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COMMON FIXED POINT RESULTS FOR INTERPOLATIVE KANNAN TYPE CONTRACTION OVER m-METRIC SPACES

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Abstract. The objective of this paper is to derive common fixed point results in m-metric spaces by using the interpolative condition proposed by Karapınar. We discuss three distinct scenarios: when the sum of the "interpolative exponents" is less than, equal to, or greater than 1. The validity of each result is supported by illustrative examples.

1. Introduction

Following Banach's famous fixed point (FP) theorem [2], FP theory has flourished across multiple dimensions and has assumed a pivotal role in various mathematical domains. In recent times, a considerable amount of research has been dedicated to developing techniques for proving FP results concerning interpolative Kannan type contractions (IKTCs). For instance, Karapınar [6] demonstrated a FP result for IKTC. Similarly, Gabba et al. [4] established this result in scenarios where the sum of "interpolative exponents" is less than 1. Moreover, Errai et al. [3] achieved such a result for the case in which the sum of "interpolative exponents" is greater than or equal to 1. Notably, all these outcomes have been proven within the realm of standard metric spaces (MSs). Furthermore, Safeer et al. [8] delve into FP outcomes concerning IKTCs within the framework of m-metric spaces (m-MSs). The concept of m-MSs was initially introduced by Asadi et al. in [1], constituting as an extension of the partial metric space (p-MS).

Key words and phrases. Common fixed point, m-interpolative Kannan type contraction (IKTC), m-metric space(m-MS).

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On the other hand Noorwali [10] initiate the study of common FP for IKTC, after that Gaba and Karapınar [5] proved the common FP results for the case when the sum of the "interpolative exponents" is less than 1.

This paper introduces a study on the existence of common FP for a pair of IKTCs within the framework of m-MSs. We explore all potential scenarios characterized by "interpolative exponents". The first section provides necessary definitions and fundamental results concerning common FPs, m-MSs, and IKTCs. In the second section, we establish three distinct results regarding common FPs for m-MSs, each under different conditions on the "interpolative exponents". Furthermore, we illustrate each result with examples in m-MSs.

Moreover, we examine our examples in standard MSs and elaborate on how the corresponding mappings fail to yield common fixed points. This underscores the significance of our established results. Additionally, we investigate similar outcomes in p-MSs which arise as specific instances of our results for m-MSs, yet represent novel discoveries in their own regard. Finally we note that our results generalize results of [5, 10].

2. Preliminaries

Definition 2.1 ([9]). A partial metric on a nonempty set Υ is a function $p : \Upsilon \times \Upsilon \to \mathbb{R}^+$ such that for all $\varrho_1, \varrho_2, \varrho_3 \in \Upsilon$

(p_1) $p(\varrho_1, \varrho_2) = p(\varrho_1, \varrho_1) = p(\varrho_2, \varrho_2) \Leftrightarrow \varrho_1 = \varrho_2$;
(p_2) $p(\varrho_1, \varrho_1) \leq p(\varrho_1, \varrho_2)$;
(p_3) $p(\varrho_1, \varrho_2) = p(\varrho_2, \varrho_1)$;
(p_4) $p(\varrho_1, \varrho_2) \leq p(\varrho_1, \varrho_3) + p(\varrho_3, \varrho_2) - p(\varrho_3, \varrho_3)$.

A partial MS is a pair (Υ, p) such that Υ is nonempty set and p is a partial metric on Υ.

Definition 2.2 ([1]). Let Υ be a nonempty set. Then m-metric is a function $m : \Upsilon \times \Upsilon \to \mathbb{R}^+$ satisfying the following conditions:

(m_1) $m(\varrho_1, \varrho_2) = m(\varrho_1, \varrho_1) = m(\varrho_2, \varrho_2) \Leftrightarrow \varrho_1 = \varrho_2$;
(m_2) $m_{\varrho_1\varrho_2} \leq m(\varrho_1, \varrho_2)$ *where* $m_{\varrho_1\varrho_2} := \min\{m(\varrho_1, \varrho_1), m(\varrho_2, \varrho_2)\}$;
(m_3) $m(\varrho_1, \varrho_2) = m(\varrho_2, \varrho_1)$;
(m_4) $(m(\varrho_1, \varrho_2) - m_{\varrho_1\varrho_2}) \leq (m(\varrho_1, \varrho_3) - m_{\varrho_1\varrho_3}) + (m(\varrho_3, \varrho_2) - m_{\varrho_3\varrho_2})$

for all $\varrho_1, \varrho_2, \varrho_3 \in \Upsilon$. The pair (Υ, m) is called m-MS.

Lemma 2.1 ([1]). *Every p-MS (Υ, p) is a m-MS.*

The converse of the above result may not hold, as we can see in Example 6 provided by Karapinar et al. in [7].

