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SOME RESULTS ON NORMAL ALMOST CONTACT MANIFOLDS WITH B-METRIC

NÜLİFER ÖZDEMIR[1], ŞIRIN AKTAY[2], AND MEHMET SOLGUN[3]

Abstract. In this study, normal almost contact manifolds with B-metric are considered. Almost complex manifolds with Norden metric are obtained by multiplying almost contact manifolds with B-metric by warped product (by using a function on real numbers). New examples of normal almost complex manifolds with Norden metric are derived. Furthermore, curvature properties of the almost complex manifolds with Norden metric obtained from almost contact manifolds with B-metric are investigated.

1. Introduction

In this work, relations between almost contact manifolds with B-metric and almost complex manifolds with Norden metric are investigated. An almost contact manifold with B-metric is obtained from an almost complex manifold with Norden metric using a method similar to that in [4]. The classifications of almost contact manifolds with B-metric and almost complex manifolds with Norden metric are made by using the covariant derivative of their fundamental tensors. Classification of almost contact manifolds with B-metric and almost complex manifolds with Norden metric can be found in [2, 3], respectively. Relations between the almost contact manifolds with B-metric and almost complex manifolds with Norden metric are investigated in [8, 9] with respect to these classifications.

In this manuscript, we study the warped product of almost contact manifolds with B-metric and $\mathbb{R}$. After presenting necessary preliminary informations, we obtain an

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almost complex structure on the product of an almost contact manifold with B-metric with $\mathbb{R}$. Then, we define a metric which is indefinite of signature $(n+1, n+1)$ on the product manifold. The product manifold is an almost complex manifold with Norden metric. We write the covariant derivative of the metric and the almost complex structure of the almost complex manifold with Norden metric in terms of the covariant derivative of the metric of the almost contact manifold with B-metric. We, then, state the relations between normal classes of almost contact manifolds with B-metric and almost complex manifolds with Norden manifolds.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to have an almost contact structure (φ, ξ, η), if this manifold admits an endomorphism φ of the tangent bundle, a vector field ξ and its dual 1-form η such that the conditions

$$\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

are satisfied for an arbitrary vector field X. If a manifold admits an almost contact structure, it is called an almost contact manifold. If (M, φ, ξ, η) is an almost contact manifold endowed with a pseudo-Riemannian metric g of signature $(n+1, n)$ such that

$$g(\varphi(X), \varphi(Y)) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y, then M is called an almost contact manifold with B-metric. From equations (2.1) and (2.2), one can easily see that

$$\eta(\varphi(X)) = 0, \quad \varphi(\xi) = 0, \quad \eta(X) = g(\xi, X), \quad g(\varphi(X), Y) = g(X, \varphi(Y)),$$

for all vector fields X, Y. The tensor $\tilde{g}$ given by

$$\tilde{g}(X, Y) = g(X, \varphi(Y)) + \eta(X)\eta(Y),$$

is a B-metric associated with the metric g. Let ∇ be the Levi-Civita connection of the pseudo-metric g. For all vector fields X, Y, Z on M, we define the structure tensor α of type $(0, 3)$ as

$$\alpha(X, Y, Z) = g\left(\left(\nabla_X \varphi\right)(Y), Z\right).$$

It is not difficult to see that the tensor α has following properties:

$$\alpha(X, Y, Z) = \alpha(X, Z, Y),$$

$$\alpha(X, \varphi(Y), \varphi(Z)) = \alpha(X, Y, Z) - \eta(Y)\alpha(X, \xi, Z) - \eta(Z)\alpha(X, Y, \xi), \tag{2.3}$$

$$\alpha(X, \xi, \xi) = 0. \tag{2.4}$$

The following 1-forms are defined as

$$\theta(X) = g^{ij}\alpha(E_i, E_j, X), \quad \theta^*(X) = g^{ij}\alpha(E_i, \varphi(E_j), X), \quad w(X) = \alpha(\xi, \xi, X),$$

where $\{E_1, \ldots, E_{2n}, \xi\}$ is a local frame, X is a vector field and (g^{ij}) is the inverse matrix of (g_{ij}).

Using the properties above, the space of covariant derivatives of the endomorphism φ are defined as

$$\begin{aligned}\mathcal{F} = \Big\{&\alpha \in \otimes_3^0 M \,:\, \alpha(X,Y,Z) = \alpha(X,Z,Y),\\ &\alpha(X,\varphi(Y),\varphi(Z)) = \alpha(X,Y,Z) - \eta(Y)\alpha(X,\xi,Z) - \eta(Z)\alpha(X,Y,\xi)\}\end{aligned} \tag{2.5}$$

The space $\mathcal{F}$ decomposes into eleven subspaces

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_{11},$$

which are orthogonal and invariant under the action of $G \times I$ where I is the identity on Span$\{\xi\}$ and $G = GL(n,\mathbb{C}) \cap O(n,n)$ [2]. An almost contact manifold M is called normal if the corresponding almost complex structure J on the even dimensional product manifold $M \times \mathbb{R}$ is integrable, i.e., the Nijenhuis torsion $[J,J]$ is identically zero [1, 7], or equivalently $N = [\varphi,\varphi] + d\eta \otimes \xi = 0$, or equivalently

$$\alpha(X,Y,\xi) = \alpha(Y,X,\xi), \tag{2.6}$$

$$\mathfrak{S}_{X,Y,Z}\{\alpha(X,Y,\varphi(Z)) - \alpha(X,\varphi(Y),\xi)\eta(Z)\} = 0, \tag{2.7}$$

see [10]. In this study, we consider only the classes of normal almost contact manifolds with B-metric. The class of the normal contact manifolds with B-metric is $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$ [6], and defining relations of this subspaces are:

