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ON PRIME LABELING OF SOME UNION GRAPHS

S. K. PATEL[1] AND JAYESH VASAVA[1]

Abstract. The cycle C_n is a well-known example of a prime graph and also it is quite easy to establish that the graph $C_n^{(k)}$ which is the one point union of k copies of the cycle C_n, is a prime graph. In this paper we investigate prime labeling for graphs which are either union of $C_n^{(k)}$'s or union of cycle graphs.

1. Introduction

We consider only finite, simple and undirected graphs. For a graph G, its vertex and edge sets are denoted by $V(G)$ and $E(G)$ respectively and further, $|V(G)|$ and $|E(G)|$ denote their cardinalities. We follow Gross and Yellen [4] for graph theoretic terminology and notations and [1] for elementary number theory results. We begin with the definition of prime labeling which was originated by Entringer and was discussed in a paper by Tout et al.[8].

Definition 1.1. A bijection $f : V(G) \to \{1, 2, 3, \ldots, n\}$ is said to be a prime labeling of a graph G with n vertices, if $f(u)$ and $f(v)$ are relatively prime numbers (i.e., $\gcd(f(u), f(v)) = 1$) whenever u are v are adjacent vertices of G. A graph that admits a prime labeling is called a prime graph.

Since the introduction of prime labeling about thirty five years ago, varieties of graphs have been studied for prime labeling. In the recent years, some of the variants of prime labeling have also been introduced and studied extensively. See for instance [7] and [5], where prime cordial labeling and neighborhood-prime labeling are introduced and studied respectively. A brief summary of the results regarding prime labeling and its variants is available in the dynamic survey of graph labeling maintained by

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Gallian [3]. In this paper, we mainly investigate prime labeling for graphs which are union of $C_n^{(k)}$ (defined below).

Definition 1.2. The graph $C_n^{(k)}$ (where $k \geq 2$) is known as the one point union of k copies of the cycle C_n and it is obtained from the k copies of the cycle C_n by identifying exactly one vertex of each of these k copies of C_n.

The graph $C_n^{(k)}$ consists of $k(n-1)+1$ vertices and kn edges as can be seen in the graph of $C_4^{(3)}$ below. In the past four decades, such graphs have been studied

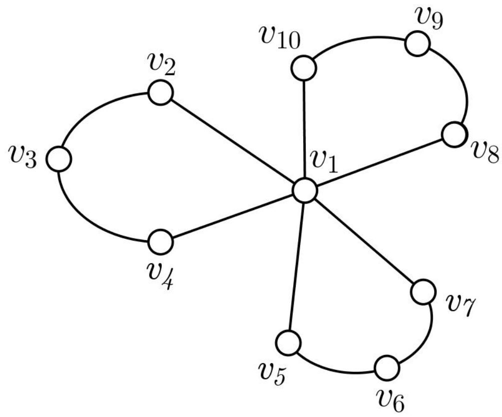

FIGURE 1. Graph of $C_4^{(3)}$

for the various types of labeling, but here our aim is to study the union of such graphs for prime labeling. Just like C_n, it is quite trivial to show that $C_n^{(k)}$ is a prime graph for all n and k. But things get non-trivial when we think about graphs that are union of C_n or $C_n^{(k)}$. It is known that $C_n \bigcup C_m$ is a prime graph if and only if either n is even or m is even. We derive a similar result for the graph $C_n^{(j)} \bigcup C_m^{(k)}$ in this paper, although technically it is much more difficult as compared to the case of union of cycles. Further, we also derive results about prime labeling of the graphs $C_{2n}^{(2)} \bigcup C_{2m}^{(2)} \bigcup C_k^{(2)}$, $C_{2n}^{(2)} \bigcup C_{2m+1}^{(2)} \bigcup C_{2k+1}^{(2)}$ and $C_{2n} \bigcup C_{2n} \bigcup C_{2n} \bigcup C_{2n} \bigcup C_{2m} \bigcup C_k$.

Now before moving to the section of main results, we state a lemma which is useful in showing that certain graphs are not prime.

Lemma 1.1. *Let $\beta_0(G)$ denote the independence number (i.e., the maximum cardinality of an indepenedent set) of G. If $\beta_0(G) < \left\lfloor \frac{|V(G)|}{2} \right\rfloor$, then G is not a prime graph (where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x).*

The proof of this lemma is not very difficult and it is available in [2].