Definition 2.3 ([1]). Let (Υ, m) be a m-MS. Then

1. a sequence (ϱ_n) in (Υ, m) converges to a point $\varrho \in \Upsilon$ if and only if

$$\lim_{n\to+\infty}(m(\varrho_n, \varrho) - m_{\varrho_n,\varrho}) = 0;$$

2. a sequence (ϱ_n) in (Υ, m) is called m-Cauchy sequence if

$$\lim_{n,j\to+\infty}(m(\varrho_n, \varrho_j) - m_{\varrho_n,\varrho_j})$$

and

$$\lim_{n,j\to+\infty}(M_{\varrho_n,\varrho_j} - m_{\varrho_n,\varrho_j}),$$

exists (and are finite), where $M_{\varrho_n,\varrho_j} = \max\{m(\varrho_n, \varrho_n), m(\varrho_j, \varrho_j)\}$;

3. The space (Υ, m) is said to be complete if every m-Cauchy sequence (ϱ_n) in Υ converges to a point in Υ.

Lemma 2.2 ([1])**.** *Assume that $\varrho_n \to \varrho$ and $\kappa_n \to \kappa$ as $n \to +\infty$ in a m-MS (Υ, m). Then*

$$\lim_{n\to+\infty}(m(\varrho_n, \kappa_n) - m_{\varrho_n,\kappa_n}) = m(\varrho, \kappa) - m_{\varrho,\kappa}.$$

In [6], Karapınar introduce the following IKTC.

Definition 2.4 ([6])**.** Let (Υ, d) be a MS. A self mapping $T : \Upsilon \to \Upsilon$ is said to be an interpolative Kannan type contraction (IKTC), if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(T\varrho, T\kappa) \leq \lambda d(\varrho, T\varrho)^{\alpha} d(\kappa, T\kappa)^{1-\alpha},$$

for all $\varrho, \kappa \in \Upsilon$ with $\varrho \neq T\varrho,\ \ \kappa \neq T\kappa$.

We term α as an "interpolative exponent".

The following result by Karapınar is proved in [6].

Theorem 2.1 ([6])**.** *Let (Υ, d) be a complete MS and T be an IKTC. Then T has a unique FP .*

In [4], Gabba et al. defined the following IKTC.

Definition 2.5. Let (Υ, d) be a MS, a self mapping $T : \Upsilon \to \Upsilon$ is called (λ, α, β)-IKTC if there exist $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, such that

$$d(T\varrho, T\kappa) \leq \lambda d(\varrho, T\varrho)^{\alpha} d(\kappa, T\kappa)^{\beta},$$

for all $\varrho, \kappa \in \Upsilon$ with $\varrho \neq T\varrho$, $\kappa \neq T\kappa$.

Moreover, they proved the following FP theorem.

Theorem 2.2 ([4])**.** *Let (Υ, d) be a complete MS such that $d(\varrho, \kappa) \geq 1$ for all $\varrho, \kappa \in \Upsilon$ with $\varrho \neq \kappa$. Let $T : \Upsilon \to \Upsilon$ be a (λ, α, β)-IKTC. Then T has a FP.*

Errai et al. [3] proved the following FP result for IKTC for the case $\alpha + \beta > 1$ with $\alpha, \beta \in (0, 1)$.

Theorem 2.3 ([3])**.** *Let (Υ, d) be a complete MS and T a self mapping on Υ such that*

$$d(T\varrho, T\kappa) \leq \lambda d(\varrho, T\varrho)^{\alpha} d(\kappa, T\kappa)^{\beta},$$

for all $\varrho, \kappa \in \Upsilon$ with $\varrho \neq T\varrho$ and $\kappa \neq T\kappa$, and where $\lambda \in (0, 1)$ and $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta \geq 1$. If there exists $\varrho \in \Upsilon$ such that $d(\varrho, T\varrho) \leq 1$, then T has a FP in Υ.

Note that all above results of interpolative contractions have been proved in ordinary MS (Υ, d). Also Safeer et al. [8] extend these results into the structure of m-MSs.

On the other hand, the common FP of any two self mappings R, T is a point $\varrho \in \Upsilon$ such that $R\varrho = \varrho = T\varrho$. The Noorwali [10] initiate the study of common FP for IKTC and proved the following result.

Theorem 2.4 ([10])**.** *Let (Υ, d) be a complete MS, $R, T : \Upsilon \to \Upsilon$ be two self mappings. Assume that there are some $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ such that the condition*

$$d(R\varrho, T\kappa) \leq \lambda d(\varrho, R\varrho)^{\alpha} d(\kappa, T\kappa)^{1-\alpha}$$

is satisfied for all $\varrho, \kappa \in \Upsilon$ such that $\varrho \neq R\varrho, \kappa \neq T\kappa$. Then R and T have a common FP.

Moreover, Gabba and Karapınar [5] proved the common FP result for the case when the sum of the "interpolative exponents" is less than one and their result is elaborated as follows.