$$\begin{aligned}\mathcal{F}_1 :\, &\alpha(X,Y,Z) = \frac{1}{2n}\Big\{g(X,\varphi(Y))\theta(\varphi(Z)) + g(X,\varphi(Z))\theta(\varphi(Y))\\ &+ g(\varphi(X),\varphi(Y))\theta(\varphi^2(Z)) + g(\varphi(X),\varphi(Z))\theta(\varphi^2(Y))\Big\},\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathcal{F}_2 :\, &\alpha(\xi,Y,Z) = \alpha(X,\xi,Z) = 0, \quad \theta = 0,\\ &\alpha(X,Y,\varphi(Z)) + \alpha(Y,Z,\varphi(X)) + \alpha(Z,X,\varphi(Y)) = 0,\end{aligned} \tag{2.9}$$

$$\mathcal{F}_4 :\alpha(X,Y,Z) = -\frac{\theta(\xi)}{2n}\left\{g(\varphi(X),\varphi(Y))\eta(Z) + g(\varphi(X),\varphi(Z))\eta(Y)\right\}, \tag{2.10}$$

$$\mathcal{F}_5 :\alpha(X,Y,Z) = -\frac{\theta^*(\xi)}{2n}\left\{g(X,\varphi(Y))\eta(Z) + g(X,\varphi(Z))\eta(Y)\right\}, \tag{2.11}$$

$$\begin{aligned}\mathcal{F}_6 :\, &\alpha(X,Y,Z) = \alpha(X,Y,\xi)\eta(Z) + \alpha(X,Z,\xi)\eta(Y),\\ &\alpha(X,Y,\xi) = \alpha(Y,X,\xi) = -\alpha(\varphi(X),\varphi(Y),\xi), \quad \theta(\xi) = \theta^*(\xi) = 0.\end{aligned} \tag{2.12}$$

If a smooth manifold N has a tensor field J (almost complex structure) and a pseudo-Riemannian metric h satisfying the conditions

- $J^2(X) = -X$,
- $h(J(X),J(Y)) = -h(X,Y)$,

for all vector fields X,Y on N, then the manifold N is called an almost complex manifold with a Norden metric [3]. The metric h is necessarily indefinite of signature (n,n). An almost complex manifold with a Norden metric has even dimension (dim$N = 2n$) and the structure group of the tangent bundle reduces to the group

$GL(n,\mathbb{C})\cap O(n,n)$. The tensor $\tilde{h}$ given by $\tilde{h}(X,Y)=h(J(X),Y)$ for any vector field X,Y is symmetric:

$$h(J(X),Y)=-h(J^2(X),J(Y))=h(X,J(Y)).$$

The structure tensor F of type $(0,3)$ on M is defined as

$$F(X,Y,Z)=h\left(\left(\nabla_X J\right)(Y),Z\right).$$

The tensor F has the following properties:

$$F(X,Y,Z)=F(X,Z,Y),$$
$$F\left(X,J(Y),J(Z)\right)=F(X,Y,Z).$$

In addition, for any vector field X on N the 1-form $\tilde{\theta}$ associated with F is defined as

$$\tilde{\theta}(X)=h^{ij}F(E_i,E_j,X),$$

where $\{E_1,E_2,\ldots,E_{2n}\}$ is a frame field on N and h^{ij} is the inverse matrix of h.

Then the subspace W of $\otimes_3^0 N$ is defined as follows:

$$W:=\left\{\alpha\in\otimes_3^0 N\mid\alpha(X,Y,Z)=\alpha(X,J(Y),J(Z))=\alpha(X,Z,Y)\right\},$$

where X,Y,Z are vector fields on N. According to the symmetries of W, this space splits into the direct sum $W=W_1\oplus W_2\oplus W_3$. The subspaces W_i are invariant and irreducible under the group $GL(n,\mathbb{C})\cap O(n,n)$. The defining relations for invariant subspaces are the following.

(a) Kaehler manifolds with a Norden metric:

$$F(X,Y,Z)=0. \tag{2.13}$$

(b) Class W_1: Conformally Kaehlerian manifolds with a Norden metric:

$$\begin{aligned} F(X,Y,Z)=&\frac{1}{2n}\left\{h(X,Y)\tilde{\theta}(Z)+h(X,Z)\tilde{\theta}(Y)\right.\\ &\left.+h(X,J(Y))\tilde{\theta}(J(Z))+h(X,J(Z))\tilde{\theta}(J(Y))\right\}. \end{aligned} \tag{2.14}$$

(c) Class W_2: Special complex manifolds with a Norden metric

$$F(X,Y,J(Z))+F(Y,Z,J(X))+F(Z,X,J(Y))=0, \tag{2.15}$$

and $\tilde{\theta}=0$.

(d) Class W_3: Quasi-Kaehlerian manifolds with Norden metric

$$F(X,Y,Z)+F(Y,Z,X)+F(Z,X,Y)=0. \tag{2.16}$$

(e) Class $W_1\oplus W_2$: Complex manifolds with Norden metric

$$F(X,Y,J(Z))+F(Y,Z,J(X))+F(Z,X,J(Y))=0,$$

or equivalently $N=0$.

(f) Class $W_2\oplus W_3$: Semi-Kaehlerian manifolds with Norden metric

$$\tilde{\theta}=0. \tag{2.17}$$

(g) Class $W_1 \oplus W_3$:

$$
\begin{aligned}
F(X,Y,Z)+F(Y,Z,X)+F(Z,X,Y) = &\frac{1}{n}\left\{h(X,Y)\tilde{\theta}(Z)+h(X,Z)\tilde{\theta}(Y)\right.\\
&+h(Y,Z)\tilde{\theta}(X)+h(X,J(Y))\tilde{\theta}(J(Z))\\
&+h(X,J(Z))\tilde{\theta}(J(Y))\\
&\left.+h(Y,J(Z))\tilde{\theta}(J(X))\right\}.
\end{aligned}
\tag{2.18}
$$

(h) Class $W_1 \oplus W_2 \oplus W_3$: No relation.

3. Almost Complex Manifolds with a Norden Metric from Almost Contact Manifolds with B-metric

In this section, first, we define an almost complex structure on the product of an almost contact manifold with B-metric with $\mathbb{R}$. We write a metric on the product manifold depending on a function σ where $\sigma : M \times \mathbb{R} \to \mathbb{R}$ only depends on t. Then, we obtain an almost complex manifold with Norden metric and we give the relations between covariant derivatives.

