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STABILITY OF CAUCHY-JENSEN TYPE FUNCTIONAL EQUATION IN $(2,\alpha)$-BANACH SPACES

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Abstract. In this paper, we investigate some stability and hyperstability results for the following Cauchy-Jensen functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x)$$

in $(2,\alpha)$-Banach spaces using Brzdęk and Ciepliński's fixed point approach.

1. Introduction

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and the set of real numbers by $\mathbb{R}$. By $\mathbb{N}_m$, $m\in\mathbb{N}$, we will denote the set of all natural numbers greater than or equal to m.

Let $\mathbb{R}_+=[0,\infty)$ be the set of nonnegative real numbers. We write B^A to mean the family of all functions mapping from a nonempty set A into a nonempty set B and we use the notation E_0 for the set $E\backslash\{0\}$.

The method of the proof of the main result corresponds to some observations in [12] and the main tool in it is a fixed point. The problem of the stability of functional equations was first raised by Ulam [30]. This included the following question concerning the stability of group homomorphisms.

Let $(G_1,*_1)$ be a group and let $(G_2,*_2)$ be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d(h(x*_1 y),h(x)*_2 h(y))<\delta,$$

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for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$, with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, we say that the equation of homomorphism

$$h(x *_1 y) = h(x) *_2 H(y)$$

is stable.

Hyers [19] provided the first partial answer to Ulam's question and obtained the result of stability where G_1 and G_2 are Banach spaces.

Aoki [5], Bourgin [7] considered the problem of stability with unbounded Cauchy differences. Later, Rassias [25, 26] used a direct method to prove a generalization of Hyers result (cf. Theorem 1.1).

The following theorem is the most classical result concerning the Hyers-Ulam stability of the Cauchy equation

$$T(x + y) = T(x) + T(y).$$

Theorem 1.1. *Let E_1 be a normed space, E_2 be a Banach space and $f : E_1 \to E_2$ be a function. If f satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta\Big(\|x\|^p + \|y\|^p\Big), \tag{1.1}$$

for some $\theta \geq 0$, for some $p \in \mathbb{R}$, with $p \neq 1$, and for all $x, y \in E_1 - \{0_{E_1}\}$, then there exists a unique additive function $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p, \tag{1.2}$$

for each $x \in E_1 - \{0_{E_1}\}$.

It is due to Aoki [5] (for $0 < p < 1$, see also [24]), Gajda [17] (for $p > 1$) and Rassias [26] (for $p < 0$, see also [27, page 326] and [7]). Also, Brzdęk [8] showed that estimation (1.2) is optimal for $p \geq 0$ in the general case. Recently, Brzdęk [10] showed that Theorem 1.1 can be significantly improved. Namely, in the case $p < 0$, each $f : E_1 \to E_2$ satisfying (1.1) must actually be additive, and the assumption of completeness of E_2 is not necessary.

Regrettably, if we restrict the domain of f, this result will not remain valid (see the further detail in [14]). Nowadays, a lot of papers concerning the stability and the hyperstability of the functional equation in various spaces have been appeared (see in [1, 2, 4, 9, 11, 22, 28, 29] and references therein).

Let us recall first (see, for instance, [16]) some definitions.

We need to recall some basic facts concerning 2-normed spaces and some preliminary results.

Definition 1.1. By a linear 2-normed space, we mean a pair $(X, \|\cdot,\cdot\|)$ such that X is at least a two-dimensional real linear space and

$$\|\cdot,\cdot\| : X \times X \to \mathbb{R}_+$$

is a function satisfying the following conditions:

(a) $\|x,y\| = 0$ if and only if x and y are linearly dependent;
(b) $\|x,y\| = \|y,x\|$ for $x,y \in X$;
(c) $\|x,y+z\| \le \|x,y\| + \|x,z\|$ for $x,y,z \in X$;
(d) $\|\lambda x,y\| = |\lambda|\|x,y\|$, $\lambda \in \mathbb{R}$, and $x,y \in X$.

A generalized version of a linear 2-normed spaces is the $(2,\alpha)$-normed space defined in the following manner.