Theorem 2.5 ([5])**.** *Let (Υ, d) be a complete MS and (R, T) be a (λ, α, β)-IKTC pair. Then R and T have a common FP in Υ.*

The (λ, α, β)-IKTC pair is defined as follows.

Definition 2.6 ([5])**.** Let (Υ, d) be a MS and $R, T : \Upsilon \to \Upsilon$ be two self mappings. We shall call (R, T) a (λ, α, β)-IKTC pair, if there exist $\lambda \in [0, 1), \alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ such that

$$d(R\varrho, T\kappa) \leq \lambda d(\varrho, R\varrho)^{\alpha} d(\kappa, T\kappa)^{\beta},$$

for all $\varrho, \kappa \in \Upsilon$ with $\varrho \neq R\varrho, \kappa \neq T\kappa$.

3. Main Results

Definition 3.1. Let (Υ, m) be a m-MS, $R, T : \Upsilon \to \Upsilon$ be two self mappings on Υ. We call (R, T) a m-IKTC pair. If there exists $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$m(R\varrho, T\kappa) \leq \lambda m(\varrho, R\varrho)^{\alpha} m(\kappa, T\kappa)^{1-\alpha} \tag{3.1}$$

holds for all $\varrho, \kappa \in \Upsilon$ with $\varrho \neq R\varrho, \kappa \neq T\kappa$ and $m(\varrho, R\varrho) \neq 0, m(\kappa, T\kappa) \neq 0$.

Theorem 3.1. *Let (Υ, m) be a complete m-MS and (R, T) be a m-IKTC pair. Then R and T have a common FP in Υ.*

Proof. Let $\varrho_0 \in \Upsilon$, define a sequence (ϱ_n) in Υ such that $\varrho_{2n+1} = R\varrho_{2n}$ and $\varrho_{2n+2} = T\varrho_{2n+1}$. If there exists a natural number n_0 such that $\varrho_{n_0} = \varrho_{n_0+1} = \varrho_{n_0+2}$, then ϱ_{n_0} is the common FP of R and T. Consider there does not exist any three identical terms in the sequence (ϱ_n). Then by (3.1),

$$\begin{aligned}
m(\varrho_{2n+1}, \varrho_{2n+2}) &= m(R\varrho_{2n}, T\varrho_{2n+1}) \\
&\leq \lambda m(\varrho_{2n}, R\varrho_{2n})^{\alpha} m(\varrho_{2n+1}, T\varrho_{2n+1})^{1-\alpha} \\
&= \lambda m(\varrho_{2n}, \varrho_{2n+1})^{\alpha} m(\varrho_{2n+1}, \varrho_{2n+2})^{1-\alpha},
\end{aligned}$$

and $m(\varrho_{2n+1}, \varrho_{2n+2})^{\alpha} \le \lambda m(\varrho_{2n}, \varrho_{2n+1})^{\alpha}$, i.e.,

$$m(\varrho_{2n+1}, \varrho_{2n+2}) \le \lambda^{1/\alpha} m(\varrho_{2n}, \varrho_{2n+1}) \le \lambda m(\varrho_{2n}, \varrho_{2n+1}).$$

Therefore,

$$m(\varrho_{2n+1}, \varrho_{2n+2}) \le \lambda m(\varrho_{2n}, \varrho_{2n+1}). \tag{3.2}$$

Consequently, for all $n \in \mathbb{N}$ we have

$$m(\varrho_n, \varrho_{n+1}) \le \lambda m(\varrho_{n-1}, \varrho_n). \tag{3.3}$$

So,

$$m(\varrho_n, \varrho_{n+1}) \le \lambda m(\varrho_{n-1}, \varrho_n) \le \lambda^2 m(\varrho_{n-2}, \varrho_{n-1}) \le \cdots \le \lambda^n m(\varrho_0, \varrho_1).$$

Thus,

$$m(\varrho_n, \varrho_{n+1}) \le \lambda^n m(\varrho_0, \varrho_1),$$

by taking limit as $n \to +\infty$,

$$\limsup_{n\to+\infty} m(\varrho_n, \varrho_{n+1}) \le \limsup_{n\to+\infty} \lambda^n m(\varrho_0, \varrho_1) = 0.$$

Hence, $\lim_{n\to+\infty} m(\varrho_n, \varrho_{n+1}) = 0$. By definition of m-metric

$$\lim_{n\to+\infty} m_{\varrho_n, \varrho_{n+1}} \le \lim_{n\to+\infty} m(\varrho_n, \varrho_{n+1}) = 0,$$

thus $\lim_{n\to+\infty} m_{\varrho_n, \varrho_{n+1}} = \min\{m(\varrho_n, \varrho_n), m(\varrho_{n+1}, \varrho_{n+1})\} = 0$. As a result

$$\lim_{n\to+\infty} m(\varrho_n, \varrho_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} m(\varrho_{n+1}, \varrho_{n+1}) = 0.$$