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact manifold with B-metric and consider the product manifold $M \times \mathbb{R}$. A vector field on the manifold $M \times \mathbb{R}$ is of the form $\left(X, a\frac{d}{dt}\right)$ where t is the coordinate of $\mathbb{R}$ and a is a smooth function on $M \times \mathbb{R}$. The almost complex structure J on $M \times \mathbb{R}$ is defined by

$$
J\left(X, a\frac{d}{dt}\right) = \left(\varphi(X) - ae^{-\sigma}\xi, e^{\sigma}\eta(X)\frac{d}{dt}\right). \tag{3.1}
$$

Then $J^2 = -I$. In addition, we define a pseudo-Riemannian metric on $M \times \mathbb{R}$ with signature $(n+1, n+1)$ by

$$
h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) := e^{2\sigma}g(X,Y) - ab. \tag{3.2}
$$

One can easily see that

$$
h\left(J\left(X, a\frac{d}{dt}\right), J\left(Y, b\frac{d}{dt}\right)\right) = -h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right). \tag{3.3}
$$

Hence $(M \times \mathbb{R}, J, h)$ is an almost complex manifold with Norden metric. Let ∇ be the Levi-Civita covariant derivative of the pseudo-Riemannian metric g on M. Levi-Civita covariant derivative of the metric h on $M \times \mathbb{R}$ is obtained using the Kozsul formula as

$$
\nabla_{\left(X, a\frac{d}{dt}\right)}\left(Y, b\frac{d}{dt}\right) = \left(\nabla_X Y + \frac{d\sigma}{dt}(aY + bX),\ \left\{X[b] + a\frac{db}{dt} + e^{2\sigma}\frac{d\sigma}{dt}g(X,Y)\right\}\frac{d}{dt}\right).
$$

Note that the covariant derivative on the product manifold $M \times \mathbb{R}$ will also be denoted with the same symbol ∇. Also, covariant derivative of the almost complex structure

J is calculated as

$$\begin{aligned}\left(\nabla_{(X,a\frac{d}{dt})}J\right)\left(Y,b\frac{d}{dt}\right)=&\Big(\left(\nabla_X\varphi\right)(Y)-be^{-\sigma}\nabla_X\xi-b\frac{d\sigma}{dt}\varphi(X)\\&+e^{\sigma}\frac{d\sigma}{dt}\left(\eta(Y)X+g(X,Y)\xi\right),\\&\left\{-2be^{\sigma}\frac{d\sigma}{dt}\eta(X)+e^{\sigma}\left(\nabla_X\eta\right)(Y)+e^{2\sigma}\frac{d\sigma}{dt}g(X,\varphi(Y))\right\}\frac{d}{dt}\Big),\end{aligned}$$

for any vector field $\left(X,a\frac{d}{dt}\right)$, $\left(Y,b\frac{d}{dt}\right)$ and $\left(Z,c\frac{d}{dt}\right)$ on $M\times\mathbb{R}$. It follows that

$$\begin{aligned}F\left(\tilde{X},\tilde{Y},\tilde{Z}\right)=&h\left(\left(\nabla_{\tilde{X}}J\right)(\tilde{Y}),\tilde{Z}\right)\\=&e^{2\sigma}\alpha(X,Y,Z)+2bce^{\sigma}\frac{d\sigma}{dt}\eta(X)\\&-e^{\sigma}\left\{bg\left(\nabla_X\xi,Z\right)+cg\left(\nabla_X\xi,Y\right)\right\}\\&+e^{3\sigma}\frac{d\sigma}{dt}\left\{\eta(Y)g(X,Z)+\eta(Z)g(X,Y)\right\}\\&-e^{2\sigma}\frac{d\sigma}{dt}\left\{bg(X,\varphi(Z))+cg(X,\varphi(Y)\right\},\end{aligned}\tag{3.4}$$

is obtained. If we take $\tilde{X}=(\xi,0)$, $\tilde{Y}=\tilde{Z}=\left(0,\frac{d}{dt}\right)$, then $F\left(\tilde{X},\tilde{Y},\tilde{Z}\right)=2e^{\sigma}\frac{d\sigma}{dt}$ is different than zero for non-constant σ. Thus, F is not equal to zero for any function σ. Since

$$\nabla_{(X,a\frac{d}{dt})}(\xi,0)=\left(\nabla_X\xi+\frac{d\sigma}{dt}a\xi,e^{2\sigma}\frac{d\sigma}{dt}\eta(X)\frac{d}{dt}\right)\neq 0,$$

$(\xi,0)$ is not parallel even if ξ is parallel. In addition, if ξ is Killing, $(\xi,0)$ is also Killing:

$$\begin{aligned}h\left(\nabla_{(X,a\frac{d}{dt})}(\xi,0),\left(Y,b\frac{d}{dt}\right)\right)&=e^{2\sigma}g(\nabla_X\xi,Y)+ae^{2\sigma}\frac{d\sigma}{dt}\eta(Y)-be^{2\sigma}\frac{d\sigma}{dt}\eta(Y)\\&=-h\left(\nabla_{(Y,b\frac{d}{dt})}(\xi,0),\left(X,a\frac{d}{dt}\right)\right).\end{aligned}\tag{3.5}$$

Note that

$$h\left(\nabla_{(X,a\frac{d}{dt})}\left(0,\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right)=e^{\sigma}\frac{d\sigma}{dt}g(X,Y)=g\left(\nabla_{(Y,b\frac{d}{dt})}\left(0,\frac{d}{dt}\right),\left(X,a\frac{d}{dt}\right)\right).$$

Let $\{e_1,\ldots,e_{2n},\xi\}$ be a local pseudo-orthonormal frame field on M. Then one can obtain an orthonormal frame field on $M\times\mathbb{R}$ as follows:

$$\left\{\left(e^{-\sigma}e_1,0\right),\ldots,\left(e^{-\sigma}e_{2n},0\right),\left(e^{-\sigma}\xi,0\right),\left(0,\frac{d}{dt}\right)\right\}.$$

