\lambda;\alpha)$ if it fulfills the subordinations:

$$\left|\arg\left(1-\mu)\left(1+\frac{z^{2-\gamma}f''(z)}{\left(zf'(z)\right)^{1-\gamma}}\right)+\mu\frac{\left(\left(zf'(z)\right)'\right)^\lambda}{f'(z)}\right)\right|<\frac{\alpha\pi}{2}$$

and

$$\left|\arg\left((1-\mu)\left(1+\frac{w^{2-\gamma}g''(w)}{\left(wg'(w)\right)^{1-\gamma}}\right)+\mu\frac{\left(\left(wg'(w)\right)'\right)^\lambda}{g'(w)}\right)\right|<\frac{\alpha\pi}{2},$$

where $g(w)=f^{-1}(w)$.

**Definition 2.6.** For $0\le\beta<1$, $0\le\mu\le1$, $\gamma\ge0$ and $\lambda\ge1$, a function $f\in\Sigma$ is said to be in the family $\mathcal{M}^*_\Sigma(\mu,\gamma,\lambda;\beta)$ if it fulfills the subordinations:

$$\operatorname{Re}\left((1-\mu)\left(1+\frac{z^{2-\gamma}f''(z)}{\left(zf'(z)\right)^{1-\gamma}}\right)+\mu\frac{\left(\left(zf'(z)\right)'\right)^\lambda}{f'(z)}\right)>\beta$$

and

$$\operatorname{Re}\left((1-\mu)\left(1+\frac{w^{2-\gamma}g''(w)}{\left(wg'(w)\right)^{1-\gamma}}\right)+\mu\frac{\left(\left(wg'(w)\right)'\right)^\lambda}{g'(w)}\right)>\beta,$$

where $g(w) = f^{-1}(w)$.

In the next theorems, we provide the Fekete-Szegö type inequalities for the functions of the families $\mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ and $\mathcal{F}_\Sigma(\mu, \gamma, \lambda; \vartheta)$.

**Theorem 2.3.** *For $\eta \in \mathbb{R}$, let $f \in \mathcal{R}_\Sigma(\mu, \gamma, \lambda; \vartheta)$ be of the form* (1.1). *Then,*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}; \\ |\eta - 1| \leq \frac{\left|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}{2\mathfrak{B}_1^2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}, \\ \frac{2\mathfrak{B}_1^3|\eta-1|}{\left|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}; \\ |\eta - 1| \geq \frac{\left|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}{2\mathfrak{B}_1^2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}. \end{cases}$$

*Proof.* It follows from (2.15) and (2.16) that

$$\begin{aligned} & a_3 - \eta a_2^2 \\ =& \frac{\mathfrak{B}_1(x_2 - y_2)}{4\left[(1-\mu)(\gamma+2) + \mu(3\lambda-1)\right]} + (1-\eta)\, a_2^2 \\ =& \frac{\mathfrak{B}_1(x_2 - y_2)}{4\left[(1-\mu)(\gamma+2) + \mu(3\lambda-1)\right]} \\ +& \frac{\mathfrak{B}_1^3(x_2 + y_2)\,(1-\eta)}{2\left(\mathfrak{B}_1^2\left[(1-\mu)(\gamma+2)(\gamma+1) + 2\mu\lambda\,(2\lambda-1)\right] + 2\left[(1-\mu)(\gamma+1) + \mu(2\lambda-1)\right]^2(\mathfrak{B}_1 - \mathfrak{B}_2)\right)} \\ =& \frac{\mathfrak{B}_1}{2}\left[\left(\Upsilon(\eta) + \frac{1}{2\left[(1-\mu)(\gamma+2) + \mu(3\lambda-1)\right]}\right) x_2 \right. \\ & \left. + \left(\Upsilon(\eta) - \frac{1}{2\left[(1-\mu)(\gamma+2) + \mu(3\lambda-1)\right]}\right) y_2\right], \end{aligned}$$

where

$$\Upsilon(\eta) = \frac{\mathfrak{B}_1^2\,(1-\eta)}{\mathfrak{B}_1^2\left[(1-\mu)(\gamma+2)(\gamma+1) + 2\mu\lambda\,(2\lambda-1)\right] + 2\left[(1-\mu)(\gamma+1) + \mu(2\lambda-1)\right]^2(\mathfrak{B}_1 - \mathfrak{B}_2)}.$$

According to Lemma 1.1 and (2.1), we find that

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}, & 0 \leq |\Upsilon(\eta)| \leq \frac{1}{2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}, \\ 2\mathfrak{B}_1\,|\Upsilon(\eta)|\,, & |\Upsilon(\eta)| \geq \frac{1}{2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}. \end{cases}$$

After some computations, we obtain

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}; \\ |\eta-1| \le \frac{\left|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}{2\mathfrak{B}_1^2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}, \\ \frac{2\mathfrak{B}_1^3|\eta-1|}{\left|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}; \\ |\eta-1| \ge \frac{\left|\mathfrak{B}_1^2[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)]+2[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}{2\mathfrak{B}_1^2[(1-\mu)(\gamma+2)+\mu(3\lambda-1)]}. \end{cases}$$

□

Putting $\eta = 1$ in Theorem 2.3, we obtain the following result.