Thus, for any $n, j \in \mathbb{N}$ with $n \ge j$

$$\lim_{n,j\to+\infty} (M_{\varrho_n, \varrho_j} - m_{\varrho_n, \varrho_j}) = 0$$

and by triangular inequality of m-metric

$$\lim_{n,j\to+\infty} (m(\varrho_n, \varrho_j) - m_{\varrho_n, \varrho_j}) = 0.$$

Thus, by definition (ϱ_n) is a Cauchy sequence in m-MS Υ, since Υ is m-complete so there exists $\varrho \in \Upsilon$ such that (ϱ_n) converges to ϱ in Υ w.r.t. the convergence of m-metric. Thus, by definition

$$\lim_{n\to+\infty} (m(\varrho_n, \varrho) - m_{\varrho_n, \varrho}) = 0.$$

Also, (ϱ_{2n+1}) and (ϱ_{2n+2}) converge to the same limit ϱ. Now for any $n \in \mathbb{N}$ and by using the relation (3.1) for $R = T$, we get

$$\begin{aligned} m(\varrho_{2n+1}, R\varrho) &= m(R\varrho_{2n}, R\varrho) \\ &\le \lambda m(\varrho_{2n}, R\varrho_{2n})^{\alpha} m(\varrho, R\varrho)^{1-\alpha} \\ &= \lambda m(\varrho_{2n}, \varrho_{2n+1})^{\alpha} m(\varrho, R\varrho)^{1-\alpha}. \end{aligned}$$

By taking limit as $n \to +\infty$ on both sides and using the m_2 condition of m-metric, we get

$$\lim_{n\to+\infty} (m(\varrho_{2n+1}, R\varrho) - m_{\varrho_{2n+1}, R\varrho}) = 0.$$

So, (ϱ_{2n+1}) converges to $R\varrho$ in m-metric, i.e. $\varrho_{2n+1} = R\varrho_{2n} \to R\varrho$. Also

$$\begin{aligned} m(\varrho_{2n+2}, R\varrho) &= m(T\varrho_{2n+1}, R\varrho) \\ &\leq \lambda m(\varrho_{2n+1}, T\varrho_{2n+1})^{\alpha} m(\varrho, R\varrho)^{1-\alpha} \\ &= \lambda m(\varrho_{2n+1}, \varrho_{2n+2})^{\alpha} m(\varrho, R\varrho)^{1-\alpha}. \end{aligned}$$

By taking limit as $n \to +\infty$ on both sides and using the m_2 condition of m-metric, we get

$$\lim_{n\to+\infty}(m(\varrho_{2n+2}, R\varrho) - m_{\varrho_{2n+2},R\varrho}) = 0.$$

So, (ϱ_{2n+2}) converges to $R\varrho$ in m-metric, i.e., $\varrho_{2n+2} = T\varrho_{2n+1} \to R\varrho$. Thus, (ϱ_n) converges to $R\varrho$ as well.

Case I. If n is even, then $\varrho_{2n+2} = T\varrho_{2n+1} \to R\varrho$ and $\varrho_{2n+2} \to \varrho$, so (ϱ_n) converges to both $R\varrho$ and ϱ. Thus, by using Lemma 2.2,

$$0 = \lim_{n\to+\infty}(m(\varrho_{2n+2}, \varrho_{2n+2}) - m_{\varrho_{2n+2},\varrho_{2n+2}}) = m(\varrho, R\varrho) - m_{\varrho,R\varrho}.$$

Also, we have $\lim_{n\to+\infty} m(\varrho_{2n+2}, \varrho_{2n+2}) = 0$, because $\lim_{n\to+\infty} m(\varrho_n, \varrho_n) = 0$ and

$$\begin{aligned} 0 &= \lim_{n\to+\infty}(m(\varrho_{2n+2}, \varrho_{2n+2}) - m_{\varrho_{2n+2},\varrho_{2n+2}}) \\ &= \lim_{n\to+\infty}(m(T\varrho_{2n+1}, T\varrho_{2n+1}) - m_{\varrho_{2n+2},T\varrho_{2n+1}}) \\ &= m(R\varrho, R\varrho) - m_{\varrho,R\varrho}. \end{aligned}$$

Moreover,

$$0 = \lim_{n\to+\infty}(m(\varrho_{2n+2}, \varrho_{2n+2}) - m_{\varrho_{2n+2},T\varrho_{2n+1}}) = m(\varrho, \varrho) - m_{\varrho,R\varrho}.$$

Thus by combining, we have

$$m(\varrho, \varrho) = m(\varrho, R\varrho) = m(R\varrho, R\varrho) = m_{\varrho,R\varrho},$$

by m_1 condition of m-metric we have $\varrho = R\varrho$.

