EVEN VERTEX EQUITABLE EVEN LABELING FOR CYCLE RELATED GRAPHS

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Abstract. Let $G$ be a graph with $p$ vertices and $q$ edges and $A = \{0, 2, 4, \ldots, q+1\}$ if $q$ is odd or $A = \{0, 2, 4, \ldots, q\}$ if $q$ is even. A graph $G$ is said to be an even vertex equitable even labeling if there exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling $f^* : E(G) \rightarrow A$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$ such that for all $a$ and $b$ in $A$, $|v_{f}(a) - v_{f}(b)| \leq 1$ and the induced edge labels are $2, 4, \ldots, 2q$, where $v_{f}(a)$ be the number of vertices $v$ with $f(v) = a$ for $a \in A$. A graph that admits even vertex equitable even labeling is called an even vertex equitable even graph. In this paper, we prove that the graphs $C_m \odot P_n$, $C_n(Q_m)$ if $n \equiv 0, 3 \pmod{4}$, $\left[ P_n; C_m^{(2)} \right]$ if $m \equiv 0 \pmod{4}$, $C_m \ast_e C_n$ and the graph obtained by duplicating an arbitrary vertex and edge of a cycle $C_n$ admit an even vertex equitable even labeling.

1. Introduction

All graphs considered here are simple, finite, connected and undirected. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. We follow the basic notations and terminology of graph theory as in [3]. A labeling of a graph is a map that carries the graph elements to the set of numbers, usually to the set of non-negative or positive integers. If the domain is the set of vertices the labeling is called vertex labeling. If the domain is the set of edges, then we speak about edge labeling. If the labels are assigned to both vertices and edges then the labeling is called total labeling. For a dynamic survey of various graph labeling, we refer to Gallian [2].

Lourdusamy et al. introduced the concept of vertex equitable labeling in [13]. Let $G$ be a graph with $p$ vertices and $q$ edges and let $A = \{0, 1, 2, \ldots, \frac{q}{2} \}$. A vertex
vertex labeling

Let admit an even vertex equitable even labeling.

In [13], Lourdusamy et al. proved that graphs like path, bistar, comb graph, cycle, equitable labeling. In addition, they proved that if graph $K_1$ for $n \leq 3$ (mod 4), the wheel $W_n$, the complete graph $K_n$ if $n \geq 3$ and triangular cactus with $q \equiv 6, 9$ (mod 12) are not vertex equitable graphs. In addition, they proved that if $G$ is a graph with $p$ vertices and $q$ edges, $q$ is even and $p < \left[ \frac{q}{2} \right] + 2$ then $G$ is not vertex equitable graph.

In [4–11], Jeyanthi et al. proved that $T_p$-tree, $T \odot \tilde{K}_n$, where $T$ is a $T_p$-tree with even number of vertices, $B(n, n + 1)$, the caterpillar $S(x_1, x_2, \ldots, x_n)$, crown, $P_n^2$, $T \odot P_n$, $T \odot 2P_n$, $T \odot C_n$ with $n \equiv 0, 3$ (mod 4) and $T \odot C_n$ with $n \equiv 0, 3$ (mod 4), tadpoles, $C_m \oplus C_n$, armed crowns, $[P_m; C_n^q]$, $[P_m \partial K_{1,n}]$, jewel graph $J_n$, jelly fish graph $(JF)_n$, balanced lobster graph $BL(n, 2, m)$, $L_n \odot \tilde{K}_m$, $\{L_n \partial K_{1,m}\}$, $DA(T_n) \odot K_1$, $DA(T_n) \odot 2K_1$, $DA(Q_n) \odot K_1$, $DA(Q_n) \odot 2K_1$, $S^*(P_n \odot K_1)$, $S^*(B(n, n))$, $S^*(P_n \times P_2)$, $S^*(Q_n)$, $S(D(T_n))$, $S(D(Q_n))$, $S(DA(T_n))$, $S(DA(Q_n))$, $DA(Q_m) \odot nK_1$, $DA(T_m) \odot nK_1$, $KY(m, n)$, $P(2.QS_n)$, $P(m.QS_n)$, $C(n.QS_m)$, $NQ(m)$ and $K_{1,n} \times P_2$ admit vertex equitable labeling.

Motivated by the concept of vertex equitable labeling, Lourdusamy et al. introduced the concept of even vertex equitable even labeling in [12]. In [12], they proved that path, comb, complete bipartite, cycle, $K_2 + mK_1$, bistar, ladder, $S(L_n)$, $S(B_{n,n})$ and $L_n \odot K_1$ admit an even vertex equitable even labeling. In this paper, we prove that the graphs $C_m \oplus P_n$, $C_m(Q_m)$ if $n \equiv 0, 3$ (mod 4), $[P_m; C_m^{(2)}]$ if $m \equiv 0$ (mod 4), $C_m *_e C_n$ and the graph obtained by duplicating an arbitrary vertex and edge of a cycle $C_n$ admit an even vertex equitable even labeling.