Using this frame, the 1-form $\tilde{\theta}$, associated with F given in [3], is evaluated as

$$\tilde{\theta}\left(X,a\frac{d}{dt}\right)=\theta(X)-ae^{-\sigma}\theta^*(\xi)+w(X)+2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X).\tag{3.6}$$

In addition, we write the curvature tensor $\widetilde{R}$ on the product manifold $M \times \mathbb{R}$ with respect to the curvature tensor R on M. Let $\widetilde{X} = \left(X, a\frac{d}{dt}\right)$, $\widetilde{Y} = \left(Y, b\frac{d}{dt}\right)$, $\widetilde{Z} = \left(Z, c\frac{d}{dt}\right)$ be vector fields on the product manifold $M \times \mathbb{R}$, then we have

$$
\begin{aligned}
\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \Bigg(& R(X,Y)Z + c\left(\left(\frac{d\sigma}{dt}\right)^2 + \frac{d^2\sigma}{dt^2}\right)(aY - bX) \\
& + e^{2\sigma}\left(\frac{d\sigma}{dt}\right)^2 (g(Y,Z)X - g(X,Z)Y), \\
& e^{2\sigma}\left(\left(\frac{d\sigma}{dt}\right)^2 + \frac{d^2\sigma}{dt^2}\right) g(aY - bX, Z)\frac{d}{dt}\Bigg).
\end{aligned}
\tag{3.7}
$$

As a result, Ricci curvature can be calculated as

$$
\begin{aligned}
\tilde{Q}(\tilde{X},\tilde{Y}) = & Q(X,Y) - ab(2n+1)\left(\left(\frac{d\sigma}{dt}\right)^2 + \frac{d^2\sigma}{dt^2}\right) \\
& + e^{2\sigma}\left(\frac{d\sigma}{dt}\right)^2 (2n+1)g(X,Y) + e^{2\sigma}\frac{d^2\sigma}{dt^2}g(X,Y).
\end{aligned}
\tag{3.8}
$$

In addition, we can evaluate the scalar curvature as

$$
\tilde{s} = e^{-2\sigma}s + (2n+1)(2n+2)\left(\frac{d\sigma}{dt}\right)^2 + 2(2n+1)\frac{d^2\sigma}{dt^2}. \tag{3.9}
$$

Let M be an almost contact manifold with B-metric with zero scalar curvature. Then we can construct an almost complex manifold with Norden metric with scalar curvature $k > 0$ with the appropriate choice of the function σ, see Example (3.1). If we take $s = 0$, then the solution of the differential equation

$$
k = (2n+1)(2n+2)\left(\frac{d\sigma}{dt}\right)^2 + 2(2n+1)\frac{d^2\sigma}{dt^2}, \tag{3.10}
$$

takes the form

$$
\sigma(t) = \frac{1}{n+1}\ln\left[\cosh\left(\sqrt{2k}\left(\frac{\sqrt{n+1}}{2\sqrt{2n+1}}t - \sqrt{(2n+1)(n+1)}c_1\right)\right)\right] + c_2, \tag{3.11}
$$

where $c_1, c_2 \in \mathbb{R}$.

If the almost contact manifold with B-metric M is Einstein, that is $Q(X,Y) = \lambda g(X,Y)$, then the almost complex manifold with Norden metric $M \times \mathbb{R}$ is Einstein if and only if

$$
\frac{\lambda}{2n} = e^{2\sigma}\frac{d^2\sigma}{dt^2}. \tag{3.12}
$$

If $K = (2n+1)\left(\left(\frac{d\sigma}{dt}\right)^2 + \frac{d^2\sigma}{dt^2}\right)$, then we have $\tilde{Q}(\tilde{X},\tilde{Y}) = Kh(\tilde{X},\tilde{Y})$.

Differential equation (3.12) has the solution

$$\sigma(t) = \ln\left(\frac{1}{2}e^{-\sqrt{c_1}(t+c_2)}\lambda + \frac{e^{\sqrt{c_1}(t+c_2)}}{4nc_1}\right),$$

where $c_1, c_2 \in \mathbb{R}$ and $c_1 > 0$. If Einstein constant $\lambda > 0$, domain of the function σ is the set of all real numbers. Hence the product manifold $M \times \mathbb{R}$ is Einstein with Einstein constant $K = (2n+1)c_1 > 0$, since $c_1 = \left(\left(\frac{d\sigma}{dt}\right)^2 + \frac{d^2\sigma}{dt^2}\right)$.

If $\lambda < 0$, it can be easily seen that domain of the function σ is $(t_0, +\infty)$, where $t_0 = \frac{1}{2\sqrt{c_1}}\ln(-2nc_1\lambda) - c_2$. Then the product manifold $M \times (t_0, +\infty)$ is Einstein with Einstein constant $K = (2n+1)c_1 > 0$. If $\lambda = 0$, then solution of the equation (3.12) is $\sigma(t) = c_1 t + c_2$, where $c_1, c_2 \in \mathbb{R}$. In this case, $K = (2n+1)c_1^2$ is obtained. Hence, for all cases, we obtain Einstein product manifold with positive Einstein constant.

Now the relations between classes of the product manifold $M \times \mathbb{R}$ and the classes of the almost contact manifold with B-metric M are investigated.