Case II. If n is odd, then $\varrho_{2n+1} = R\varrho_{2n} \to R\varrho$ and $\varrho_{2n+1} \to \varrho$, we have

$$0 = \lim_{n\to+\infty}(m(\varrho_{2n+1}, \varrho_{2n+1}) - m_{\varrho_{2n+1},\varrho_{2n+1}}) = m(\varrho, R\varrho) - m_{\varrho,R\varrho}.$$

Also,

$$\begin{aligned} 0 &= \lim_{n\to+\infty}(m(\varrho_{2n+1}, \varrho_{2n+1}) - m_{\varrho_{2n+1},\varrho_{2n+1}}) \\ &= \lim_{n\to+\infty}(m(R\varrho_{2n}, R\varrho_{2n}) - m_{\varrho_{2n+1},R\varrho_{2n}}) \\ &= m(R\varrho, R\varrho) - m_{\varrho,R\varrho}. \end{aligned}$$

Moreover, $0 = \lim_{n\to+\infty}(m(\varrho_{2n+1}, \varrho_{2n+1}) - m_{\varrho_{2n+1},R\varrho_{2n}}) = m(\varrho, \varrho) - m_{\varrho,R\varrho}$. Thus, by combining, we have $m(\varrho, \varrho) = m(\varrho, R\varrho) = m(R\varrho, R\varrho) = m_{\varrho,R\varrho}$, by m_1 condition of m-metric we have $\varrho = R\varrho$. Consequently, $\varrho = R\varrho$. By using similar steps we can get $\varrho = T\varrho$, thus ϱ is the common FP for T and R. □

Corollary 3.1. *If we take $R = T$, then Theorem* 3.2 *of* [8] *becomes the special case of our result in Theorem* 3.1.

Example 3.1. Let $\Upsilon = [1/8, 4]$ and the m-metric on Υ is defined as follows:

$$m(\varrho, \kappa) = \begin{cases} \varrho, & \varrho = \kappa, \\ \varrho + \kappa, & \varrho \neq \kappa. \end{cases} \tag{3.4}$$

Let $R, T : \Upsilon \to \Upsilon$ be self mappings, such that

$$R\varrho = \begin{cases} 1/2, & \varrho \in [1/8, 2], \\ 1/(\varrho + 3), & \varrho \in (2, 4], \end{cases} \qquad T\varrho = \begin{cases} 1/2, & \varrho \in [1/8, 2], \\ 1/2\varrho, & \varrho \in (2, 4]. \end{cases}$$

We discuss the following cases for $\alpha = 1/2$ and $\lambda = 17/18$.

Case 1. If $\varrho, \kappa \in [1/8, 2]$, then for $\varrho \neq 1/2$ and $\kappa \neq 1/2$, we have,

$$\begin{aligned} m(R\varrho, T\kappa) &= m(1/2, 1/2) = 1/2 \leq (17/18)\,(1/8 + 1/2) \\ &\leq \lambda(\varrho + 1/2)^{1/2}(\kappa + 1/2)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 2. If $\varrho \in [1/8, 2]$ and $\kappa \in (2, 4]$, then for $\varrho \neq 1/2$, we have

$$\begin{aligned} m(R\varrho, T\kappa) &= m(1/2, 1/2\kappa) \leq 1/2 + 1/4 \leq (17/18)(1/8 + 1/2)^{1/2}(2 + 1/4)^{1/2} \\ &\leq \lambda(\varrho + 1/2)^{1/2}(\kappa + 1/2\kappa)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 3. If $\varrho \in (2, 4]$ and $\kappa \in [1/8, 2]$ then for $\kappa \neq 1/2$, we have

$$\begin{aligned} m(R\varrho, T\kappa) &= m(1/(\tau + 3), 1/2) \leq 1/2 + 1/5 \leq (17/18)(2 + 1/5)^{1/2}(1/8 + 1/2)^{1/2} \\ &\leq \lambda(\varrho + 1/(\varrho + 3))^{1/2}(\kappa + 1/2)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 4. If $\varrho, \kappa \in (2, 4]$, then

$$\begin{aligned} m(R\varrho, T\kappa) &= m(1/2\tau, 1/(\kappa + 3)) \leq 1/4 + 1/5 \leq (17/18)(2 + 1/4)^{1/2}(2 + 1/5)^{1/2} \\ &\leq (17/18)(\varrho + 1/2\tau)^{1/2}(\kappa + 1/(\kappa + 3))^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Hence, (R, T) is a m-IKTC, so by Theorem 3.1, R and T have a common FP and it is actually $\varrho = 1/2$.

Remark 3.1. If we use the standard metric $d(\varrho, \kappa) = |\varrho - \kappa|$ instead of the m-metric (3.4), then Case 2 and 3 of the above example do not satisfy the required IKTC for the pair (R, T) across many different combinations of ϱ and κ. One combination where Case 2 fails to satisfy IKTC is when $\kappa = 3 \in (2, 4]$ and $\varrho \in (4481/9826, 5345/9826) \subset [1/8, 2]$. Therefore, the common fixed point results of the standard MS, as elaborated in [10], do not apply to our given pair (R, T).