**Definition 1.1.** [13] Let $G$ be a graph with $p$ vertices and $q$ edges and let $A = \{0, 1, 2, \ldots, \left[ \frac{q}{2} \right] \}$. A vertex labeling $f : V(G) \rightarrow A$ induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$. For $a \in A$, let $v_f(a)$ be the number of vertices $v$ with $f(v) = a$. A graph $G$ is said to be vertex equitable if there exists a vertex labeling $f$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$.

**Definition 1.2.** [12] Let $G$ be a graph with $p$ vertices and $q$ edges and $A = \{0, 2, 4, \ldots, q + 1\}$ if $q$ is odd or $A = \{0, 2, 4, \ldots, q\}$ if $q$ is even. A graph $G$ is said to be an even vertex equitable even labeling if there exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \ldots, 2q$.
where \( v_f(a) \) be the number of vertices \( v \) with \( f(v) = a \) for \( a \in A \). A graph that admits even vertex equitable even labeling is called an even vertex equitable even graph.

**Definition 1.3.** [6] An armed crown is a cycle attached with paths of equal lengths at each vertex of the cycle. It is denoted by \( C_m \ominus P_n \), where \( P_n \) is a path of length \( n - 1 \).

**Definition 1.4.** [14] The graph \( C_n(Q_m) \) is defined as isomorphic quadrilateral snake attached with each vertex of cycle \( C_n \). 'm' is the number of \( C_4 \) attached in the quadrilateral snake.

**Definition 1.5.** [1] Let \( G \) be a graph with fixed vertex \( v \) and let \( [P_n; G] \) be the graph obtained from \( n \) copies of \( G \) by joining \( v_{ij} \) and \( v_{i+1j} \) by means of an edge, for some \( j \) and \( 1 \leq i \leq n - 1 \).

**Definition 1.6.** [6] Let \( G_1 \) and \( G_2 \) be two graphs. The identification of \( G_1 \) and \( G_2 \) is a graph obtained by identifying an edge of \( G_1 \), with an edge of \( G_2 \) and is denoted by \( G_1 \ast_e G_2 \).

**Definition 1.7.** [6] Let \( G \) be a graph and \( v \) be any vertex of \( G \). A new vertex \( v' \) is said to be duplication of \( v \) if all the vertices which are adjacent to \( v \) are adjacent to \( v' \). The graph obtained by duplication \( v \) is denoted by \( D(G, v') \).

**Definition 1.8.** [6] Let \( G \) be a graph and \( e \) be any edge of \( G \). A new edge \( e' \) is said to be duplication of an edge \( e \) if all the edges which are incident to \( e \) in \( G \) are incident to \( e' \). The graph obtained by duplication \( e \) is denoted by \( D(G, e') \).

2. Main Results

**Theorem 2.1.** The graph \( C_m \ominus P_n \) is an even vertex equitable even graph.

*Proof.* Let \( u_{11}, u_{21}, \ldots, u_{m1} \) be the vertices of the cycle \( C_m \) and \( u_{11}u_{12} \ldots u_{1n}, u_{21}u_{22}u_{23} \ldots u_{2n}, \ldots, u_{m1}u_{m2} \ldots u_{mn} \) be the vertices of the path \( P_n \) attached with \( u_{i1} \) by identifying \( u_{ij} \) with \( u_{i1} \) for \( 1 \leq i \leq m, 1 \leq j \leq n \). Then \( C_m \ominus P_n \) is of order \( mn \) and size \( mn \).

Define \( f : V(C_m \ominus P_n) \rightarrow A = \{0, 2, \ldots, mn + 1, \quad \text{if } mn \text{ is odd},
0, 2, \ldots, mn, \quad \text{if } mn \text{ is even}, \)

as follows.

**Case 1.** \( m \equiv 0, 3 \pmod{4} \).

For \( 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \),

\[
f(u_{ij}) = \begin{cases} 
  n(i-1) + j - 1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\
  n(i-1) + j, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n,
\end{cases}
\]

\[
f(u_{ij}) = \begin{cases} 
  ni - j + 1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\
  ni - j, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n.
\end{cases}
\]
For $\left\lfloor \frac{m}{2} \right\rfloor + 1 \leq i \leq m$,

$$f(u_{ij}) = \begin{cases} 
  n(i - 1) + j + 1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\
  n(i - 1) + j, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n,
\end{cases}$$

$$f(u_{ij}) = \begin{cases} 
  ni - j + 1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\
  ni - j + 2, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n.
\end{cases}$$