Definition 1.2. Let α be a fixed real number with $0 < \alpha \le 1$, and let X be a linear space over K with $\dim X > 1$. A function

$$\|\cdot,\cdot\|_\alpha : X \cdot X \to \mathbb{R}$$

is called a $(2,\alpha)$-norm on X if and only if it satisfies the following conditions:
(a) $\|x,y\|_\alpha = 0$ if and only if x and y are linearly dependent;
(b) $\|x,y\|_\alpha = \|y,x\|_\alpha$ for $x,y \in X$;
(c) $\|x,y+z\|_\alpha \le \|x,y\|_\alpha + \|x,z\|_\alpha$ for $x,y,z \in X$;
(d) $\|\beta x,y\|_\alpha = |\beta|^\alpha\|x,y\|_\alpha$ for $\beta \in \mathbb{R}$ and $x,y \in X$.
The pair $(X, \|\cdot,\cdot\|_\alpha)$ is called a $(2,\alpha)$-normed space.

Example 1.1. For $x = (x_1,x_2)$, $y = (y_1,y_2) \in E = \mathbb{R}^2$, the Euclidean $(2,\alpha)$-norm $\|x,y\|_\alpha$ is defined by

$$\|x,y\|_\alpha = |x_1y_2 - x_2y_1|^\alpha,$$

where α is a fixed real number with $0 < \alpha \le 1$.

Definition 1.3. A sequence $\{x_k\}$ in a $(2,\alpha)$-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\|_\alpha = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \to x$, with $k \to \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 1.4. A sequence $\{x_k\}$ in a $(2,\alpha)$-normed space X is said to be a *Cauchy sequence* with respect to the $(2,\alpha)$-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\|_\alpha = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the $(2,\alpha)$-norm. Any complete $(2,\alpha)$-normed space is said to be a $(2,\alpha)$-Banach space.

Next, it is easily seen that we have the following property.

Lemma 1.1. *If X is a linear $(2,\alpha)$-normed space, $x, y_1, y_2 \in X$, y_1, y_2 are linearly independent, and $\|x,y_1\|_\alpha = \|x,y_2\|_\alpha = 0$, then $x = 0$.*

Let us yet recall a lemma from [23].

Lemma 1.2. *If X is a linear $(2,\alpha)$-normed space and $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence of elements of X, then*

$$\lim_{n\to\infty}\|x_n,y\|_\alpha=\|\lim_{n\to\infty}x_n,y\|_\alpha=0,\quad y\in X.$$

Let E,Y be normed spaces. A function $f:E\to Y$ is Cauchy-Jensen provided it satisfies the functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x), \tag{1.3}$$

and we can say that $f:E\to Y$ is Cauchy-Jensen on E_0 if it satisfies (1.3) for all $x,y\in E_0$ such that $\frac{x+y}{2}\neq 0$ and $\frac{x-y}{2}\neq 0$. Recently, interesting results concerning the Cauchy-Jensen functional equation (1.3) have been obtained in [3, 6, 18, 20, 21].

In 2018, Brzdęk and Ciepliński [12] proved a new fixed point theorem in 2-Banach spaces and showed its applications to the Ulam stability of some single-variable equations and the most important functional equation in several variables. And they extended the fixed point result to the n-normed spaces in [13].

The main purpose of this paper is to establish the stability result concerning the functional equation (1.3) in $(2,\alpha)$-Banach spaces using fixed point theorem which was prove by Brzdęk and Ciepliński [12]. Before approaching our main results, we present the fixed point theorem concerning $(2,\alpha)$-Banach spaces which is given in [15]. To present it, we use the following three hypotheses.

(H1) E is a nonempty set, $\left(Y,\|\cdot,\cdot\|_\alpha\right)$ is a $(2,\alpha)$-Banach space, Y_0 is a subset of Y containing two linearly independent vectors, $j\in\mathbb{N}$, $f_i:E\to E$, $g_i:Y_0\to Y_0$, and $L_i:E\times Y_0\to\mathbb{R}_+$ for $i=1,\dots,j$.

(H2) $\mathcal{T}:Y^E\to Y^E$ is an operator satisfying the inequality

$$\left\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x),y\right\|_\alpha\leq\sum_{i=1}^{j}L_i(x,y)\left\|\xi\big(f_i(x)\big)-\mu\big(f_i(x)\big),g_i(y)\right\|_\alpha,$$

for all $\xi,\mu\in Y^E$, $x\in E$, $y\in Y_0$.