The following example asserts that the common fixed point is not always unique.

Example 3.2. Let $\Upsilon = [0, +\infty)$ and the mapping $m : \Upsilon \times \Upsilon \to \mathbb{R}^+$ be defined as $m(\varrho, \kappa) = |\varrho - \kappa| + a$, where "$a$" is any non-negative real number. Let R, T be the self mappings defined on Υ as follows:

$$R\varrho = \begin{cases} 1, & \varrho \in [0, 1/2), \\ \varrho, & \varrho \in [1/2, 200), \\ 1/\varrho^2, & \varrho \in [200, +\infty). \end{cases} \qquad T\varrho = \begin{cases} 1, & \varrho \in [0, 1/2), \\ \varrho, & \varrho \in [1/2, 200), \\ e^{-2\varrho}, & \varrho \in [200, +\infty). \end{cases}$$

Now we discuss following cases to prove that (R, T) is m-IKTC for $\alpha = 1/2$ and $\lambda = 3/4$.

Case 1. If $\varrho, \kappa \in [0, 1/2)$, then for all $a \in [0, 3/2]$ following relation holds:

$$\begin{aligned} m(R\varrho, T\kappa) &= a \leq (3/4)(a + 1/2) \\ &\leq \lambda(|\varrho - 1| + a)^{1/2}(|\kappa - 1| + a)^{1/2} \\ &= \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 2. If $\varrho \in [0, 1/2)$ and $\kappa \in [200, +\infty)$, then for all $0 \leq a \leq 253$, the following relation holds:

$$\begin{aligned} m(R\varrho, T\kappa) &= |1 - e^{-2\kappa}| + a \leq 1 + a \leq (3/4)(1/2 + a)^{1/2}(200 - e^{-400} + a)^{1/2} \\ &\leq \lambda(|\varrho - 1| + a)^{1/2}(|\kappa - e^{-2\kappa}| + a)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 3. If $\varrho \in [200, +\infty)$ and $\kappa \in [0, 1/2)$, then for all $0 \leq a \leq 253$, the following relation holds:

$$\begin{aligned} m(R\varrho, T\kappa) &= |1/\varrho^2 - 1| + a \leq 1 + a \leq (3/4)(200 - (1/200^2) + a)^{1/2}(1/2 + a)^{1/2} \\ &\leq \lambda(|\varrho - 1/\varrho^2| + a)^{1/2}(|\kappa - 1| + a)^{1/2} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Case 4. If $\varrho, \kappa \in [200, +\infty)$, then for all $0 \leq a \leq 600$, the following relation holds:

$$\begin{aligned} m(R\varrho, T\kappa) &= |1/\varrho^2 - e^{-2\kappa}| + a \leq (1/200^2) + a \\ &\leq (3/4)(200 - (1/200^2) + a)^{1/2}(200 - e^{-400} + a)^{1/2} \\ &\leq (3/4)(|\varrho - 1/\varrho^2| + a)^{1/2}(|\kappa - e^{-2\kappa}| + a)^{1/2} = \lambda m x, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/2}. \end{aligned}$$

Hence, from all the above cases we conclude that the interpolative condition of Definition 3.1 holds when $a \in [0, 3/2]$. Thus for such values of a, by Theorem 3.1, R and T have common FPs and they actually are all the points in interval $[1/2, 200)$.

Remark 3.2. Given that our previous example remains valid for $a \in [0, 3/2]$, when $a = 0$, the corresponding m-metric aligns with the standard metric on the real line. However, for $a \neq 0$, the results derived in [10] do not apply to our specified pair (R, T), as they were established solely for standard metric spaces. In such instances, our results concerning the m-metric will prove effective for identifying common fixed points.

Definition 3.2. Let (Υ, m) be a m-MS and $R, T : \Upsilon \to \Upsilon$ be two self mappings. We call (R, T) a (λ, α, β)-m-IKTC, if there exist $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ such that

$$m(R\varrho, T\kappa) \leq \lambda m(\varrho, R\varrho)^{\alpha} m(\kappa, T\kappa)^{\beta}, \tag{3.5}$$

for all $\varrho, \kappa \in \Upsilon$ with $\varrho \neq R\varrho$, $\kappa \neq T\kappa$ and $m(\varrho, R\varrho) \geq 1$, $m(\kappa, T\kappa) \neq 0$.