Theorem 3.1. *If $(M, \varphi, \xi, \eta, g)$ is a cosymplectic almost contact B-metric manifold, then the product manifold $M \times \mathbb{R}$ is of the class W_1 for all non-constant σ functions.*

Proof. Since $(M, \varphi, \xi, \eta, g)$ is a cosymplectic almost contact B-metric manifold we have

$$\tilde{\theta}(\tilde{X}) = 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X), \quad \tilde{\theta}(J(\tilde{X})) = -2a(n+1)\frac{d\sigma}{dt}.$$

Implying that $\tilde{\theta} \neq 0$ and the product manifold is not of the class W_2. Since $\alpha(X, Y, Z) = 0$ for all vector fields X, Y, Z from (3.4), we have

$$\begin{aligned} F\left(\tilde{X}, \tilde{Y}, \tilde{Z}\right) =& e^{3\sigma}\frac{d\sigma}{dt}\left\{\eta(Y)g(X,Z) + \eta(Z)g(X,Y)\right\} \\ &- e^{2\sigma}\frac{d\sigma}{dt}\left\{bg(X,\varphi(Z)) + cg(X,\varphi(Y))\right\} + 2bce^{\sigma}\frac{d\sigma}{dt}\eta(X), \end{aligned}$$

which is equivalent to the right hand side of the defining relation (2.14) of a conformally Kaehlerian manifold by direct calculation. Thus for a non-constant function σ, the product manifold is a conformally Kaehlerian manifold with Norden metric (class W_1). $\square$

Theorem 3.2. *If $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_1$, then the product manifold $M \times \mathbb{R}$ is of the class $W_1 \oplus W_2$ for all non-constant σ functions.*

Proof. Since $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_1$, we have

$$\theta(\xi) = 0, \quad \theta^*(\xi) = 0,$$

for all vector fields X on M and ξ is parallel [6]. Then, we have

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = \theta(X) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X).$$

Hence we say that $\tilde{\theta} \neq 0$ and the product manifold $M \times \mathbb{R}$ is not of the class W_2.

When M is of the class $\mathcal{F}_1$, if we take $\tilde{X} = \left(0, \frac{d}{dt}\right)$, $\tilde{Y} = (Y, 0)$ and $\tilde{Z} = (\xi, 0)$ in equation (2.14), the left hand side of (2.14) becomes $F(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$, whereas the right hand side of (2.14) is

$$-\frac{1}{2(n+1)} e^{\sigma} \theta(\varphi(Y)).$$

Thus, the equation (2.14) is not satisfied and the product manifold $M \times \mathbb{R}$ is not in the class W_1. Since

$$\begin{aligned} F\left(\tilde{X}, \tilde{Y}, J(\tilde{Z})\right) =& e^{2\sigma}\alpha(X, Y, \varphi(Z)) - e^{2\sigma}\eta(Z)g\left(\nabla_X \xi, Y\right) \\ & - e^{\sigma}\left\{c\alpha(X, Y, \xi) - b\alpha(X, Z, \xi)\right\} \\ & + e^{3\sigma}\frac{d\sigma}{dt}\left\{\eta(Y)g(X, \varphi(Z)) - \eta(Z)g(X, \varphi(Y))\right\} \\ & - e^{2\sigma}\frac{d\sigma}{dt}\left\{bg(\varphi(X), \varphi(Z)) - cg(\varphi(X), \varphi(Y))\right\} \\ & + 2e^{2\sigma}\frac{d\sigma}{dt}\eta(X)\left\{b\eta(Z) - c\eta(Y)\right\}, \end{aligned} \tag{3.13}$$

it can be checked that the normality condition (2.15) is satisfied when the manifold M belongs to class $\mathcal{F}_1$. As a result the product manifold $M \times \mathbb{R}$ is of the class $W_1 \oplus W_2$. □

Theorem 3.3. *If $(M, \varphi, \xi, \eta, g)$ is in $\mathcal{F}_2$, then the product manifold $M \times \mathbb{R}$ is in $W_1 \oplus W_2$ for all non-constant σ functions.*

Proof. Since $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_2$, the equation (2.9) yields

$$\theta(X) = 0, \quad \theta^*(X) = 0,$$

for all vector fields X on M and ξ is parallel [6]. Then,

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X).$$

Hence, $\tilde{\theta} \neq 0$, as a result $M \times \mathbb{R}$ is not in W_2.

When M is of the class $\mathcal{F}_2$, if we take $\tilde{X} = \tilde{Y} = \left(0, \frac{d}{dt}\right)$ and $\tilde{Z} = (\xi, 0)$ in equation (2.14), the left hand side of equation (2.14) is $F(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$, and the right hand side of the equation (2.14) is $-e^{\sigma}\frac{d\sigma}{dt}$. Thus, the equation (2.14) is satisfied if and only if σ is constant. So, $M \times \mathbb{R}$ does not belong to W_1 for non-constant σ. In addition, one can easily check that normality condition (2.15) is satisfied when the manifold M is of the class $\mathcal{F}_2$. To sum up $M \times \mathbb{R}$ is in $W_1 \oplus W_2$ for non-constant σ. □

Theorem 3.4. *If $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_4$, then the product manifold $M \times \mathbb{R}$ is of the class $W_1 \oplus W_2$ for all non-constant σ functions.*

Proof. Since $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_4$, the equation (2.10) gives

$$\theta(X) = \eta(X)\theta(\xi), \quad \theta^*(\xi) = 0, \quad \omega(X) = 0, \quad \nabla_X \xi = \frac{\theta(\xi)}{2n}. \tag{3.14}$$

From equations (3.4) and (3.14), we get

$$\begin{aligned} F(\tilde{X}, \tilde{Y}, \tilde{Z}) = &- e^{2\sigma}\left(\frac{\theta(\xi)}{2n} + e^{\sigma}\frac{d\sigma}{dt}\right)\{\eta(Y)g(\varphi(X), \varphi(Z)) + \eta(Z)g(\varphi(X), \varphi(Y))\} \\ &- e^{\sigma}\left(\frac{\theta(\xi)}{2n} + e^{\sigma}\frac{d\sigma}{dt}\right)\{bg(X, \varphi(Z)) + cg(X, \varphi(Y))\} \\ &+ 2e^{3\sigma}\frac{d\sigma}{dt}\eta(X)\eta(Y)\eta(Z) + 2bce^{\sigma}\frac{d\sigma}{dt}\eta(X) \end{aligned}$$

and

$$\begin{aligned} F(\tilde{X}, \tilde{Y}, J(\tilde{Z})) = &e^{2\sigma}\left(\frac{\theta(\xi)}{2n} + e^{\sigma}\frac{d\sigma}{dt}\right)\{\eta(Y)g(X, \varphi(Z)) - \eta(Z)g(X, \varphi(Y))\} \\ &- e^{\sigma}\left(\frac{\theta(\xi)}{2n} + e^{\sigma}\frac{d\sigma}{dt}\right)\{bg(\varphi(X), \varphi(Z)) - cg(\varphi(X), \varphi(Y))\} \\ &- 2e^{2\sigma}\frac{d\sigma}{dt}\eta(X)\left(c\eta(Y) - b\eta(Z)\right)\eta(Y). \end{aligned}$$