(H3) $\Lambda:\mathbb{R}_+^{E\times Y_0}\to\mathbb{R}_+^{E\times Y_0}$ is an operator defined by

$$\Lambda\delta(x,y):=\sum_{i=1}^{j}L_i(x,y)\delta\big(f_i(x),g_i(y)\big),\quad \delta\in\mathbb{R}_+^{E\times Y_0},\ x\in E,\ y\in Y_0.$$

Theorem 1.2 ([15]). *Let hypotheses* (H1)-(H3) *hold and functions $\varepsilon:E\times Y_0\to\mathbb{R}_+$ and $\varphi:E\to Y$ fulfill the following two conditions:*

$$\left\|\mathcal{T}\varphi(x)-\varphi(x),y\right\|_\alpha\leq\varepsilon(x,y),\quad x\in E,\ y\in Y_0,$$

$$\varepsilon^*(x,y):=\sum_{n=0}^{\infty}\left(\Lambda^n\varepsilon\right)(x,y)<\infty,\quad x\in E,\ y\in Y_0.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ for which

$$\left\|\varphi(x)-\psi(x),y\right\|_\alpha\leq\varepsilon^*(x,y),\quad x\in E,\ y\in Y_0.$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n \varphi)(x), \quad x \in E.$$

2. Main Results

In this section, we prove some stability results for the Cauchy-Jensen equation (1.3) in $(2,\alpha)$-Banach spaces by using Theorem 1.2. In what follows $(Y, \|\cdot,\cdot\|_\alpha)$ is a real $(2,\alpha)$-Banach space.

Theorem 2.1. *Let E be a normed space, $(Y, \|\cdot,\cdot\|_\alpha)$ be a real $(2,\alpha)$-Banach space, α be a fixed real number, with $0 < \alpha \le 1$, Y_0 be a subset of Y containing two linearly independent vectors and $h_1, h_2 : E_0 \times Y_0 \to \mathbb{R}_+$ be two functions such that*

$$\mathcal{U} := \{n \in \mathbb{N} \colon b_n := \lambda_1(2+n)\lambda_2(2+n) + \lambda_1(1+n)\lambda_2(1+n) < 1\} \neq \emptyset,$$

where

$$\lambda_i(n) := \inf\left\{t \in \mathbb{R}_+ \colon h_i(nx,z) \le t\, h_i(x,z),\, x \in E_0, z \in Y_0\right\},$$

for all $n \in \mathbb{N}$, where $i = 1,2$. Assume that $f : E \to Y$ satisfies the inequality

$$\left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x), z \right\|_\alpha \le h_1(x,z)h_2(y,z), \tag{2.1}$$

for all $x, y \in E_0$, $z \in Y_0$ such that $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$, then there exists a unique Cauchy-Jensen function $F : E \to Y$ such that

$$\left\| f(x) - F(x), z \right\|_\alpha \le \lambda_0 h_1(x,z)h_2(x,z), \tag{2.2}$$

for all $x \in E_0$, $z \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}} \left\{ \frac{\lambda_1(2+n)\lambda_2(n)}{1-b_n} \right\}.$$

Proof. Replacing x by $(2+m)x$ and y by mx, where $x \in E_0$ and $m \in \mathbb{N}$, in inequality (2.1), we get

$$\left\| f\Big((2+m)x\Big) - f((1+m)x) - f(x), z \right\|_\alpha \le h_1((2+m)x,z)h_2(mx,z), \tag{2.3}$$

for all $x \in E_0$, $z \in Y_0$. For each $m \in \mathbb{N}$, we define the operator $\mathcal{T}_m : Y^{E_0} \to Y^{E_0}$ by

$$\mathcal{T}_m \xi(x) := \xi\Big((2+m)x\Big) - \xi((1+m)x), \quad \xi \in Y^{E_0},\, x \in E_0.$$

Further put

$$\varepsilon_m(x,z) := h_1((2+m)x,z)h_2(mx,z), \quad x \in E_0,\, z \in Y_0, \tag{2.4}$$

and observe that

$$\varepsilon_m(x,z) = h_1((2+m)x,z)h_2(mx,z) \le \lambda_1(2+m)\lambda_2(m)h_1(x,z)h_2(x,z), \tag{2.5}$$

for all $x \in E_0$, $z \in Y_0$, $m \in \mathbb{N}$. Then the inequality (2.3) takes the form

$$\left\| \mathcal{T}_m f(x) - f(x), z \right\|_\alpha \le \varepsilon_m(x,z), \quad x \in E_0,\, z \in Y_0.$$