Theorem 3.2. *Let (Υ, m) be a complete m-MS and (R, T) be (λ, α, β)-m-IKTC. Then R and T have a common FP.*

Proof. Let $\varrho_0 \in \Upsilon$, we construct the iterating sequence (ϱ_n) such that $\varrho_{2n+1} = R\varrho_{2n}$ and $\varrho_{2n+2} = T\varrho_{2n+1}$. Thus,

$$\begin{aligned} m(\varrho_{2n+1}, \varrho_{2n+2}) &= m(R\varrho_{2n}, T\varrho_{2n+1}) \\ &\leq \lambda m(\varrho_{2n}, R\varrho_{2n})^{\alpha} m(\varrho_{2n+1}, T\varrho_{2n+1})^{\beta} \\ &= \lambda m(\varrho_{2n}, \varrho_{2n+1})^{\alpha} m(\varrho_{2n+1}, \varrho_{2n+2})^{\beta}, \\ m(\varrho_{2n+1}, \varrho_{2n+2})^{1-\beta} &\leq \lambda m(\varrho_{2n}, \varrho_{2n+1})^{\alpha}, \end{aligned}$$

since $\alpha < 1 - \beta$ and $m(\varrho_{2n}, \varrho_{2n+1}) \geq 1$, so we have

$$\begin{aligned} m(\varrho_{2n+1}, \varrho_{2n+2})^{1-\beta} &\leq \lambda m(\varrho_{2n}, \varrho_{2n+1})^{1-\beta}, \\ m(\varrho_{2n+1}, \varrho_{2n+2}) &\leq \lambda m(\varrho_{2n}, \varrho_{2n+1}). \end{aligned}$$

The rest of the proof follows the similar procedure as in Theorem 3.1. To avoid the repetition, we leave it for the interested reader to dig out the details. $\square$

Corollary 3.2. *If we take $R = T$, then Theorem* 3.6 *of* [8] *becomes the special case of our result in Theorem* 3.2.

Example 3.3. Let $\Upsilon = [0, +\infty)$ and m-metric on Υ be defined as in (3.4), define self mappings $R, T : \Upsilon \to \Upsilon$ as follows:

$$R\varrho = \begin{cases} \varrho, & \varrho \in [0, 5], \\ 1/\varrho, & h \in (5, +\infty), \end{cases} \qquad T\varrho = \begin{cases} \varrho, & \varrho \in [0, 5], \\ 1/\ln \varrho, & \varrho \in (5, +\infty). \end{cases}$$

We discuss the required case to confirm that (R, T) is $(2/3, 1/2, 1/4)$-m-IKTC used in Theorem 3.2. For any $\varrho, \kappa \in (5, +\infty)$, we have

$$\begin{aligned} m(R\varrho, T\kappa) &\leq (1/5 + 1/\ln 5) \leq (2/3)(5 + 1/5)^{1/2}(5 + 1/\ln 5)^{1/4} \\ &\leq (2/3)(\varrho + 1/\varrho)^{1/2}(\kappa + 1/\ln \kappa)^{1/4} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{1/4}. \end{aligned}$$

Consequently, (R, T) satisfies the required m-IKTC of Theorem 3.2, so every $\varrho \in [0, 5]$ is the common FP of R and T.

Remark 3.3. In the case of the discrete metric $d(\varrho, \kappa) = 1$ if $\varrho \neq \kappa$ and zero if $\varrho = \kappa$, the IKTC in the above example is not satisfied for the pair (R, T).

Theorem 3.3. *Let (Υ, m) be a complete m-MS, $R, T : \Upsilon \to \Upsilon$ be two self mappings and let there exists $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta > 1$ such that*

$$m(R\varrho, T\kappa) \leq \lambda m(\varrho, R\varrho)^{\alpha} m(\kappa, T\kappa)^{\beta}, \tag{3.6}$$

for all $\varrho, \kappa \in \Upsilon$ with $\varrho \neq R\varrho$, $\kappa \neq T\kappa$ and $m(\varrho, R\varrho) \neq 0$, $m(\kappa, T\kappa) \neq 0$. If there exist $\varrho_0 \in \Upsilon$ such that $m(\varrho_0, R\varrho_0) \leq 1$, then R and T have common FP in Υ.

Proof. Since $\varrho_0 \in \Upsilon$ such that $m(\varrho_0, R\varrho_0) \leq 1$, we construct a sequence (ϱ_n) in Υ such that $\varrho_{2n+1} = R_{2n}$ and $\varrho_{2n+2} = T\varrho_{2n+1}$. So,

$$\begin{aligned} m(\varrho_1, \varrho_2) =& m(R\varrho_0, T\varrho_1) \leq \lambda m(\varrho_0, R\varrho_0)^{\alpha} m(\varrho_1, T\varrho_1)^{\beta} = \lambda m(\varrho_0, R\varrho_0)^{\alpha} m(\varrho_1, \varrho_2)^{\beta}, \\ m(\varrho_1, \varrho_2)^{1-\beta} \leq& \lambda m(\varrho_0, R\varrho_0)^{\alpha}, \\ m(\varrho_1, \varrho_2) \leq& \lambda^{1/1-\beta} m(\varrho_0, R\varrho_0)^{\alpha/1-\beta} \leq \lambda, \end{aligned}$$

because $\alpha/(1-\beta) > 1$ and $m(\varrho_0, R\varrho_0) \leq 1$. Similarly, by mathematical induction, the relation $m(\varrho_n, \varrho_{n+1}) \leq \lambda^n$ holds for all natural numbers $n \in \mathbb{N}$. Thus, by taking limit we get $\lim_{n\to+\infty} m(\varrho_n, \varrho_{n+1}) = 0$. Also, by m_2 condition of m-metric, we have

$$\lim_{n\to+\infty} m_{\varrho_n, \varrho_{n+1}} = 0,$$

and thus

$$\lim_{n\to+\infty} m(\varrho_n, \varrho_n) = 0, \quad \lim_{n\to+\infty} m(\varrho_{n+1}, \varrho_{n+1}) = 0.$$