One can see that the normality condition (2.15) is satisfied. Hence the product manifold $M \times \mathbb{R}$ is of the class $W_1 \oplus W_2$. In addition, we have

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = \eta(X)\left(\theta(\xi) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\right).$$

If we take $\tilde{X} = (X, 0)$, $\tilde{Y} = \left(Y, \frac{d}{dt}\right)$ and $\tilde{Z} = \left(Z, \frac{d}{dt}\right)$, then the equation (2.14) is not satisfied. Hence $M \times \mathbb{R}$ is not of the class W_1. If $\theta(\xi)$ is constant and the function σ has the property that

$$e^{\sigma}\frac{d\sigma}{dt} = -\frac{\theta(\xi)}{2(n+1)},$$

then the product manifold $M \times \mathbb{R}$ is of the class W_2. □

Theorem 3.5. *If $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_5$, then the product manifold $M \times \mathbb{R}$ is of the class $W_1 \oplus W_2$ for all non-constant σ functions.*

Proof. If $(M, \varphi, \xi, \eta, g)$ belongs to $\mathcal{F}_5$, then from (2.11) we have

$$\theta(X) = 0, \quad \theta^*(X) = \theta^*(\xi)\eta(X), \quad w(X) = 0,$$

for all vector fields X on M [6]. Therefore,

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = -e^{-\sigma}a\theta^*(\xi) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X) \neq 0,$$

since $\tilde{\theta}\left(0, \frac{d}{dt}\right) = -e^{-\sigma}\theta^*(\xi)$. Therefore $M \times \mathbb{R}$ is not in W_2. Replacing $\tilde{X} = \left(0, \frac{d}{dt}\right)$, $\tilde{Y} = (\xi, 0)$ and $\tilde{Z} = (\xi, 0)$ in defining relation (2.14) of the class W_1, we have

$$\theta^*(\xi) = 0.$$

This is a contradiction since $\theta^*(\xi) \neq 0$ is in the class $\mathcal{F}_5$ for non-constant function σ. In addition, if the manifold M is of the class $\mathcal{F}_5$, then one can easily check that equation (2.15) is satisfied on $M \times \mathbb{R}$. Hence, if the manifold M is of the class $\mathcal{F}_5$, the product manifold is of the class $W_1 \oplus W_2$. □

Theorem 3.6. *If $(M, \varphi, \xi, \eta, g)$ is in $\mathcal{F}_6$, then the product manifold $M \times \mathbb{R}$ is in $W_1 \oplus W_2$ for all non-constant σ functions.*

Proof. Since $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_6$, by (2.12) we obtain

$$\theta(X) = 0, \quad \theta^*(X) = 0, \quad w(X) = 0,$$

for all vector fields X on M [6]. Then,

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X) \neq 0,$$

since, for instance, $\tilde{\theta}(\xi, 0) = 2(n+1)e^{\sigma}\frac{d\sigma}{dt}$ is not equal to zero for non-constant function σ. Choosing $\tilde{X} = \left(0, \frac{d}{dt}\right)$, $\tilde{Y} = (Y, 0)$ and $\tilde{Z} = \left(0, \frac{d}{dt}\right)$ in the defining relation (2.14) of the class W_1, we have

$$e^{\sigma}\frac{d\sigma}{dt}\eta(Y) = 0.$$

This is a contradiction for a non-constant function σ. One can also check that if the manifold M is of the class $\mathcal{F}_6$, then the equation (2.15) is satisfied on $M \times \mathbb{R}$. Hence, if the manifold M is of the class $\mathcal{F}_6$, the product manifold is of the class $W_1 \oplus W_2$. □

Now we show that if the product manifold $M \times \mathbb{R}$ is normal, then so is M.

Theorem 3.7. *If the product manifold $M \times \mathbb{R}$ is of the class W_1, then the almost contact manifold with B-metric is cosymplectic.*

Proof. If the product manifold $M \times \mathbb{R}$ is of the class W_1, the defining relation (2.14) is satisfied for all vector fields $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$. Take $\tilde{X} = \left(0, \frac{d}{dt}\right)$ and $\tilde{Y} = \tilde{Z} = (\xi, 0)$, we get $\theta^*(\xi) = 0$. Replace $\tilde{X} = \left(0, \frac{d}{dt}\right)$ and $\tilde{Y} = (\xi, 0)$ and $\tilde{Z} = (Z, 0)$ in (2.14) to get $\theta(\varphi(Z)) = -w(\varphi(Z))$, and hence $\theta(Z) + w(Z) = \eta(Z)\theta(\xi)$. For $\tilde{X} = (X, 0)$, $\tilde{Y} = (Y, 0)$ and $\tilde{Z} = (Z, 0)$, we get

$$\alpha(X, Y, Z) = \frac{\theta(\xi)}{2(n+1)}\left\{\eta(Y)g(X, Z) + \eta(Z)g(X, Y)\right\}. \tag{3.15}$$

Note that $\alpha(X, \varphi(Y), \varphi(Z)) = 0$. From the equation (3.15) we have

$$\theta(X) = \frac{n}{n+1}\theta(\xi)\eta(X),$$

for all vector field on M. Then, we obtain $\theta(\xi) = 0$. Hence, we get $\alpha(X, Y, Z) = 0$. □