2. MAIN RESULTS

Let $G = C_n \bigcup C_m$. If n amd m both are odd, then $\beta_0(G) = \frac{n+m}{2} - 1 < \frac{n+m}{2} = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ and so in view of Lemma 1.1, G is not a prime graph. However, if n is even and

$\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_m\}$ are the vertex sets of C_n and C_m respectively then it is easy to see that $f : V(G) \to \{1, 2, \ldots, n+m\}$ defined by

$$\begin{aligned} f(v_i) &= i+1, \quad 1 \le i \le n, \\ f(u_1) &= 1, \\ f(u_i) &= i+n, \quad 2 \le i \le m, \end{aligned}$$

is a prime labeling of $G = C_n \bigcup C_m$. Thus $C_n \bigcup C_m$ is a prime graph if and only if either n or m is even. Here we prove the same result for the graph $C_n^{(j)} \bigcup C_m^{(k)}$.

Theorem 2.1. *If n and m both are odd, then $C_n^{(j)} \bigcup C_m^{(k)}$ is not a prime graph.*

Proof. Let G denote the graph $C_n^{(j)} \bigcup C_m^{(k)}$. Since n and m are odd, the independence numbers of the cycles C_n and C_m are $\frac{n-1}{2}$ and $\frac{m-1}{2}$ respectively and further it may be verified that

$$\beta_0(G) = j\left(\frac{n-1}{2}\right) + k\left(\frac{m-1}{2}\right).$$

But $|V(G)| = j(n-1) + k(m-1) + 2$ and so

$$\left\lfloor \frac{|V(G)|}{2} \right\rfloor = \frac{j(n-1)+k(m-1)}{2} + 1.$$

Thus,

$$\beta_0(G) < \left\lfloor \frac{|V(G)|}{2} \right\rfloor.$$

Therefore in view of Lemma 1.1, we conclude that G is not a prime graph . □

Theorem 2.2. *$C_{2n}^{(j)} \bigcup C_m^{(k)}$ is a prime graph for all n and m.*

Proof. Let $G = C_{2n}^{(j)} \bigcup C_m^{(k)}$. Let the vertices of the h^{th} cycle of $C_{2n}^{(j)}$ be

$$\{v_1, v_{(2n-1)(h-1)+2}, v_{(2n-1)(h-1)+3}, \ldots, v_{(2n-1)(h-1)+2n}\},$$

where $h = 1, 2, 3, \ldots, j$, and the vertices of the l^{th} cycle of $C_m^{(k)}$ be

$$\{v_{j(2n-1)+2}, v_{j(2n-1)+(m-1)(l-1)+3}, v_{j(2n-1)+(m-1)(l-1)+4}, \ldots, v_{j(2n-1)+(m-1)(l-1)+m+1}\},$$

where $l = 1, 2, 3, \ldots, k$. We prove the theorem by considering two cases as under.

Case 1: $k(m-1)$ is odd.

Let p be a randomly chosen prime number lying strictly between $\frac{k(m-1)+1}{2}$ and $k(m-1)+1$, which exists due to Bertrand's postulate which states that for any positive integer $N > 1$, there is a prime number lying strictly between N and $2N$. With the help of this number p, we now define a prime labeling $g : V(G) \to \{1, 2, 3, \ldots, j(2n-1) + k(m-1) + 2\}$ as follows:

$$\begin{aligned} g(v_1) &= 1, \\ g(v_i) &= i + k(m-1) + 1, \quad i = 2, 3, 4, \ldots, j(2n-1)+1, \\ g(v_{j(2n-1)+2}) &= p, \end{aligned}$$

$$g(v_{i+j(2n-1)+2}) = i + p, \quad i = 1, 2, 3, \ldots, k(m-1) - p + 2,$$
$$g(v_{i+j(2n-1)+k(m-1)-p+4}) = i + 1, \quad i = 1, 2, 3, \ldots, p-2.$$

The definition of g is illustrated in Figure 2. Note that if two vertices of G are adjacent,