**Case 2.** $m \equiv 1, 2 \pmod{4}$ and $n$ is odd.

For $1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$,

$$f(u_{ij}) = \begin{cases} 
  ni - j, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\
  ni - j + 1, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n,
\end{cases}$$

$$f(u_{ij}) = \begin{cases} 
  n(i - 1) + j, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\
  n(i - 1) + j - 1, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n.
\end{cases}$$

For $i = \left\lceil \frac{m}{2} \right\rceil + 1$,

$$f(u_{ij}) = \begin{cases} 
  n(i - 1) + j + 2, & \text{if } j \text{ is odd and } 1 \leq j \leq n, \\
  n(i - 1) + j - 1, & \text{if } j \text{ is even and } 1 \leq j \leq n.
\end{cases}$$

For $i = \left\lceil \frac{m}{2} \right\rceil + 2$,

$$f(u_{ij}) = \begin{cases} 
  ni - j + 2, & \text{if } j \text{ is odd and } 1 \leq j \leq n - 1, \\
  ni - j + 1, & \text{if } j \text{ is even and } 1 \leq j \leq n - 1, \\
  ni - j, & \text{if } j = n.
\end{cases}$$

For $\left\lceil \frac{m}{2} \right\rceil + 3 \leq i \leq m$, 

$$f(u_{ij}) = \begin{cases} 
  n(i - 1) + j, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\
  n(i - 1) + j + 1, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n,
\end{cases}$$

$$f(u_{ij}) = \begin{cases} 
  ni - j + 2, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\
  ni - j + 1, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n.
\end{cases}$$

**Case 3.** $m \equiv 2 \pmod{4}$ and $n$ is even.

For $1 \leq i \leq \frac{m}{2}$,

$$f(u_{ij}) = \begin{cases} 
  n(i - 1) + j - 1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\
  n(i - 1) + j, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n,
\end{cases}$$

$$f(u_{ij}) = \begin{cases} 
  ni - j + 1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\
  ni - j, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n.
\end{cases}$$
For $i = \frac{m}{2} + 1$,

$$f(v_{ij}) = \begin{cases} 
ni + 2j, & \text{if } j = 1, \\
ni - j + 2, & \text{if } j \text{ is even and } 2 \leq j \leq n, \\
ni - j + 1, & \text{if } j \text{ is odd and } 2 \leq j \leq n.
\end{cases}$$

For $i = \frac{m}{2} + 2$,

$$f(v_{ij}) = \begin{cases} 
n(i - 1) + j + 1, & \text{if } j \text{ is odd and } 1 \leq j \leq n - 1, \\
n(i - 1) + j + 2, & \text{if } j \text{ is even and } 1 \leq j \leq n - 1, \\
\frac{nm}{2}, & \text{if } j = n.
\end{cases}$$

For $\frac{m}{2} + 3 \leq i \leq m$,

$$f(v_{ij}) = \begin{cases} 
ni - j + 1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\
ni - j + 2, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n, \\
n(i - 1) + j + 1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\
n(i - 1) + j, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n.
\end{cases}$$

**Case 4.** $m \equiv 1 \pmod{4}$ and $n$ is even.

For $1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$,

$$f(v_{ij}) = \begin{cases} 
ni - j + 1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\
ni - j, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n, \\
n(i - 1) + j + 1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\
n(i - 1) + j, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n.
\end{cases}$$

For $\left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m$,

$$f(v_{ij}) = \begin{cases} 
n(i - 1) + j + 1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\
n(i - 1) + j, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n, \\
ni - j + 1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\
ni - j + 2, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n.
\end{cases}$$

In above four cases, it can be verified that the induced edge labels of $C_m \odot P_n$ are 2, 4, . . . , 2mn and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $C_m \odot P_n$ is an even vertex equitable even graph.

**Theorem 2.2.** The graph $C_n(Q_m)$ is an even vertex equitable even graph if $n \equiv 0, 3 \pmod{4}$.

**Proof.** Let $G = C_n(Q_m)$. Let

$$V(G) = \{u_i : 1 \leq i \leq n\} \cup \{u_{ij}, v_{ij}, w_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\}$$

Thus, $G$ is an even vertex equitable even graph. \hfill \qed
and $E(G) = \{u_iu_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_1u_n\} \cup \{u_iv_{i+1}, u_iw_{i+1} : 1 \leq i \leq n\} \cup \{v_{ij}w_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\} \cup \{u_{ij}v_{ij+1}, u_{ij}w_{ij+1} : 1 \leq i \leq n; 1 \leq j \leq m-1\}$. Then $G$ is of order $3mn + n$ and size $4mn + n$.