Theorem 3.8. *If the product manifold $M \times \mathbb{R}$ is of the class W_2, then the almost contact manifold with B-metric is of the class $\mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_6$.*

Proof. Since the product manifold $M \times \mathbb{R}$ is of the class W_2, $\tilde{\theta}(X, a\frac{d}{dt}) = 0$ for all vector fields $(X, a\frac{d}{dt})$. We, thus, obtain

$$\tilde{\theta}\left(0, \frac{d}{dt}\right) = -e^{-\sigma}\theta^*(\xi) = 0.$$

Hence, $\theta^*(\xi) = 0$ and

$$\tilde{\theta}\,(\xi, 0) = \theta(\xi) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt} = 0.$$

Since equation (2.15) is satisfied in the class W_2, if one takes $\tilde{X} = \left(0, \frac{d}{dt}\right)$, $\tilde{Y} = (Y, 0)$ and $\tilde{Z} = (Z, 0)$, then the following equation is obtained:

$$\alpha(Y, Z, \xi) = \alpha(Z, Y, \xi).$$

Since $0 = \alpha(X, \xi, \xi) = \alpha(\xi, X, \xi) = \alpha(\xi, \xi, X)$, we have $\nabla_\xi \xi = 0$. In addition, in the class W_2 we get

$$0 = \tilde{\theta}\left(X, \frac{d}{dt}\right) = \theta(X) + (2n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X),$$

and $\theta(\varphi(X)) = 0$. Moreover, taking $\tilde{X} = (X, 0)$, $\tilde{Y} = (Y, 0)$, $\tilde{Z} = (Z, 0)$ in equation (2.15), we obtain $\mathfrak{S}_{XYZ}\,\{\alpha(X, Y, \varphi(Z) - \alpha(X, \varphi(Y), \xi)\eta(Z)\} = 0$. Thus, equations (2.6) and (2.7) hold and the manifold M is normal $(\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6)$ [10]. Since $\theta^*(\xi) = 0$ and $\theta(\varphi(X)) = 0$, M is in $\mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_6$. $\square$

Theorem 3.9. *If the product manifold $M \times \mathbb{R}$ is of the class $W_1 \oplus W_2$, then the manifold M is normal almost contact manifold with B-metric (the class $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$).*

Proof. If the product manifold $M \times \mathbb{R}$ is of the class $W_1 \oplus W_2$, the defining relation (2.15) is satisfied for all vector fields $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$. If we take $\tilde{X} = (X, 0)$, $\tilde{Y} = (Y, 0)$ and $\tilde{Z} = (Z, 0)$, we obtain

$$\mathfrak{S}_{XYZ}\,(\alpha(X, Y, \varphi(Z) - \alpha(X, \varphi(Y), \xi)\eta(Z)) = 0.$$

In addition, if we take $\tilde{X} = \left(0, \frac{d}{dt}\right)$, $\tilde{Y} = (Y, 0)$ and $\tilde{Z} = (Z, 0)$, we get

$$\alpha(Y, Z, \xi) = \alpha(Z, Y, \xi).$$

Hence, manifold M is normal almost contact manifold with B-metric from (2.6) and (2.7) [10]. $\square$

Example 3.1. Consider the Lie group G of dimension 5 with a basis of left-invariant vector fields $\{e_1, e_2, e_3, e_4, e_5\}$ defined by the non-zero brackets

$$\begin{aligned}
[e_1, e_5] &= \lambda_1 e_1 + \lambda_2 e_2 + \lambda_1 e_3 + \lambda_4 e_4,\\
[e_2, e_5] &= -\lambda_2 e_1 - \lambda_1 e_2 - \lambda_4 e_3 - \lambda_1 e_4,\\
[e_3, e_5] &= -\lambda_1 e_1 - \lambda_4 e_2 + \lambda_1 e_3 + \lambda_2 e_4,\\
[e_4, e_5] &= \lambda_4 e_1 + \lambda_1 e_2 - \lambda_2 e_3 - \lambda_1 e_4.
\end{aligned}$$

One can define an invariant almost contact structure with B-metric on G as

$$\begin{gathered}
g(e_0, e_0) = g(e_1, e_1) = g(e_2, e_2) = 1,\\
g(e_3, e_3) = g(e_4, e_4) = -1, \quad g(e_i, e_j) = 0, \quad i \neq j,\\
e_5 = \xi, \quad \varphi(e_1) = e_3, \quad \varphi(e_2) = e_4.
\end{gathered}$$

This almost contact structure with B-metric on G has zero scalar curvature [5]. Then we can construct an almost complex structure with Norden metric on $G \times \mathbb{R}$ having any scalar curvature $k > 0$ from the equation (3.9).

For example, from (3.11), the function

$$\sigma(t) = \frac{1}{3} \ln \left(\cosh \left(\frac{\sqrt{6}}{2\sqrt{5}} t \right) \right),$$

satisfies the differential equation (3.10):

$$1 = 30 \left(\frac{d\sigma}{dt} \right)^2 + 10 \frac{d^2\sigma}{dt^2}.$$

Thus the scalar curvature k of the product manifold $G \times \mathbb{R}$ is $k = 1$ from (3.9).

Similarly for

$$\sigma(t) = \frac{1}{3} \ln \left(\cosh \left(\frac{\sqrt{3}}{\sqrt{5}} t \right) \right),$$

the scalar curvature of $G \times \mathbb{R}$ is 2.