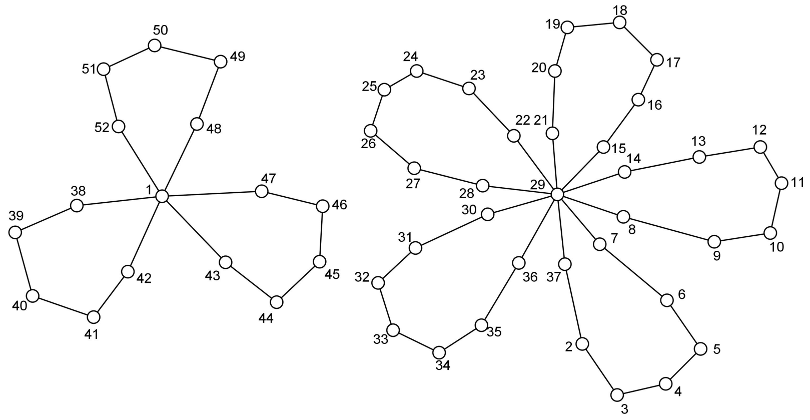

FIGURE 2. Prime Labeling of $C_6^{(3)} \bigcup C_8^{(5)}$

then either both of them are vertices of $C_{2n}^{(j)}$ or both are vertices of $C_m^{(k)}$. It is easy to see that any two adjacent vertices of $C_{2n}^{(j)}$ have relatively prime labels because either these two labels are consecutive integers or one of the two labels is equal to 1. Also note that unless one of the vertices is either $v_{j(2n-1)+2}$ or $v_{j(2n-1)+k(m-1)-p+4}$, any two adjacent vertices of the $C_m^{(k)}$ have consecutive labels. Further, suppose any vertex say u of $C_m^{(k)}$ is adjacent to the vertex $v_{j(2n-1)+2}$, then $g(u)$ and $g(v_{j(2n-1)+2})$ are relatively prime because $g(v_{j(2n-1)+2}) = p$ where as $g(u) < 2p$. Finally, if a vertex u (which is different from $v_{j(2n-1)+2}$) of the $C_m^{(k)}$ is adjacent to the vertex $v_{j(2n-1)+k(m-1)-p+4}$ then either $u = v_{j(2n-1)+k(m-1)-p+3}$ or $u = v_{j(2n-1)+k(m-1)-p+5}$. But $g(v_{j(2n-1)+k(m-1)-p+3})$ and $g(v_{j(2n-1)+k(m-1)-p+4})$ are consecutive integers where as $g(v_{j(2n-1)+k(m-1)-p+5}) = 2$ is relatively prime to $g(v_{j(2n-1)+k(m-1)-p+4}) = k(m-1) + 2$ because we have assumed $k(m-1)$ to be odd. Thus g defines a prime labeling. Now we consider the second case.

Case 2: $k(m-1)$ is even.

Let p be a randomly chosen prime number lying strictly between $\frac{k(m-1)+2}{2}$ and $k(m-1)+2$. Here we consider two subcases.

Sub-case 1: $p - 1 \not\equiv 0 \pmod 3$.

Define $f : V(G) \to \{1, 2, \ldots, j(2n-1) + k(m-1) + 2\}$ as

$$f(v_1) = 1,$$
$$f(v_i) = i + k(m-1) + 2, \quad i = 2, 3, \ldots, 2n,$$
$$f(v_{2n+1}) = 2,$$
$$f(v_i) = i + k(m-1) + 1, \quad i = 2n+2, 2n+3, \ldots, j(2n-1) + 1,$$

$$\begin{aligned}
f(v_{j(2n-1)+2}) &= p, \\
f(v_{i+j(2n-1)+2}) &= i+p, \quad i = 1,2,\ldots,k(m-1)-p+3, \\
f(v_{i+j(2n-1)+k(m-1)-p+5}) &= i+3, \quad i = 1,2,\ldots,p-4, \\
f(v_{j(2n-1)+k(m-1)+2}) &= 3.
\end{aligned}$$

The definition of f is illustrated in Figure 3. The definition of f clearly suggests

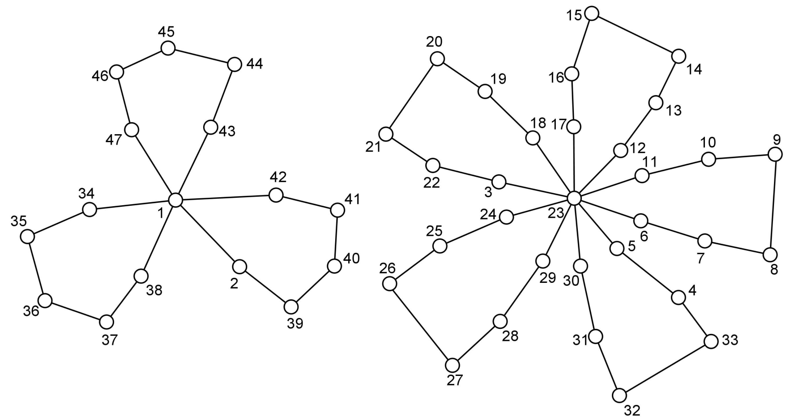

FIGURE 3. Prime Labeling of $C_6^{(3)} \bigcup C_7^{(5)}$

that unless one of the vertices is v_{2n+1}; the labels of any two adjacent vertices of $C_{2n}^{(j)}$ are either consecutive integers or one of the labels is equal to 1. Further, the two neighbors of v_{2n+1} are v_1 and v_{2n+2} whose labels are 1 and $2n + k(m-1) + 3$ respectively. But $k(m-1)$ is assumed to be even and hence $2n + k(m-1) + 3$ is an odd number where as $f(v_{2n+1}) = 2$. Thus, we conclude that any two adjacent vertices of $C_{2n}^{(j)}$ have relatively prime labels under f. Now suppose u and v are any two adjacent vertices of $C_m^{(k)}$. If one of them say $u = v_{j(2n-1)+2}$, then $f(u)$ and $f(v)$ are relatively prime because $f(u = v_{j(2n-1)+2}) = p$ and $f(v) < 2p$. On the other hand if u and v are adjacent and both are different from the vertex $v_{j(2n-1)+2}$, then they have relatively prime labels since either they are consecutive integers or else one of the following two possibilities occur:

$$\begin{aligned}
&\gcd(f(v_{j(2n-1)+k(m-1)-p+5}), f(v_{j(2n-1)+k(m-1)-p+6})) = \gcd(k(m-1)+3, 4) = 1, \\
&\gcd(f(v_{j(2n-1)+k(m-1)+1}), f(v_{j(2n-1)+k(m-1)+2})) = \gcd(p-1, 3) = 1.
\end{aligned}$$

So we are done in this subcase.
Sub-case 2: $p - 1 \equiv 0 \pmod 3$ (and hence $p + 1 \not\equiv 0 \pmod 3$).

When $p - 1 \equiv 0 \pmod 3$, we observe that the function f defined above is no more a prime labeling of G because

$$\gcd(f(v_{j(2n-1)+k(m-1)+1}), f(v_{j(2n-1)+k(m-1)+2})) = \gcd(p-1, 3) = 3.$$

To eliminate this problem we modify f by defining $F : V(G) \to \{1, 2, 3, \dots, j(2n-1)+k(m-1)+2\}$ as

$$F(v_i) = \begin{cases} f(v_i), & 1 \le i \le j(2n-1)+2, \\ f(v_{j(2n-1)+k(m-1)+2}), & i = j(2n-1)+3, \\ f(v_{i-1}), & j(2n-1)+4 \le i \le j(2n-1)+k(m-1)+2. \end{cases}$$

The definition of F is illustrated in Figure 4. Note that under F, the label 3 is

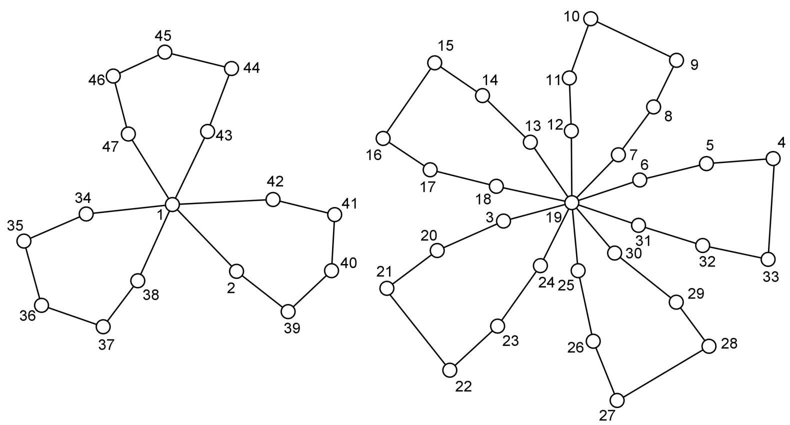

FIGURE 4. Prime Labeling of $C_6^{(3)} \bigcup C_7^{(5)}$

adjacent to p and $p+1$ but not to $p-1$. Since the detailed verification that F, is a prime labeling of G is almost similar to that of f in Sub-case 1, we do not discuss it here. ☐

In view of Theorem 2.1 and Theorem 2.2, we conclude that the necessary and the sufficient condition for the graph $C_n^{(j)} \bigcup C_m^{(k)}$ to be prime is that either n is even or m is even. Our next result gives a necessary condition for a graph to be prime when it is union of three or more $C_n^{(2)}$'s.

Theorem 2.3. *Let* $G = \left(\bigcup_{k=1}^{N} C_{n_k}^{(2)} \right) \bigcup \left(\bigcup_{j=1}^{M} C_{m_j}^{(2)} \right)$*, where each* n_k *is an odd integer and each* m_j *is an even integer. Then* G *is not a prime graph if* $M \le N-2$.