Moreover, for any $n, j \in \mathbb{N}$ with $n \geq j$, we have

$$\lim_{n,j\to+\infty} (M_{\varrho_n, \varrho_j} - m_{\varrho_n, \varrho_j}) = 0,$$

by triangular inequality of m-metric

$$\lim_{n,j\to+\infty} (m(\varrho_n, \varrho_j) - m_{\varrho_n, \varrho_j}) = 0.$$

Thus (ϱ_n) is a m-Cauchy sequence in Υ, since Υ is complete so it converges to some $\varrho \in \Upsilon$. Now

$$\begin{aligned} m(\varrho_{2n+1}, R\varrho) &\leq \lambda m(\varrho_{2n}, R\varrho_{2n})^{\alpha} m(\varrho, R\varrho)^{\beta} \\ &= \lambda m(\varrho_{2n}, \varrho_{2n+1})^{\alpha} m(\varrho, R\varrho)^{\beta}, \\ &\leq \lambda^{1+\alpha 2n} m(\varrho, R\varrho), \end{aligned}$$

thus by applying limit, we get $\lim_{n\to+\infty} m(\varrho_{2n+1}, R\varrho) = 0$ and then by m_2 condition of m-metric we have

$$\lim_{n\to+\infty} (m(\varrho_{2n+1}, R\varrho) - m_{\varrho_{2n+1}, R\varrho}) = 0,$$

by definition (ϱ_{2n+1}) converges to $R\varrho$. On similar steps, (ϱ_{2n+2}) converges to $R\varrho$, thus by combining both the arguments, we get the sequence (ϱ_n) also converges to $R\varrho$. Moreover, by using the similar arguments as in Case I and Case II of Theorem 3.1, we get $\varrho = R\varrho$.

Also, for $T\varrho$ by following the similar procedure as mentioned above for $\varrho = R\varrho$, we get $\varrho = T\varrho$. Consequently, ϱ is the common FP for R and T. □

Corollary 3.3. *If we take $R = T$, then Theorem* 3.8 *of* [8] *becomes the special case of our result in Theorem* 3.3.

Example 3.4. Let $\Upsilon = [0,2]$ and m-metric on Υ be defined as in (3.4) and define self mappings $R, T : \Upsilon \to \Upsilon$ as follows:

$$R\varrho = \begin{cases} \varrho, & \varrho \in [0,1), \\ e^{-\varrho}, & \varrho \in [1,2], \end{cases} \qquad T\varrho = \begin{cases} \varrho, & \varrho \in [0,1), \\ 1/\varrho^2, & \varrho \in [1,2]. \end{cases}$$

We discuss the following cases to confirm that for $\alpha = 1/2, \beta = 3/4$ and $\lambda = 3/4$ the pair (R, T) is m-IKTC pair used in Theorem 3.3. For any $\varrho, \kappa \in [1,2]$, we have

$$\begin{aligned}(R\varrho, T\kappa) &\leq e^{-1} + 1 \leq (17/18)(1+e^{-1})^{1/2}(1+1)^{3/4} \\ &\leq \lambda(\varrho + e^{-\varrho})^{1/2}(\kappa + 1/\kappa^2)^{3/4} = \lambda m(\varrho, R\varrho)^{1/2} m(\kappa, T\kappa)^{3/4}.\end{aligned}$$

Moreover, $e^{-\varrho} + 1/\kappa^2 \leq (17/18)(1+e^{-1})^{3/4}(1+1)^{1/2}$. Thus by Theorem 3.3, the self mappings R and T have common FPs for all $\varrho \in [0,1)$.

Furthermore, in the case of the standard MS with $d(\varrho, \kappa) = |\varrho - \kappa|$, the IKTC for the pair (R, T) does not work when $\kappa = 1$. Therefore, our results in the m-MS are the ones applicable for such pairs to determine the common FP.

Remark 3.4. By Lemma 2.1, every p-MS is also a m-MS. Consequently, similar results of common FPs (Theorem 3.1, Theorem 3.2 and Theorem 3.3) for p-MSs naturally hold across all possible scenarios: when the sum of the 'interpolative exponents' is equal to 1, less than 1, and greater than 1.

Remark 3.5. Since every ordinary metric d is a p-metric, our Theorem 3.1 and Theorem 3.2 generalize the corresponding results of [5, 10], respectively.

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