Example 3.2. Let $\mathbb{R}^{2n+1} = \{(u^1, \dots, u^n, v^1, \dots, v^n, t) \mid u^i, v^i, t \in \mathbb{R}\}$. Consider the cosymplectic almost contact structure with B-metric given in [2] by

$$\begin{gathered}
\xi = \frac{\partial}{\partial t}, \quad \eta = dt,\\
\varphi \left(\frac{\partial}{\partial u^i} \right) = \frac{\partial}{\partial v^i}, \quad \varphi \left(\frac{\partial}{\partial v^i} \right) = -\frac{\partial}{\partial u^i}, \quad \varphi \left(\frac{\partial}{\partial t} \right) = 0,\\
g(X, X) = -\delta_{ij} \lambda^i \lambda^j + \delta_{ij} \mu^i \mu^j + \nu^2,
\end{gathered}$$

where $X = \lambda^i \frac{\partial}{\partial u^i} + \mu^i \frac{\partial}{\partial v^i} + \nu \frac{\partial}{\partial t}$ and δ_{ij} are the Kronecker's symbols. We obtain infinitely many conformally Kaehlerian structure with Norden metric on $\mathbb{R}^{2n+1} \times \mathbb{R}$

from Theorem 3.1 for any non-constant function σ. For instance, choose $\sigma(t) = t$. Then, for

$$\begin{aligned} X =& x_i \frac{\partial}{\partial u_i} + \tilde{x}_i \frac{\partial}{\partial v_i} + \bar{x} \frac{d}{dt}, \\ Y =& y_i \frac{\partial}{\partial u_i} + \tilde{y}_i \frac{\partial}{\partial v_i} + \bar{y} \frac{d}{dt}, \\ Z =& z_i \frac{\partial}{\partial u_i} + \tilde{z}_i \frac{\partial}{\partial v_i} + \bar{z} \frac{d}{dt} \end{aligned}$$

and $\tilde{X} = (X, a\frac{d}{dt})$, $\tilde{Y} = (Y, b\frac{d}{dt})$, $\tilde{Z} = (Z, c\frac{d}{dt})$, from the proof of Theorem 3.1, we have

$$\begin{aligned} F\left(\tilde{X}, \tilde{Y}, \tilde{Z}\right) =& e^{3t} \left\{\bar{y} g(X, Z) + \bar{z} g(X, Y)\right\} \\ &- e^{2t} \frac{d\sigma}{dt} \left\{b g(X, \varphi(Z)) + c g(X, \varphi(Y))\right\} + 2bce^t \bar{x}, \end{aligned}$$

and $\tilde{\theta}(\tilde{X}) = 2(n+1)e^t \bar{x}$, $\tilde{\theta}(J(\tilde{X})) = -2a(n+1)$. Theorem 3.1 implies that F satisfies the defining relation (2.14) of W_1.

Example 3.3. Let $\mathbb{R}^{2n+2} = \{(u^1, \ldots, u^{n+1}; v^1, \ldots, v^{n+1}) \mid u^i, v^i \in \mathbb{R}\}$ and consider $\mathbb{R}^{2n+2}$ as a complex Riemannian manifold with the canonical complex structure J and the metric g defined by

$$g(x, x) = -\delta_{ij} \lambda^i \lambda^j + \delta_{ij} \mu^i \mu^j,$$

where $x = \lambda^i \frac{\partial}{\partial u^i} + \mu^i \frac{\partial}{\partial v^i}$. Let Z denote the position vector of the point p.

We consider the unit time-like sphere $S^{2n+1} : g(Z, Z) = -1$ of the metric g given in [2]. The characteristic vector field ξ on S^{2n+1} is given by

$$\xi = \lambda Z + \mu J Z, \quad g(Z, \xi) = 0, \quad g(\xi, \xi) = 1.$$

For each p in S^{2n+1}, setting $g(J\xi, Z) = \tan t$, $t \in (-\pi/2, \pi/2)$, it is obtained that

$$\xi = (\sin t) Z + (\cos t) J Z, \quad J\xi = -(\cos t) Z + (\sin t) J Z.$$

For any $x \in T_p S^{2n+1}$, let φx be the projection of the vector Jx into $T_p S^{2n+1}$ with respect to $J\xi$. Then, one has the unique decomposition $Jx = \varphi x + \eta(x) J\xi$, where η is a 1-form in $T_p S^{2n+1}$. It is shown that $(S^{2n+1}, \varphi, \xi, \eta, g)$ is an almost contact manifold with B-metric in the class $\mathcal{F}_4 \oplus \mathcal{F}_5$, that is

$$\begin{aligned} \alpha(X, Y, Z) =& -\frac{\theta(\xi)}{2n} \left\{g(\varphi(X), \varphi(Y)) \eta(Z) + g(\varphi(X), \varphi(Z)) \eta(Y)\right\} \\ &- \frac{\theta^*(\xi)}{2n} \left\{g(X, \varphi(Y)) \eta(Z) + g(X, \varphi(Z)) \eta(Y)\right\}. \end{aligned}$$

For any choice of a non-constant function σ, we obtain infinitely many almost complex manifolds with Norden metrics on $S^{2n+1} \times \mathbb{R}$ of the class $W_1 \oplus W_2$ from Theorem 3.4 and Theorem 3.5.

For instance, let $\sigma(t) = t$. Since $(S^{2n+1}, \varphi, \xi, \eta, g)$ is in $\mathcal{F}_4 \oplus \mathcal{F}_5$, we have $\theta(X) = \eta(X)\theta(\xi)$, and from equation (3.6), we get

$$\tilde{\theta}\left(0, \frac{d}{dt}\right) = -\frac{\theta^*(\xi)}{e^t} \neq 0,$$

which implies that the structure is not in W_2. Similar to the proof of Theorem (3.4), by direct calculation, it can be seen that the defining relation (2.14) of the class W_1 is not satisfied and the defining relation of $W_1 \oplus W_2$ holds. Thus the product manifold $S^{2n+1} \times \mathbb{R}$ is in the class $W_1 \oplus W_2$.

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[1]Department of Mathematics,
Eskişehir Technical University,
Eskişehir, 26470, Türkiye
Email address: nozdemir@eskisehir.edu.tr

[2]Department of Mathematics,
Eskişehir Technical University,
Eskişehir, 26470, Türkiye
Email address: sirins@eskisehir.edu.tr

[3]Department of Mathematics,
Bilecik Şeyh Edebali University,
Bilecik, 11230, Türkiye
Email address: mehmet.solgun@bilecik.edu.tr