Proof. Since the independence number of each $C_{n_k}^{(2)}$ and each $C_{m_j}^{(2)}$ is $n_k - 1$ and m_j respectively, we have

$$\beta_0(G) = \sum_{k=1}^{N} (n_k - 1) + \sum_{j=1}^{M} m_j$$

$$= \sum_{k=1}^{N} n_k + \sum_{j=1}^{M} m_j - N. \tag{2.1}$$

Also,

$$\begin{aligned}|V(G)| &= (2n_1-1)+(2n_2-1)+\cdots+(2n_N-1)\\ &\quad +(2m_1-1)+(2m_2-1)+\cdots+(2m_M-1)\\ &= \sum_{k=1}^{N}(2n_k)+\sum_{j=1}^{M}(2m_j)-(N+M).\end{aligned}$$

So if $\lceil x\rceil$ denotes the smallest integer $\geq x$ and $\lfloor x\rfloor$ denotes the greatest integer $\leq x$, then

$$\left\lfloor \frac{|V(G)|}{2} \right\rfloor = \sum_{i=1}^{N} n_k + \sum_{j=1}^{M} m_j - \left\lceil \frac{(N+M)}{2} \right\rceil. \tag{2.2}$$

Since $M \leq N-2$, it follows from (2.1) and (2.2) that

$$\beta_0(G) < \left\lfloor \frac{|V(G)|}{2} \right\rfloor .$$

Therefore G is not a prime graph due to Lemma 1.1. □

It is known that $C_{2n} \bigcup C_{2m} \bigcup C_k$ is a prime graph for all n, m and k [6]. We prove the same for the one point union of cycles in our next theorem.

Theorem 2.4. $C_{2n}^{(2)} \bigcup C_{2m}^{(2)} \bigcup C_k^{(2)}$ *is a prime graph for all* n, m *and* k.

Proof. Let $G = C_{2n}^{(2)} \bigcup C_{2m}^{(2)} \bigcup C_k^{(2)}$. Let $\{v_1, v_2, v_3, \ldots, v_{2n}\}$ and $\{v_1, v_{2n+1}, v_{2n+2}, \ldots, v_{4n-1}\}$ be sets of vertices of two cycles of $C_{2n}^{(2)}$, $\{v_{4n}, v_{4n+1}, v_{4n+2}, \ldots, v_{4n+2m-1}\}$ and $\{v_{4n}, v_{4n+2m}, v_{4n+2m+1}, \ldots, v_{4n+4m-2}\}$ be sets of vertices of two cycles of $C_{2m}^{(2)}$ and, $\{v_{4n+4m-1}, v_{4n+4m}, v_{4n+4m+1}, \ldots, v_{4n+4m+k-2}\}$ and $\{v_{4n+4m-1}, v_{4n+4m+k-1}, v_{4n+4m+k}, \ldots, v_{4n+4m+2k-3}\}$ be sets of vertices of two cycles of $C_k^{(2)}$.

Case 1: $n + 2m \not\equiv 0 \pmod 3$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 4n+4m+2k-3\}$ as

$$\begin{aligned} f(v_1) &= 2n+3,\\ f(v_i) &= i+2, \quad i = 2,3,\ldots,2n \quad \text{and} \quad 4n+1, 4n+2, \ldots, 4n+2m-2,\\ f(v_i) &= i+3, \quad i = 2n+1, 2n+2, \ldots, 4n-1,\\ f(v_i) &= i+1, \quad i = 4n+2m, 4n+2m+1, \ldots, 4n+4m-2,\\ f(v_{4n}) &= 2,\\ f(v_{4n+2m-1}) &= 3,\\ f(v_{4n+4m-1}) &= 1,\\ f(v_i) &= i, \quad i = 4n+4m, 4n+4m+1, \ldots, 4n+4m+2k-3.\end{aligned}$$

The definition of f is illustrated in Figure 5. Here the labels of any two adjacent

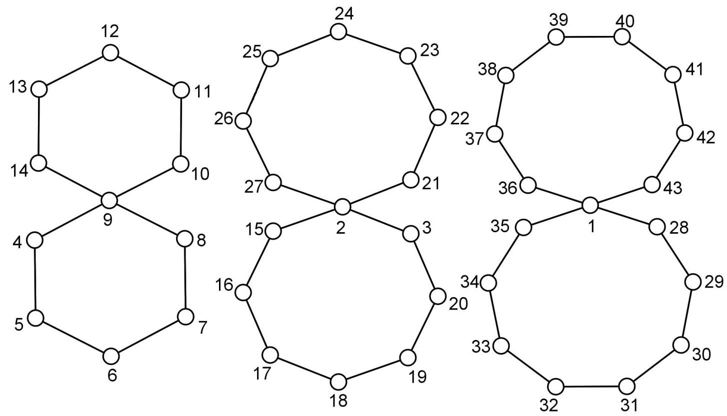

FIGURE 5. Prime Labeling of $C_6^{(2)} \bigcup C_8^{(2)} \bigcup C_9^{(2)}$

vertices of the $C_k^{(2)}$ are either consecutive integers or one of the labels is equal to 1. Further, for the vertices of the $C_{2n}^{(2)}$, we observe that