Furthermore, for every $x \in E_0$, $z \in Y^{E_0}$, $\xi, \mu \in Y^{E_0}$, we obtain

$$\begin{aligned}\Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), z\Big\|_\alpha &= \Big\|\xi\Big((2+m)x\Big) - \xi((1+m)x) \\ &\qquad - \mu\Big((2+m)x\Big) + \mu((1+m)x), z\Big\|_\alpha \\ &\le \Big\|(\xi-\mu)\Big((2+m)x\Big), z\Big\|_\alpha + \Big\|(\xi-\mu)((1+m)x), z\Big\|_\alpha.\end{aligned}$$

This brings us to define the operator $\Lambda_m : \mathbb{R}_+^{E_0 \times Y_0} \to \mathbb{R}_+^{E_0 \times Y_0}$ by

$$\Lambda_m\delta(x,z) := \delta\Big((2+m)x, z\Big) + \delta((1+m)x, z), \quad \delta \in \mathbb{R}_+^{E_0\times Y_0},\ x \in E_0,\ z \in Y_0.$$

For each $m \in \mathbb{N}$ the above operator has the form described in (H2) with $f_1(x) = (2+m)x$, $f_2(x) = (1+m)x$, $g_1(z) = g_2(z) = z$ and $L_1(x) = L_2(x) = 1$ for all $x \in E_0$. By mathematical induction on $n \in \mathbb{N}_0$, we prove that

$$(\Lambda_m^n \varepsilon_m)(x,z) \le \lambda_1(2+m)\lambda_2(m) b_m^n h_1(x,z)h_2(x,z), \tag{2.6}$$

for all $x \in E_0$ and $z \in Y_0$, where

$$b_m = \lambda_1(2+m)\lambda_2(2+m) + \lambda_1(1+m)\lambda_2(1+m).$$

From (2.4) and (2.5), we obtain that the inequality (2.6) holds for $n = 0$. Next, we will assume that (2.6) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}(\Lambda_m^{k+1}\varepsilon_m)(x,z) &= \Lambda_m\left((\Lambda_m^k\varepsilon_m)(x,z)\right) \\ &= (\Lambda_m^k\varepsilon_m)\Big((2+m)x, z\Big) + (\Lambda_m^k\varepsilon_m)((1+m)x, z) \\ &\le \lambda_1(2+m)\lambda_2(m)b_m^k h_1((2+m)x,z)h_2((2+m)x,z) \\ &\quad + \lambda_1(2+m)\lambda_2(m)b_m^k h_1((1+m)x,z)h_2((1+m)x,z) \\ &= \lambda_1(2+m)\lambda_2(m)b_m^{k+1}h_1(x,z)h_2(x,z)\end{aligned}$$

for all $x \in E_0$, $z \in Y_0$, $m \in \mathcal{U}$. This shows that (2.6) holds for $n = k+1$. Now we can conclude that the inequality (2.6) holds for all $n \in \mathbb{N}_0$. Hence, we obtain

$$\begin{aligned}\varepsilon_m^*(x,z) &= \sum_{n=0}^{\infty}(\Lambda_m^n\varepsilon_m)(x,z) \\ &\le \sum_{n=0}^{\infty}\lambda_1(2+m)\lambda_2(m)b_m^n h_1(x,z)h_2(x,z) \\ &= \frac{\lambda_1(2+m)\lambda_2(m)}{1-b_m}h_1(x,z)h_2(x,z) < \infty,\end{aligned}$$

for all $x \in E_0$, $z \in Y_0$, $m \in \mathcal{U}$. Therefore, according to Theorem 1.2 with $\varphi = f$, we get that the limit

$$F_m(x) := \lim_{n\to\infty}\Big(\mathcal{T}_m^n f\Big)(x)$$

exists for each $x \in E_0$ and $m \in \mathcal{U}$, and

(2.7)
$$\left\| f(x) - F_m(x), z \right\|_\alpha \le \frac{\lambda_1(2+m)\lambda_2(m)h_1(x,z)h_2(x,z)}{1-b_m}, \quad x \in E_0,\ z \in Y_0,\ m \in \mathcal{U}.$$