**Case 1.** $n \equiv 0 \pmod{4}$.
Define $f : V(G) \rightarrow A = \{0, 2, \ldots, 4mn + n\}$ as follows:

$$f(u_i) = \begin{cases} 
(4m + 1)(i - 1), & \text{if } i \text{ is odd and } 1 \leq i \leq \frac{n}{2}, \\
(4m + 1)(i - 1) + 2, & \text{if } i \text{ is odd and } \frac{n}{2} + 1 \leq i \leq n, \\
(4m + 1)i, & \text{if } i \text{ is even and } 1 \leq i \leq n. 
\end{cases}$$

For $1 \leq j \leq m$,

$$f(u_{ij}) = \begin{cases} 
(4m + 1)(i - 1) + 4j, & \text{if } i \text{ is odd and } 1 \leq i \leq \frac{n}{2}, \\
(4m + 1)(i - 1) + 4j + 2, & \text{if } i \text{ is odd and } \frac{n}{2} + 1 \leq i \leq n, \\
(4m + 1)i - 4j, & \text{if } i \text{ is even and } 1 \leq i \leq n, 
\end{cases}$$

$$f(v_{ij}) = \begin{cases} 
(4m + 1)i - 4j, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n}{2}, \\
(4m + 1)i - 4j + 2, & \text{if } i \text{ is even and } \frac{n}{2} + 1 \leq i \leq n, 
\end{cases}$$

$$f(w_{ij}) = \begin{cases} 
(4m + 1)(i - 1) + 4j, & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\
(4m + 1)i - 4j + 2, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n}{2}, \\
(4m + 1)i - 4j + 4, & \text{if } i \text{ is even and } \frac{n}{2} + 1 \leq i \leq n. 
\end{cases}$$

**Case 2.** $n \equiv 3 \pmod{4}$.
Define $f : V(G) \rightarrow A = \{0, 2, \ldots, 4mn + n + 1\}$ as follows:

$$f(u_i) = \begin{cases} 
(4m + 1)i - 1, & \text{if } i \text{ is odd and } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
(4m + 1)i + 1, & \text{if } i \text{ is odd and } \left\lceil \frac{n}{2} \right\rceil \leq i \leq n, \\
(4m + 1)(i - 1) + 1, & \text{if } i \text{ is even and } 1 \leq i \leq n. 
\end{cases}$$

For $1 \leq j \leq m$,

$$f(u_{ij}) = \begin{cases} 
(4m + 1)i - 4j - 1, & \text{if } i \text{ is odd and } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
(4m + 1)i - 4j + 1, & \text{if } i \text{ is odd and } \left\lceil \frac{n}{2} \right\rceil \leq i \leq n, \\
(4m + 1)(i - 1) - 1 + 4j & \text{if } i \text{ is even and } 1 \leq i \leq n, 
\end{cases}$$

$$f(v_{ij}) = \begin{cases} 
(4m + 1)i - 4(j - 1) - 1, & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\
(4m + 1)(i - 1) + 4j - 1, & \text{if } i \text{ is even and } 1 \leq i \leq n, 
\end{cases}$$

$$f(w_{ij}) = \begin{cases} 
(4m + 1)i - 4j + 1, & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\
(4m + 1)(i - 1) + 4(j - 1) + 1, & \text{if } i \text{ is even and } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
(4m + 1)(i - 1) + 4j + 1, & \text{if } i \text{ is even and } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. 
\end{cases}$$
In both the cases, it can be verified that the induced edge labels of $C_n(Q_m)$ are $2, 4, \ldots, 8mn + 2n$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $C_n(Q_m)$ is an even vertex equitable even graph for $n \equiv 0, 3 \pmod{4}$. \hfill \Box

**Theorem 2.3.** The graph $[P_n; C_m^{(2)}]$ is an even vertex equitable even graph if $m \equiv 0 \pmod{4}$.

**Proof.** Let $u_1, u_2, \ldots, u_n$ be the vertices of a path $P_n$ and the vertex $u_i (1 \leq i \leq n)$ is attached with the center vertex of the $i^{th}$ copy of $C_m^{(2)}$. Let $u_i = v_1$ (center vertex in the $i^{th}$ copy of $C_m^{(2)}$). Let $v_{ij} \text{ and } v_{ij}'$, for $1 \leq i \leq n, 2\leq j \leq m$, be the remaining vertices in the $i^{th}$ copy of $C_m^{(2)}$. Then $[P_n; C_m^{(2)}]$ is of order $2mn - n$ and size $2mn + n - 1$.