$$\gcd(f(v_1), f(v_2)) = \gcd(2n+3, 4) = 1,$$
$$\gcd(f(v_1), f(v_{4n-1})) = \gcd(2n+3, 4n+2) = \gcd(2n+3, 2n+1) = 1$$

and except these two pairs, the labels of any other pair of adjacent vertices of $C_{2n}^{(2)}$ are consecutive integers. Further, unless one of the vertices is v_{4n} or $v_{4n+2m-1}$, any two adjacent vertices of the $C_{2m}^{(2)}$ are also consecutive integers. Next, the vertices that are adjacent to v_{4n} are $v_{4n+1}, v_{4n+2m-1}, v_{4n+2m}$ and $v_{4n+4m-2}$ whose labels are $4n+3, 3, 4n+2m+1$ and $4n+4m-1$ respectively, which are all odd numbers, where as $f(v_{4n}) = 2$. Finally, the vertices that are adjacent to $v_{4n+2m-1}$ are v_{4n} and $v_{4n+2m-2}$ whose labels are 2 and $4n+2m$ respectively. But $f(v_{4n+2m-1}) = 3$ and since $n+2m \not\equiv 0 \pmod 3$, we have $\gcd(3, 4n+2m) = 1$. Thus, f is a prime labeling of G if $n+2m \not\equiv 0 \pmod 3$. Note that this f is not a prime labeling when $n+2m \equiv 0 \pmod 3$. So we need to make some changes in f for the resulting function g to be prime labeling for that case.

Case 2: $n+2m \equiv 0 \pmod 3$.

Define $g : V(G) \to \{1, 2, 3, \ldots, 4n+4m+2k-3\}$ as

$$g(w) = f(w), \quad w \neq v_{4n+2m-1}, v_{4n+2m},$$
$$g(v_{4n+2m-1}) = f(v_{4n+2m}),$$
$$g(v_{4n+2m}) = f(v_{4n+2m-1}).$$

The definition of g is illustrated in Figure 6. Note that the labels of v_{4n+2m} and $v_{4n+2m+1}$ are 3 and $4n+2m+2$ respectively. Since $n+2m \equiv 0 \pmod 3$, they are relatively prime. The detailed verification that the given function g defines prime labeling on graph G is similar to that of in Case 1. □

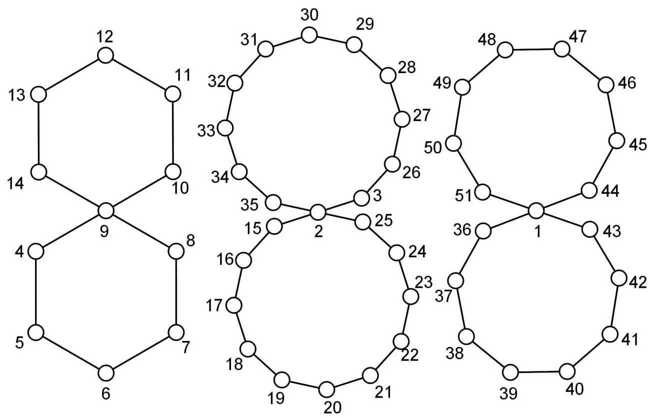

FIGURE 6. Prime Labeling of $C_6^{(2)} \bigcup C_{12}^{(2)} \bigcup C_9^{(2)}$

In view of Lemma 1.1, it is easy to establish that if a graph G is union of three cycles out of which two are odd, then G is not prime. However, if $G = C_{2n}^{(2)} \bigcup C_{2m+1}^{(2)} \bigcup C_{2k+1}^{(2)}$, then $\beta_0(G) = \left\lfloor \frac{|V(G)|}{2} \right\rfloor = 2(n+m+k)$, and so there is a hope for positive results in this case. Our next result gives some of these positive results.

Theorem 2.5. *Let $G = C_{2n}^{(2)} \bigcup C_{2m+1}^{(2)} \bigcup C_{2k+1}^{(2)}$. Then G is a prime graph in each of the following cases:*

(i) $n \equiv 0 \pmod 3$ *and* $m \equiv 0 \pmod 3$ *or* $n \equiv 2 \pmod 3$ *and* $m \equiv 2 \pmod 3$*;*
(ii) $n \equiv 1 \pmod 3$ *and* $m \equiv 2 \pmod 3$*;*
(iii) $n \equiv 1 \pmod 3$ *and* $2m \equiv 1 \pmod 3$ *or* $n \equiv 2 \pmod 3$ *and* $m \equiv 0 \pmod 3$.