To prove that F_m satisfies the functional equation (1.3), just prove the following inequality

$$\left\| (\mathcal{T}_m^n f)\Big(\frac{x+y}{2}\Big) + (\mathcal{T}_m^n f)\Big(\frac{x-y}{2}\Big) - (\mathcal{T}_m^n f)(x), z \right\|_\alpha \le b_m^n h_1(x,z)h_2(y,z), \tag{2.8}$$

for every $x, y \in E_0$, $z \in Y_0$, $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Since the case $n = 0$ is just (2.1), take $k \in \mathbb{N}$ and assume that (2.8) holds for $n = k$. Then, for each $x, y \in E_0$, $z \in Y_0$ and $m \in \mathcal{U}$, we have

$$\begin{aligned}
&\left\| (\mathcal{T}_m^{k+1} f)\Big(\frac{x+y}{2}\Big) + (\mathcal{T}_m^{k+1} f)\Big(\frac{x-y}{2}\Big) - (\mathcal{T}_m^{k+1} f)(x), z \right\|_\alpha \\
=&\Big\| \mathcal{T}_m^k f\Big((2+m)\Big(\frac{x+y}{2}\Big)\Big) - \mathcal{T}_m^k f\Big((1+m)\Big(\frac{x+y}{2}\Big)\Big) \\
&+ \mathcal{T}_m^k f\Big((2+m)\Big(\frac{x-y}{2}\Big)\Big) - \mathcal{T}_m^k f\Big((1+m)\Big(\frac{x-y}{2}\Big)\Big) \\
&- \mathcal{T}_m^k f\Big((2+m)x\Big) + \mathcal{T}_m^k f((1+m)x), z \Big\|_\alpha \\
\le&\Big\| \mathcal{T}_m^k f\Big((2+m)\Big(\frac{x+y}{2}\Big)\Big) + \mathcal{T}_m^k f\Big((2+m)\Big(\frac{x-y}{2}\Big)\Big) \\
&- \mathcal{T}_m^k f\Big((2+m)x\Big), z \Big\|_\alpha + \Big\| \mathcal{T}_m^k f\Big((1+m)\Big(\frac{x+y}{2}\Big)\Big) \\
&+ \mathcal{T}_m^k f\Big((1+m)\Big(\frac{x-y}{2}\Big)\Big) - \mathcal{T}_m^k f((1+m)x), z \Big\|_\alpha \\
\le& b_m^k h_1\Big((2+m)x, z\Big) h_2\Big((2+m)y, z\Big) + b_m^k h_1\Big((1+m)x, z\Big) h_2\Big((1+m)y, z\Big) \\
=& b_m^{k+1} h_1(x,z)h_2(y,z).
\end{aligned}$$

Thus, by using the mathematical induction on $n \in \mathbb{N}_0$, we have shown that (2.8) holds for all $x, y \in E_0$, $z \in Y_0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Letting $n \to \infty$ in (2.8), we obtain the equality

$$F_m\Big(\frac{x+y}{2}\Big) + F_m\Big(\frac{x-y}{2}\Big) = F_m(x),$$

for all $x, y \in E_0$, such that $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$, $m \in \mathcal{U}$. This implies that $F_m : E \to Y$, defined in this way, is a solution of the equation

$$F(x) = F\Big((2+m)x\Big) - F((1+m)x), \quad x \in E_0,\ m \in \mathcal{U}. \tag{2.9}$$

Next, we will prove that each Cauchy-Jensen function $F : E \to Y$ satisfying the inequality

$$\left\| f(x) - F(x), z \right\|_\alpha \le L\, h_1(x,z)h_2(x,z), \quad x \in E_0,\ z \in Y_0 \tag{2.10}$$

with some $L > 0$, is equal to F_m for each $m \in \mathcal{U}$. To this end, we fix $m_0 \in \mathcal{U}$ and $F : E \to Y$ satisfying (2.10). From (2.7), for each $x \in E$, we get

$$\begin{aligned}\left\|F(x) - F_{m_0}(x), z\right\|_\alpha \leq& \left\|F(x) - f(x), z\right\|_\alpha + \left\|f(x) - F_{m_0}(x), z\right\|_\alpha \\ \leq& L\ h_1(x,z)h_2(x,z) + \varepsilon^*_{m_0}(x,z) \\ \leq& L_0\ h_1(x,z)h_2(x,z) \sum_{n=0}^{\infty} b^n_{m_0}, \end{aligned} \tag{2.11}$$

where $L_0 := (1 - b_{m_0})L + \lambda_1(m_0)\lambda_2(m_0) > 0$ and we exclude the case that $h_1(x,z) \equiv 0$ or $h_2(x,z) \equiv 0$, which is trivial. Observe that F and F_{m_0} are solutions to equation (2.9) for all $m \in \mathcal{U}$. Next, we show that, for each $j \in \mathbb{N}_0$, we have

$$\left\|F(x) - F_{m_0}(x), z\right\|_\alpha \leq L_0\ h_1(x,z)h_2(x,z) \sum_{n=j}^{\infty} b^n_{m_0}, \qquad x \in E_0,\ z \in Y_0. \tag{2.12}$$