Define $f : V(G) \rightarrow A = \begin{cases} 0, 2, \ldots, 2mn + n, & \text{if } 2mn + n - 1 \text{ is odd}, \\ 0, 2, \ldots, 2mn + n - 1 & \text{if } 2mn + n - 1 \text{ is even}, \end{cases}$

as follows:

$$f(v_{11}) = \begin{cases} (2m + 1)i - (m + 1), & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\ (2m + 1)i - m, & \text{if } i \text{ is even and } 1 \leq i \leq n, \end{cases}$$

For $1 \leq i \leq n$ and $i$ is odd,

$$f(v_{ij}) = \begin{cases} (2m + 1)i - (m + 1) - j + 1, & \text{if } j \text{ is odd and } 2 \leq j \leq m, \\ (2m + 1)i - (m + 1) - j + 2, & \text{if } j \text{ is even and } 2 \leq j \leq m, \\ (2m + 1)i - (m + 1) - j, & \text{if } j \text{ is even and } \frac{m}{2} + 1 \leq j \leq m, \end{cases}$$

$$f(v_{ij}^{'}) = \begin{cases} (2m + 1)i - (m + 1) + j, & \text{if } j \text{ is even and } 2 \leq j \leq m, \\ (2m + 1)i - (m + 1) + j - 1, & \text{if } j \text{ is odd and } 2 \leq j \leq m, \\ (2m + 1)i - (m + 1) + j + 1, & \text{if } j \text{ is odd and } \frac{m}{2} + 1 \leq j \leq m. \end{cases}$$

For $1 \leq i \leq n$ and $i$ is even,

$$f(v_{ij}) = \begin{cases} (2m + 1)i - (m + 1) - j + 1, & \text{if } j \text{ is even and } 2 \leq j \leq m, \\ (2m + 1)i - (m + 1) - j + 2, & \text{if } j \text{ is odd and } 2 \leq j \leq m, \\ (2m + 1)i - (m + 1) - j, & \text{if } j \text{ is odd and } \frac{m}{2} + 1 \leq j \leq m, \end{cases}$$

$$f(v_{ij}^{'}) = \begin{cases} (2m + 1)i - (m + 1) + j, & \text{if } j \text{ is odd and } 2 \leq j \leq m, \\ (2m + 1)i - (m + 1) + j - 1, & \text{if } j \text{ is even and } 2 \leq j \leq m, \\ (2m + 1)i - (m + 1) + j + 1, & \text{if } j \text{ is even and } \frac{m}{2} + 1 \leq j \leq m. \end{cases}$$

It can be verified that the induced edge labels of $[P_n; C_m^{(2)}]$ are $2, 4, \ldots, 4mn + 2n - 2$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $[P_n; C_m^{(2)}]$ is an even vertex equitable even graph for $m \equiv 0 \pmod{4}$. \hfill \Box

**Theorem 2.4.** If $G = C_m * C_n$ is a graph obtained by identifying an edge of $C_m$ with an edge of $C_n$ then $G$ is an even vertex equitable even graph.
Proof. Let \( v_1, v_2, \ldots, v_m \) be the vertices of \( C_m \) and \( u_1, u_2, \ldots, u_n \) be the vertices of \( C_n \). Let \( G \) be a graph obtained by identifying an edge \( v_1v_m \) of \( C_m \) with an edge \( u_nu_1 \) of \( C_n \). Then \( G \) is of order \( m + n - 2 \) and size \( m + n - 1 \).

Define \( f : V(G) \to A = \{0, 2, \ldots, m + n\} \) as follows.

**Case 1.** \( m \equiv 0 \pmod{4} \).

**Subcase 1.1.** \( n \equiv 0 \pmod{4} \).

**Subcase 1.2.** \( n \equiv 3 \pmod{4} \). \( n = 3 \).

**Subcase 1.2.1.** \( n > 3 \).

**Subcase 1.3.** \( n \equiv 2 \pmod{4} \).
\[ f(v_2) = m, \]
\[ f(v_i) = \begin{cases} i - 3, & \text{if } i \text{ is odd and } 3 \leq i \leq \frac{m}{2} + 1, \\ i - 1, & \text{if } i \text{ is odd and } \frac{m}{2} + 2 \leq i \leq m, \\ i - 2, & \text{if } i \text{ is even and } 3 \leq i \leq m, \end{cases} \]
\[ f(u_i) = \begin{cases} (m + n) - i + 2, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2} + 1, \\ (m + n) - i, & \text{if } i \text{ is even and } \frac{n}{2} + 2 \leq i \leq n, \\ (m + n) - i + 1, & \text{if } i \text{ is odd and } 2 \leq i \leq n. \end{cases} \]

**Subcase 1.4.** \( n \equiv 1 \pmod{4} \).

\[ f(v_1) = m, \]
\[ f(v_i) = \begin{cases} i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{m}{2}, \\ i, & \text{if } i \text{ is even and } \frac{m}{2} + 1 \leq i \leq m, \\ i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq m, \end{cases} \]
\[ f(u_i) = \begin{cases} (m + n) - i + 1, & \text{if } i \text{ is even and } 2 \leq i \leq n, \\ (m + n) - i, & \text{if } i \text{ is odd and } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases} \]