Proof. Let $\{v_1, v_2, v_3, \ldots, v_{2n}\}$ and $\{v_1, v_{2n+1}, v_{2n+2}, v_{2n+3}, \ldots, v_{4n-1}\}$ be the vertex sets of the two cycles of $C_{2n}^{(2)}$; $\{v_{4n}, v_{4n+1}, v_{4n+2}, \ldots, v_{4n+2m}\}$ and $\{v_{4n}, v_{4n+2m+1}, v_{4n+2m+2}, v_{4n+2m+3}, \ldots, v_{4n+4m}\}$ be the vertex sets of the two cycles of $C_{2m+1}^{(2)}$ and $\{v_{4n+4m+1}, v_{4n+4m+2}, v_{4n+4m+3}, \ldots, v_{4n+4m+2k+1}\}$ and $\{v_{4n+4m+1}, v_{4n+4m+2k+2}, v_{4n+4m+2k+3}, \ldots, v_{4n+4m+4k+1}\}$ be the vertex sets of the two cycles of $C_{2k+1}^{(2)}$.

Case 1: $n \equiv 0 \pmod 3$ and $m \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 4n+4m+4k+1\}$ as

$$\begin{aligned}
f(v_1) &= 2n+3,\\
f(v_i) &= i+2, \quad i = 2, 3, 4, \ldots, 2n,\\
f(v_i) &= i+3, \quad i = 2n+1, 2n+2, \ldots, 4n-1,\\
f(v_{4n}) &= 3,\\
f(v_{4n+1}) &= 2,\\
f(v_i) &= i+1, \quad i = 4n+2, 4n+3, \ldots, 4n+4m,\\
f(v_{4n+4m+1}) &= 1,\\
f(v_i) &= i, \quad i = 4n+4m+2, 4n+4m+3, \ldots, 4n+4m+4k+1.
\end{aligned}$$

For any two arbitrary vertices u and v of G, we show that $\gcd(f(u), f(v)) = 1$. If this pair of vertices is of $C_{2n}^{(2)}$ or $C_{2k+1}^{(2)}$, then this can be done as in Theorem 2.4 and so we assume that u and v are adjacent vertices of $C_{2m+1}^{(2)}$. Here if u and v are different from the vertex v_{4n} then $\gcd(f(u), f(v)) = 1$ follows because either both are consecutive integers or else one of them is equal to 2 and the other is an odd integer $4n + 3$. Finally, if say $u = v_{4n}$, then $f(u) = 3$, where as $f(v)$ is one of the four values which are $2, 4n+2m+1, 4n+2m+2$ and $4n+4m+1$. But the integer 3 is relatively prime to all of them under the assumptions of the first case and so f is a prime labeling on G. Since this function f may not be a prime labeling of G under the assumptions of second and third cases; we modify the function f to get new prime labelings in the second and the third case as shown below. As the verification is essentially the same we skip the details.

Case 2: $n \equiv 1 \pmod 3$ and $m \equiv 2 \pmod 3$.

Define $g : V(G) \to \{1, 2, 3, \ldots, 4n + 4m + 4k + 1\}$ as

$$g(v_i) = \begin{cases} f(v_i), & i \neq 4n+1, 4n+2, 4n+3, \ldots, 4n+2m, \\ f(v_{i+1}), & i = 4n+1, 4n+2, 4n+3, \ldots, 4n+2m-1, \\ f(v_{4n+1}), & i = 4n+2m. \end{cases}$$

Case 3: $n \equiv 1 \pmod 3$ and $2m \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$ and $m \equiv 0 \pmod 3$.

Define $h : V(G) \to \{1, 2, 3, \ldots, 4n + 4m + 4k + 1\}$ as

$$h(v_i) = \begin{cases} f(v_i), & i \neq 4n+1, 4n+2, 4n+3, \ldots, 4n+4m, \\ f(v_{i+1}), & i = 4n+1, 4n+2, 4n+3, \ldots, 4n+4m-1, \\ f(v_{4n+1}), & i = 4n+4m. \end{cases}$$

□

Our final result is about the union of cycles. In [6] it has been shown that $C_{2n} \bigcup C_{2m} \bigcup C_k$, $C_{2n} \bigcup C_{2n} \bigcup C_{2m} \bigcup C_{2m}$, $C_{2n} \bigcup C_{2n} \bigcup C_{2m} \bigcup C_{2k+1}$ and $C_{2n} \bigcup C_{2n} \bigcup C_{2n} \bigcup C_{2m} \bigcup C_k$ are prime graphs. Here we derive a prime labeling for the union of six cycles.