The case $j = 0$ is exactly (2.11). We fix $k \in \mathbb{N}$ and assume that (2.12) holds for $j = k$. Then, in view of (2.11), for each $x \in E_0$, $z \in Y_0$, we get

$$\begin{aligned}\left\|F(x) - F_{m_0}(x), z\right\|_\alpha =& \Big\|F\Big((2+m_0)x\Big) - F((1+m_0)x) \\ &- F_{m_0}\Big((2+m_0)x\Big) + F_{m_0}((1+m_0)x), z\Big\|_\alpha \\ \leq& \left\|F\Big((2+m_0)x\Big) - F_{m_0}\Big((2+m_0)x\Big), z\right\|_\alpha \\ &+ \left\|F((1+m_0)x) - F_{m_0}((1+m_0)x), z\right\|_\alpha \\ \leq& L_0\ h_1\Big((2+m_0)x, z\Big)h_2\Big((2+m_0)x, z\Big) \sum_{n=k}^{\infty} b^n_{m_0} \\ &+ L_0\ h_1\Big((1+m_0)x, z\Big)h_2\Big((1+m_0)x, z\Big) \sum_{n=k}^{\infty} b^n_{m_0} \\ =& L_0\ \Big(h_1\Big((2+m_0)x, z\Big)h_2\Big((2+m_0)x, z\Big) \\ &+ h_1\Big((1+m_0)x, z\Big)h_2\Big((1+m_0)x, z\Big)\Big) \sum_{n=k}^{\infty} b^n_{m_0} \\ \leq& L_0\ b_{m_0} h_1(x,z)h_2(x,z) \sum_{n=k}^{\infty} b^n_{m_0} \\ =& L_0\ h_1(x,z)h_2(x,z) \sum_{n=k+1}^{\infty} b^n_{m_0}. \end{aligned}$$

This shows that (2.12) holds for $j = k+1$. Now we can conclude that the inequality (2.12) holds for all $j \in \mathbb{N}_0$. Now, letting $j \to \infty$ in (2.12), we get

$$F = F_{m_0}. \tag{2.13}$$

Thus, we have also proved that $F_m = F_{m_0}$ for each $m \in \mathcal{U}$, which (in view of (2.7)) yields

$$\left\|f(x) - F_{m_0}(x), z\right\|_\alpha \leq \frac{\lambda_1(2+m)\lambda_2(m)h_1(x,z)h_2(x,z)}{1-b_m},$$

for all $x \in E_0, z \in Y_0,\ m \in \mathcal{U}$. This implies (2.2) with $F = F_{m_0}$ and (2.13) confirms the uniqueness of F. $\square$

Theorem 2.2. *Let E be a normed space, $(Y, \|\cdot,\cdot\|_\alpha)$ be a real $(2,\alpha)$-Banach space, α be a fixed real number with $0 < \alpha \leq 1$, Y_0 be a subset of Y containing two linearly independent vectors and $h : E_0 \times Y_0 \to \mathbb{R}_+$ be a functions such that*

$$\mathcal{U} := \{n \in \mathbb{N}\colon \beta_n := \lambda(2+n) + \lambda(1+n) < 1\} \neq \emptyset,$$

where

$$\lambda(n) := \inf\left\{t \in \mathbb{R}_+\colon h(nx,z) \leq t\, h(x,z),\ x \in E_0,\ z \in Y_0\right\},$$

for all $n \in \mathbb{N}$. Assume that $f : E \to Y$ satisfies the inequality

$$\left\|f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x), z\right\|_\alpha \leq h(x,z) + h(y,z), \tag{2.14}$$

for all $x, y \in E_0$, $z \in Y_0$ such that $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$. Then there exists a unique Cauchy-Jensen function $F : E \to Y$ such that

$$\left\|f(x) - F(x), z\right\|_\alpha \leq \lambda_0 h(x,z),$$

for all $x \in E_0, z \in Y_0$, where

$$\lambda_0 := \inf_{n \in \mathcal{U}}\left\{\frac{\lambda(2+n) + \lambda(n)}{1 - \lambda(2+n) - \lambda(1+n)}\right\}.$$