**Case 2.** \( m \equiv 3 \pmod{4} \).

**Subcase 2.1.** \( n \equiv 3 \pmod{4} \).

\[ f(v_1) = m + 1, \]
\[ f(v_i) = \begin{cases} i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \left\lceil \frac{m}{2} \right\rceil, \\ i, & \text{if } i \text{ is even and } \left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m, \\ i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq m, \end{cases} \]
\[ f(u_i) = \begin{cases} (m + n) - i + 1, & \text{if } i \text{ is even and } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ (m + n) - i, & \text{if } i \text{ is even and } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n, \\ (m + n) - i + 1, & \text{if } i \text{ is odd and } 2 \leq i \leq n. \end{cases} \]

**Subcase 2.2.** \( n \equiv 2 \pmod{4} \) and \( m \geq 3 \).

\[ f(v_1) = m - 1, \]
\[ f(v_2) = m + 1, \]
\[ f(v_i) = \begin{cases} i - 3, & \text{if } i \text{ is odd and } 3 \leq i \leq \left\lceil \frac{m}{2} \right\rceil + 1, \\ i - 1, & \text{if } i \text{ is odd and } \left\lceil \frac{m}{2} \right\rceil + 2 \leq i \leq m, \\ i - 2, & \text{if } i \text{ is even and } 3 \leq i \leq m, \end{cases} \]
\[
f(u_i) = \begin{cases} 
(m + n) - i + 2, & \text{if } i \text{ is odd and } 2 \leq i \leq n, \\
(m + n) - i + 1, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2} + 1, \\
(m + n) - i - 1, & \text{if } i \text{ is even and } \frac{n}{2} + 2 \leq i \leq n.
\end{cases}
\]

**Subcase 2.3.** \(n \equiv 1 \pmod{4}\) and \(m > 3\).

\[
f(v_1) = m - 1, \\
f(v_2) = m + 1, \\
f(v_i) = \begin{cases} 
i - 3, & \text{if } i \text{ is odd and } 3 \leq i \leq \left\lceil \frac{m}{2} \right\rceil + 1, \\
i - 1, & \text{if } i \text{ is odd and } \left\lceil \frac{m}{2} \right\rceil + 2 \leq i \leq m, \\
i - 2, & \text{if } i \text{ is even and } 3 \leq i \leq m, \\
(m + n) - i + 2, & \text{if } i \text{ is even and } 2 \leq i \leq n, \\
(m + n) - i + 1, & \text{if } i \text{ is odd and } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
(m + n) - i - 1, & \text{if } i \text{ is odd and } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}
\]

**Case 3.** \(m \equiv 2 \pmod{4}\).

**Subcase 3.1.** \(n \equiv 1 \pmod{4}\).

\[
f(v_1) = m + 2, \\
f(v_i) = \begin{cases} 
i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq m, \\
i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \left\lceil \frac{m}{2} \right\rceil + 1, \\
i, & \text{if } i \text{ is even and } \left\lceil \frac{m}{2} \right\rceil + 2 \leq i \leq m, \\
m + i + 1, & \text{if } i \text{ is odd and } 2 \leq i \leq n - 1, \\
m + i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
m + i, & \text{if } i \text{ is even and } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n - 1.
\end{cases}
\]

**Subcase 3.2.** \(n \equiv 2 \pmod{4}\).

**Subcase 3.2.1.** \(m = n\).

\[
f(v_1) = m + n, \\
f(v_i) = \begin{cases} 
i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq m, \\
i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq m, \\
m + i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2} - 1, \\
m + i, & \text{if } i \text{ is even and } \frac{n}{2} \leq i \leq n, \\
m + i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq n.
\end{cases}
\]

**Subcase 3.2.2.** \(n > m\) and \(m = 6\).

\[
f(v_1) = m + n,
\]

\[
f(v_i) = \begin{cases} 
i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq m, \\
i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq m, \\
m + i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2} - 1, \\
m + i, & \text{if } i \text{ is even and } \frac{n}{2} \leq i \leq n, \\
m + i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq n.
\end{cases}
\]

\[
f(u_i) = \begin{cases} 
(m + n) - i + 2, & \text{if } i \text{ is odd and } 2 \leq i \leq n, \\
(m + n) - i + 1, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2} + 1, \\
(m + n) - i - 1, & \text{if } i \text{ is even and } \frac{n}{2} + 2 \leq i \leq n.
\end{cases}
\]
\begin{equation}
    f(v_i) = \begin{cases}
    i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq m, \\
    i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq m,
    \end{cases}
\end{equation}

\begin{equation}
    f(u_i) = \begin{cases}
    m + i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2} - 1, \\
    m + i, & \text{if } i \text{ is even and } \frac{n}{2} \leq i \leq n, \\
    m + i - 3, & \text{if } i \text{ is odd and } 2 \leq i \leq \frac{n}{2} - 2, \\
    m + i - 1, & \text{if } i \text{ is odd and } \frac{n}{2} - 1 \leq i \leq n.
    \end{cases}
\end{equation}