Theorem 2.6. *$C_{2n} \bigcup C_{2n} \bigcup C_{2n} \bigcup C_{2n} \bigcup C_{2m} \bigcup C_k$ is a prime graph for all n, m and k.*

Proof. Let $G = C_{2n} \bigcup C_{2n} \bigcup C_{2n} \bigcup C_{2n} \bigcup C_{2m} \bigcup C_k$. Let $\{v_1, v_2, \ldots, v_{2n}\}$, $\{v_{2n+1}, v_{2n+2}, \ldots, v_{4n}\}$, $\{v_{4n+1}, v_{4n+2}, \ldots, v_{6n}\}$, $\{v_{6n+1}, v_{6n+2}, \ldots, v_{8n}\}$, $\{v_{8n+1}, v_{8n+2}, \ldots, v_{8n+2m}\}$ and $\{v_{8n+2m+1}, v_{8n+2m+2}, \ldots, v_{8n+2m+k}\}$ be the vertex sets of the six cycles of G.

Define $f : V(G) \to \{1, 2, 3, \ldots, 8n + 2m + k\}$ as

$$\begin{aligned} f(v_i) &= i + 2, \quad i = 1, 2, 3, \ldots, 2n-2, \\ f(v_{2n-1}) &= 4n + 1, \\ f(v_{2n}) &= 6n + 2, \\ f(v_i) &= i, \quad i = 2n+1, 2n+2, \ldots, 4n \quad \text{and} \end{aligned}$$

$$
\begin{aligned}
& && 8n+2m+2, 8n+2m+3, \ldots, 8n+2m+k, \\
f(v_i) &= i+1, && i = 4n+1, 4n+2, \ldots, 6n \quad \text{and} \\
& && 8n+4, 8n+5, \ldots, 8n+2m, \\
f(v_i) &= i+4, && i = 6n+1, 6n+2, \ldots, 8n, \\
f(v_{8n+1}) &= 2, \\
f(v_{8n+2}) &= 6n+3, \\
f(v_{8n+3}) &= 6n+4, \\
f(v_{8n+2m+1}) &= 1.
\end{aligned}
$$

Observe that
$\gcd(f(v_1), f(v_{2n})) = \gcd(3, 6n+2) = 1$,
$\gcd(f(v_{2n-2}), f(v_{2n-1})) = \gcd(2n, 4n+1) = 1$,
$\gcd(f(v_{2n-1}), f(v_{2n})) = \gcd(4n+1, 6n+2) = \gcd(4n+1, 2n+1) = \gcd(1, 2n+1) = 1$,
$\gcd(f(v_{2n+1}), f(v_{4n})) = \gcd(2n+1, 4n) = 1$,
$\gcd(f(v_{4n+1}), f(v_{6n})) = \gcd(4n+2, 6n+1) = \gcd(4n+2, 2n-1) = \gcd(4, 2n-1) = 1$,
$\gcd(f(v_{6n+1}), f(v_{8n})) = \gcd(6n+5, 8n+4) = \gcd(6n+5, 2n-1) = \gcd(8, 2n-1) = 1$,
$\gcd(f(v_{8n+1}), f(v_{8n+2})) = \gcd(2, 6n+3) = 1$,
$\gcd(f(v_{8n+1}), f(v_{8n+2m})) = \gcd(2, 8n+2m+1) = 1$,
$\gcd(f(v_{8n+3}), f(v_{8n+4})) = \gcd(6n+4, 8n+5) = \gcd(6n+4, 2n+1) = \gcd(1, 2n+1) = 1$,
$\gcd(f(v_{8n+2m+1}), f(v_{8n+2m+2})) = \gcd(1, 8n+2m+2) = 1$,
$\gcd(f(v_{8n+2m+1}), f(v_{8m+2m+k})) = \gcd(1, 8n+2m+k) = 1$.
Except these cases every other pair of adjacent vertices have consecutive labels and therefore f is a prime labeling on G. □

3. Conclusion

We have shown that the necessary and sufficient condition for the graph $C_n^{(j)} \bigcup C_m^{(k)}$ to be prime is that at least one of n and m is an even number. Further, it is shown that $C_{2n}^{(2)} \bigcup C_{2m}^{(2)} \bigcup C_k^{(2)}$ is a prime graph for all n, m and k and that $C_{2n}^{(2)} \bigcup C_{2m+1}^{(2)} \bigcup C_{2k+1}^{(2)}$ is a prime graph under certain assumptions on n and m. Although difficult, it will be interesting to prove this result without any assumptions on n and m. We leave it as an open problem. One may also think about generalizing or extending Theorem 2.6 by considering more than 3 variables or by considering more than six cycles.

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[1]Department of Mathematics,
Gujarat University,
Ahmedabad-380009, India
E-mail address: `skpatel27@yahoo.com`
E-mail address: `jayeshvasava1910@gmail.com`