**Corollary 2.5.** *If $f \in \mathcal{R}_\Sigma(\mu,\gamma,\lambda;\vartheta)$ is of the form (1.1), then*

$$\left|a_3-a_2^2\right| \le \frac{\mathfrak{B}_1}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}.$$

**Theorem 2.4.** *For $\eta \in \mathbb{R}$, let $f \in \mathcal{F}_\Sigma(\mu,\gamma,\lambda;\vartheta)$ be of the form (1.1). Then,*

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \frac{\mathfrak{B}_1}{3(3\mu(\lambda-1)+2)}; \\ |\eta-1| \le \frac{\left|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}{3\mathfrak{B}_1^2(3\mu(\lambda-1)+2)}, \\ \frac{\mathfrak{B}_1^3|\eta-1|}{\left|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}; \\ |\eta-1| \ge \frac{\left|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}{3\mathfrak{B}_1^2(3\mu(\lambda-1)+2)}. \end{cases}$$

*Proof.* It follows from (2.28) and (2.29) that

$$\begin{aligned} & a_3-\eta a_2^2 \\ =& \frac{\mathfrak{B}_1(x_2-y_2)}{12\,(3\mu(\lambda-1)+2)} + (1-\eta)\,a_2^2 \\ =& \frac{\mathfrak{B}_1(x_2-y_2)}{12\,(3\mu(\lambda-1)+2)} \\ &+ \frac{\mathfrak{B}_1^3(x_2+y_2)\,(1-\eta)}{4\left[\mathfrak{B}_1^2\,[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4\,(2\mu(\lambda-1)+1)^2\,(\mathfrak{B}_1-\mathfrak{B}_2)\right]} \\ =& \frac{\mathfrak{B}_1}{4}\left[\left(\Omega(\eta)+\frac{1}{3\,(3\mu(\lambda-1)+2)}\right)x_2+\left(\Omega(\eta)-\frac{1}{3\,(3\mu(\lambda-1)+2)}\right)y_2\right], \end{aligned}$$

where

$$\Omega(\eta) = \frac{\mathfrak{B}_1^2\,(1-\eta)}{\mathfrak{B}_1^2\,[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4\,(2\mu(\lambda-1)+1)^2\,(\mathfrak{B}_1-\mathfrak{B}_2)}.$$

According to Lemma 1.1 and (2.1), we find that

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{\mathfrak{B}_1}{3(3\mu(\lambda-1)+2)}, & 0 \leq |\Omega(\eta)| \leq \frac{1}{3(3\mu(\lambda-1)+2)}, \\ \mathfrak{B}_1 \left|\Omega(\eta)\right|, & |\Omega(\eta)| \geq \frac{1}{3(3\mu(\lambda-1)+2)}. \end{cases}$$

After some computations, we obtain

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{\mathfrak{B}_1}{3(3\mu(\lambda-1)+2)}; \\ |\eta-1| \leq \frac{\left|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}{3\mathfrak{B}_1^2(3\mu(\lambda-1)+2)}, \\ \frac{\mathfrak{B}_1^3|\eta-1|}{\left|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}; \\ |\eta-1| \geq \frac{\left|\mathfrak{B}_1^2[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]+4(2\mu(\lambda-1)+1)^2(\mathfrak{B}_1-\mathfrak{B}_2)\right|}{3\mathfrak{B}_1^2(3\mu(\lambda-1)+2)}. \end{cases}$$ □

Putting $\eta = 1$ in Theorem 2.4, we obtain the following result.

**Corollary 2.6.** *If $f \in \mathcal{R}_\Sigma(\mu,\gamma,\lambda;\vartheta)$ is of the form* (1.1)*, then*

$$\left|a_3-a_2^2\right| \leq \frac{\mathfrak{B}_1}{3\left(3\mu(\lambda-1)+2\right)}.$$

## 3. Conclusion

This work has introduced a new families of bi-univalent functions associated with the Bazilevič functions and the λ-pseudo functions. For these families, coefficient bounds and Fekete-Szegö inequalities have been investigated.

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