**Subcase 3.2.3.** \(n > m\) and \(m > 6\).

\begin{equation}
    f(v_1) = m + n,
\end{equation}

\begin{equation}
    f(v_i) = \begin{cases}
    i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq m, \\
    i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq m,
    \end{cases}
\end{equation}

\begin{equation}
    f(u_3) = m,
\end{equation}

\begin{equation}
    f(u_i) = \begin{cases}
    m + i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2} - 1, \\
    m + i, & \text{if } i \text{ is even and } \frac{n}{2} \leq i \leq n, \\
    m + i - 1 & \text{if } i \text{ is odd and } 5 \leq i \leq n.
    \end{cases}
\end{equation}

**Case 4.** \(m \equiv 1 \pmod{4}, n \equiv 1 \pmod{4}\) and \(m, n > 5\).

\begin{equation}
    f(v_1) = m - 1,
\end{equation}

\begin{equation}
    f(v_2) = m - 1,
\end{equation}

\begin{equation}
    f(v_3) = m + 3,
\end{equation}

\begin{equation}
    f(v_i) = \begin{cases}
    i - 3, & \text{if } i \text{ is odd and } 4 \leq i \leq m, \\
    i - 4, & \text{if } i \text{ is even and } 4 \leq i \leq \left\lceil \frac{m}{2} \right\rceil + 4, \\
    i - 2, & \text{if } i \text{ is even and } \left\lceil \frac{m}{2} \right\rceil + 5 \leq i \leq m,
    \end{cases}
\end{equation}

\begin{equation}
    f(u_i) = \begin{cases}
    (m + 1) + i, & \text{if } i \text{ is even and } 2 \leq i \leq n - 1, \\
    m + i - 2, & \text{if } i \text{ is odd and } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
    m + i, & \text{if } i \text{ is odd and } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n - 1.
    \end{cases}
\end{equation}

In above all cases, it can be verified that the induced edge labels of \(G\) are 2, 4, \ldots, \(2m + 2n - 2\) and \(|v_f(a) - v_f(b)| \leq 1\) for all \(a, b \in A\). Hence, the graph \(G\) is an even vertex equitable even graph. \(\square\)

**Theorem 2.5.** The graph obtained by duplicating an arbitrary vertex of a cycle \(C_n\) is an even vertex equitable even graph.

**Proof.** Let \(u_1, u_2, \ldots, u_n\) be the vertices of cycle \(C_n\). Let \(G = D(C_n, u')\) be the graph obtained by duplicating an arbitrary vertex \(u\) of \(C_n\). Without loss of generality take \(u = u_1\) and the duplication of \(u_1\) be \(u'_1\). Then \(G\) is of order \(n + 1\) and size \(n + 2\).
Define $f : V(G) \rightarrow A = \begin{cases} 0, 2, \ldots , n + 3, & \text{if } n + 2 \text{ is odd,} \\ 0, 2, \ldots , n + 2, & \text{if } n + 2 \text{ is even,} \end{cases}$ as follows.

**Case 1.** $n \equiv 0 \pmod{4}$ and $n > 4$.

- $f(u'_1) = n + 2$,
- $f(u_n) = n + 2$,
- $f(u_i) = \begin{cases} n - i + 2, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n-4}{2}, \\ n - i, & \text{if } i \text{ is even and } \frac{n-4}{2} + 1 \leq i \leq n - 1, \\ n - i - 1, & \text{if } i \text{ is odd and } 1 \leq i \leq n - 1. \end{cases}$

**Case 2.** $n \equiv 1 \pmod{4}$ and $n > 5$.

- $f(u'_1) = n + 1$,
- $f(u_1) = n - 1$,
- $f(u_n) = n + 3$,
- $f(u_i) = \begin{cases} n - i + 1, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n-5}{2}, \\ n - i - 1, & \text{if } i \text{ is even and } \frac{n-5}{2} + 1 \leq i \leq n - 1, \\ n - i & \text{if } i \text{ is odd and } 2 \leq i \leq n - 1. \end{cases}$

**Case 3.** $n \equiv 2 \pmod{4}$.

- $f(u'_1) = n$,
- $f(u_1) = n + 2$,
- $f(u_n) = n + 2$,
- $f(u_i) = \begin{cases} n - i, & \text{if } i \text{ is even and } 2 \leq i \leq n - 1, \\ n - i + 1, & \text{if } i \text{ is odd and } 2 \leq i \leq \frac{n-4}{2}, \\ n - i - 1, & \text{if } i \text{ is odd and } \frac{n-4}{2} + 1 \leq i \leq n - 1. \end{cases}$