Proof. Replacing x with $(2+m)x$ and y with mx, where $x \in E_0$ and $m \in \mathbb{N}$, in inequality (2.14), we get

$$\left\|f\Big((2+m)x\Big) - f\Big((1+m)x\Big) - f(x), z\right\|_\alpha \leq h((2+m)x,z) + h(mx,z), \tag{2.15}$$

for all $x \in E_0$, $z \in Y_0$. For each $m \in \mathbb{N}$, we define the operator $\mathcal{T}_m : Y^{E_0} \to Y^{E_0}$ by

$$\mathcal{T}_m\xi(x) := \xi\Big((2+m)x\Big) - \xi\Big((1+m)x\Big), \quad \xi \in Y^{E_0},\ x \in E_0.$$

Further put

$$\varepsilon_m(x,z) := h((2+m)x,z) + h(mx,z), \quad x \in E_0,\ z \in Y_0, \tag{2.16}$$

and observe that

$$\varepsilon_m(x,z) = \Big(h((2+m)x,z) + h(mx,z)\Big) \leq (\lambda(2+m) + \lambda(m))h(x,z), \quad m \in \mathbb{N}. \tag{2.17}$$

Then the inequality (2.15) takes the form

$$\left\|\mathcal{T}_m f(x) - f(x), z\right\|_\alpha \leq \varepsilon_m(x,z), \quad x \in E_0,\ z \in Y_0.$$

Furthermore, for every $x \in E_0, z \in Y_0$, $\xi, \mu \in Y^{E_0}$, we obtain

$$\begin{aligned}\Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), z\Big\|_\alpha =& \Big\|\xi\Big((2+m)x\Big) - \xi\Big((1+m)x\Big) \\ &- \mu\Big((2+m)x\Big) + \mu\Big((1+m)x\Big), z\Big\|_\alpha \\ \leq& \Big\|(\xi-\mu)\Big((2+m)x\Big), z\Big\|_\alpha + \Big\|(\xi-\mu)\Big((1+m)x\Big), z\Big\|_\alpha.\end{aligned}$$

This brings us to define the operator $\Lambda_m : \mathbb{R}_+^{E_0\times Y_0} \to \mathbb{R}_+^{E_0\times Y_0}$ by

$$\Lambda_m\delta(x,z) := \delta\Big((2+m)x, z\Big) + \delta\Big((1+m)x, z\Big), \quad \delta \in \mathbb{R}_+^{E_0\times Y_0},\ x \in E_0,\ z \in Y_0.$$

For each $m \in \mathbb{N}$ the above operator has the form described in (H2) with $f_1(x) = (2+m)x$, $f_2(x) = (1+m)x$, $g_1(z) = g_2(z) = z$ and $L_1(x) = L_2(x) = 1$ for all $x \in X$. By mathematical induction on $n \in \mathbb{N}_0$, we prove that

$$(\Lambda_m^n \varepsilon_m)(x,z) \leq (\lambda(2+m) + \lambda(m))\beta_m^n h(x,z), \tag{2.18}$$

for all $x \in E_0$ and $z \in Y_0$, where

$$\beta_m := \lambda(2+m) + \lambda(1+m).$$

From (2.16) and (2.17), we obtain that the inequality (2.18) holds for $n = 0$. Next, we will assume that (2.18) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}(\Lambda_m^{k+1}\varepsilon_m)(x,z) =& \Lambda_m\left((\Lambda_m^k\varepsilon_m)(x,z)\right) \\ =& (\Lambda_m^k\varepsilon_m)\Big((2+m)x, z\Big) + (\Lambda_m^k\varepsilon_m)\Big((1+m)x, z\Big) \\ \leq& \Big((\lambda(2+m) + \lambda(m))\beta_m^k h((2+m)x, z) \\ &+ (\lambda(2+m) + \lambda(m))\beta_m^k h((1+m)x, z)\Big) \\ =& (\lambda(2+m) + \lambda(m))\beta_m^{k+1} h(x,z),\end{aligned}$$

for all $x \in E_0$, $z \in Y_0$, $m \in \mathcal{U}$. This shows that (2.18) holds for $n = k+1$. Now we can conclude that the inequality (2.18) holds for all $n \in \mathbb{N}_0$. Hence, we obtain

$$\begin{aligned}\varepsilon_m^*(x,z) &= \sum_{n=0}^{\infty}(\Lambda_m^n\varepsilon_m)(x,z) \\ &\leq \sum_{n=0}^{\infty}(\lambda(2+m) + \lambda(m))\beta_m^n h(x,z) \\ &= \frac{(\lambda(2+m) + \lambda(m))h(x,z)}{(1-\beta_m)} < \infty,\end{aligned}$$

for all $x \in E_0$, $z \in Y_0$, $m \in \mathcal{U}$. Therefore, according to Theorem 1.2 with $\varphi = f$, we get that the limit

$$F_m(x) := \lim_{n\to\infty}\left(\mathcal{T}_m^n f\right)(x)$$

exists for each $x \in E_0$ and $m \in \mathcal{U}$, and

$$\Big\|f(x) - F_m(x), z\Big\|_\alpha \le \frac{(\lambda(2+m)+\lambda(m))h(x,z)}{(1-\beta_m)}, \quad x \in E_0,\ z \in Y_0,\ m \in \mathcal{U}.$$