**Case 4.** $n \equiv 3 \pmod{4}$.

- $f(u'_1) = n + 3$,
- $f(u_n) = n + 1$,
- $f(u_i) = \begin{cases} n - i, & \text{if } i \text{ is odd and } 1 \leq i \leq n - 1, \\ n - i + 1, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n-3}{2}, \\ n - i - 1, & \text{if } i \text{ is even and } \frac{n-3}{2} + 1 \leq i \leq n - 1. \end{cases}$

In above four cases, it can be verified that the induced edge labels of $G$ are $2, 4, \ldots , 2n + 4$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence, the graph obtained by duplicating an arbitrary vertex of a cycle $C_n$ is an even vertex equitable even graph. □
Theorem 2.6. The graph obtained by duplicating an arbitrary edge of a cycle $C_n$ is an even vertex equitable even graph.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of cycle $C_n$. Let $G = D(C_n, e')$ be the graph obtained by duplicating an arbitrary edge $e$ of $C_n$. Without loss of generality take $e = u_1u_2$ and the duplication of $e$ be $e' = u'_1u'_2$. Then $G$ is of order $n + 2$ and size $n + 3$.

Define $f : V(G) \to A = \{0, 2, \ldots, n \pm 4, 0, 2, \ldots, n + 3, n + 1\}$ as follows.

Case 1. $n \equiv 0 \pmod{4}$.

Subcase 1.1. For $n = 4$,

$$
\begin{align*}
  f(u'_1) &= 8, \\
  f(u'_2) &= 6, \\
  f(u_1) &= 0, \\
  f(u_2) &= 2, \\
  f(u_3) &= f(u_4) = 4.
\end{align*}
$$

Subcase 1.2. For $n > 4$,

$$
\begin{align*}
  f(u'_1) &= n - 2, \\
  f(u'_2) &= n, \\
  f(u_1) &= n + 2, \\
  f(u_{n-1}) &= n, \\
  f(u_n) &= n + 4, \\
  f(u_i) &= \begin{cases} 
  n - i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq n - 2, \\
  n - i, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n-2}{2}, \\
  n - i - 2, & \text{if } i \text{ is even and } \frac{n-2}{2} + 1 \leq i \leq n - 2.
\end{cases}
\end{align*}
$$

Case 2. $n \equiv 1 \pmod{4}$ and $n > 5$.

$$
\begin{align*}
  f(u'_1) &= n - 3, \\
  f(u'_2) &= f(u_{n-1}) = n + 1, \\
  f(u_1) &= f(u_n) = n + 3, \\
  f(u_i) &= \begin{cases} 
  n - i + 1, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n-3}{2}, \\
  n - i - 1, & \text{if } i \text{ is even and } \frac{n-3}{2} + 1 \leq i \leq n - 2, \\
  n - i - 2, & \text{if } i \text{ is odd and } 2 \leq i \leq n - 2.
\end{cases}
\end{align*}
$$

Case 3. $n \equiv 2 \pmod{4}$.

$$
\begin{align*}
  f(u'_1) &= n + 4,
\end{align*}
$$
\[ f(u'_2) = n, \]
\[ f(u_1) = n, \]
\[ f(u_{n-1}) = n - 2, \]
\[ f(u_n) = n + 2, \]
\[ f(u_i) = \begin{cases} n - i, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n-2}{2}, \\ n - i - 2, & \text{if } i \text{ is even and } \frac{n-2}{2} + 1 \leq i \leq n - 2, \\ n - i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq n - 2. \end{cases} \]

**Case 4.** \( n \equiv 3 \pmod{4} \) and \( n > 7 \).

\[ f(u'_1) = f(u'_2) = n - 1, \]
\[ f(u_1) = f(u_n) = n + 3, \]
\[ f(u_{n-1}) = n + 1, \]
\[ f(u_i) = \begin{cases} n - i - 1, & \text{if } i \text{ is even and } 2 \leq i \leq n - 2, \\ n - i, & \text{if } i \text{ is odd and } 2 \leq i \leq \frac{n-3}{2}, \\ n - i - 2, & \text{if } i \text{ is odd and } \frac{n-3}{2} + 1 \leq i \leq n - 2. \end{cases} \]

In above four cases, it can be verified that the induced edge labels of \( G \) are 2, 4, ..., \( 2n+6 \) and \( |v_f(a) - v_f(b)| \leq 1 \) for all \( a, b \in A \). Hence, the graph obtained by duplicating an arbitrary edge of a cycle \( C_n \) is an even vertex equitable even graph. \( \square \)

**References**


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