By a similar method in the proof of Theorem 2.1, we show that

$$\Big\|(\mathcal{T}_m^n f)\big(x+y\big) + (\mathcal{T}_m^n f)\big(x-y\big) - (\mathcal{T}_m^n f)(x), z\Big\|_\alpha \le \beta_m^n(h(x,z)+h(y,z)),$$

for every $x, y \in E_0, z \in Y_0, n \in \mathbb{N}_0$ and $m \in \mathcal{U}$. Also, the remaining reasonings are analogous as in the proof of that theorem. □

3. Applications

According to above theorems, we can obtain the following corollaries for the hyperstability results of the Cauchy-Jensen equation (1.3) in $(2,\alpha)$-Banach spaces.

Corollary 3.1. *Let E be a normed space, $(Y, \|\cdot,\cdot\|_\alpha)$ be a real $(2,\alpha)$-Banach space, α be a fixed real number with $0 < \alpha \le 1$, Y_0 be a subset of Y containing two linearly independent vectors and h_1, h_2, and $\mathcal{U}$ be as in Theorem* 2.1. *Assume that*

$$\begin{cases} \lim_{n\to\infty} \lambda_1(2+n)\lambda_2(n) = 0, \\ \lim_{n\to\infty} \lambda_1(2+n)\lambda_2(2+n) = \lim_{n\to\infty} \lambda_1(1+n)\lambda_2(1+n) = 0. \end{cases} \tag{3.1}$$

Then every function $f : E \to Y$ satisfying (2.1) *is a solution of* (1.3) *on E_0.*

Proof. Suppose that $f : E \to Y$ satisfies (2.1). Then, by Theorem 2.1, there exists a function $F : E \to Y$ satisfying (1.3) and

$$\|f(x) - F(x), z\|_\alpha \le \lambda_0 h_1(x,z)h_2(x,z),$$

for all $x \in E_0, z \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}} \left\{\frac{\lambda_1(2+n)\lambda_2(n)}{1-b_n}\right\}.$$

By (3.1), $\lambda_0 = 0$. This means that $f(x) = F(x)$ for all $x \in E_0$, whence

$$f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{x-y}{2}\Big) = f(x),$$

for all $x, y \in E_0$ such that $\frac{x+y}{2} \ne 0$ and $\frac{x-y}{2} \ne 0$, which implies that f satisfies the functional equation (1.3) on E_0. □

Corollary 3.2. *Let E be a normed space, $(Y, \|\cdot,\cdot\|_\alpha)$ be a real $(2,\alpha)$-Banach space, α be a fixed real number with $0 < \alpha \le 1$, Y_0 be a subset of Y containing two linearly independent vectors and h_1 and $\mathcal{U}$ be as in Theorem* 2.2. *Assume that*

$$\begin{cases} \lim_{n\to\infty}(\lambda_1(2+n) + \lambda_2(n)) = 0, \\ \lim_{n\to\infty}(\lambda_1(2+n) + \lambda_2(1+n)) = 0. \end{cases} \tag{3.2}$$

Then every function $f : E \to Y$ satisfying (2.14) *is a solution of* (1.3) *on E_0.*

Proof. Suppose that $f : E \to Y$ satisfies (2.14). Then, by Theorem 2.2, there exists a function $F : E \to Y$ satisfying (1.3) and

$$\|f(x) - F(x), z\|_\alpha \leq \lambda_0 h(x, z)$$

for all $x \in E_0, z \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathfrak{U}} \left\{ \frac{\lambda_1(2 + n) + \lambda_2(n)}{1 - \beta_n} \right\}.$$

By (3.2), $\lambda_0 = 0$. This means that $f(x) = F(x)$ for all $x \in E_0$, whence

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x),$$

for all $x, y \in E_0$ such that $\frac{x+y}{2} \neq 0$ and $\frac{x-y}{2} \neq 0$, which implies that f satisfies the functional equation (1.3) on E_0. $\square$

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