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EVEN VERTEX EQUITABLE EVEN LABELING FOR CYCLE RELATED GRAPHS

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Abstract. Let G be a graph with p vertices and q edges and $A = \{0, 2, 4, \ldots, q+1\}$ if q is odd or $A = \{0, 2, 4, \ldots, q\}$ if q is even. A graph G is said to be an even vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u)+f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \ldots, 2q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits even vertex equitable even labeling is called an even vertex equitable even graph. In this paper, we prove that the graphs $C_m \ominus P_n$, $C_n(Q_m)$ if $n \equiv 0, 3 \pmod 4$, $\left[P_n; C_m^{(2)}\right]$ if $m \equiv 0 \pmod 4$, $C_m *_e C_n$ and the graph obtained by duplicating an arbitrary vertex and edge of a cycle C_n admit an even vertex equitable even labeling.

1. Introduction

All graphs considered here are simple, finite, connected and undirected. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. We follow the basic notations and terminology of graph theory as in [3]. A labeling of a graph is a map that carries the graph elements to the set of numbers, usually to the set of non-negative or positive integers. If the domain is the set of vertices the labeling is called vertex labeling. If the domain is the set of edges, then we speak about edge labeling. If the labels are assigned to both vertices and edges then the labeling is called total labeling. For a dynamic survey of various graph labeling, we refer to Gallian [2].

Lourdusamy et al. introduced the concept of vertex equitable labeling in [13]. Let G be a graph with p vertices and q edges and let $A = \left\{0, 1, 2, \ldots, \left\lceil \frac{q}{2} \right\rceil\right\}$. A vertex

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labeling $f : V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv. For $a \in A$, let $v_f(a)$ be the number of vertices v with $f(v) = a$. A graph G is said to be vertex equitable if there exists a vertex labeling f such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \dots, q$.

In [13], Lourdusamy et al. proved that graphs like path, bistar, comb graph, cycle C_n for $n \equiv 0, 3 \pmod 4$, $K_{2,n}$, $C_3^{(t)}$ for $t \geq 2$, quadrilateral snake, $K_2 + mK_1$, $K_{1,n} \bigcup K_{1,n+k}$ for $1 \leq k \leq 3$, ladder and arbitrary super division of any path and cycle C_n with $n \equiv 0, 3 \pmod 4$ admit vertex equitable labeling. Also, they proved $K_{1,n}$ for $n \geq 4$, any Eulerian graph with n edges where $n \equiv 1, 2 \pmod 4$, the wheel W_n, the complete graph K_n if $n > 3$ and triangular cactus with $q \equiv 0, 6, 9 \pmod{12}$ are not vertex equitable graphs. In addition, they proved that if G is a graph with p vertices and q edges, q is even and $p < \left\lceil \frac{q}{2} \right\rceil + 2$ then G is not vertex equitable graph.

In [4–11], Jeyanthi et al. proved that T_p-tree, $T \odot \bar{K_n}$, where T is a T_p-tree with even number of vertices, $B(n, n+1)$, the caterpillar $S(x_1, x_2, \dots, x_n)$, crown, P_n^2, $T\hat{o}P_n$, $T\hat{o}2P_n$, $T\hat{o}C_n$ with $n \equiv 0, 3 \pmod 4$ and $T\tilde{o}C_n$ with $n \equiv 0, 3 \pmod 4$, tadpoles, $C_m \oplus C_n$, armed crowns, $[P_m; C_n^2]$, $\langle P_m \hat{o} K_{1,n} \rangle$, jewel graph J_n, jelly fish graph $(JF)_n$, balanced lobster graph $BL(n, 2, m)$, $L_n \odot \bar{K_m}$, $\langle L_n \hat{o} K_{1,m} \rangle$, $DA(T_n) \odot K_1$, $DA(T_n) \odot 2K_1$, $DA(Q_n) \odot K_1$, $DA(Q_n) \odot 2K_1$, $S^*(P_n \odot K_1)$, $S^*(B(n, n))$, $S^*(P_n \times P_2)$, $S^*(Q_n)$, $S(D(T_n))$, $S(D(Q_n))$, $S(DA(T_n))$, $S(DA(Q_n))$, $DA(Q_m) \odot nK_1$, $DA(T_m) \odot nK_1$, $KY(m, n)$, $P(2.QS_n)$, $P(m.QS_n)$, $C(n.QS_m)$, $NQ(m)$ and $K_{1,n} \times P_2$ admit vertex equitable labeling.

Motivated by the concept of vertex equitable labeling, Lourdusamy et al. introduced the concept of even vertex equitable even labeling in [12]. In [12], they proved that path, comb, complete bipartite, cycle, $K_2 + mK_1$, bistar, ladder, $S(L_n)$, $S(B_{n,n})$ and $L_n \odot K_1$ admit an even vertex equitable even labeling. In this paper, we prove that the graphs $C_m \ominus P_n$, $C_n(Q_m)$ if $n \equiv 0, 3 \pmod 4$, $\left[P_n; C_m^{(2)}\right]$ if $m \equiv 0 \pmod 4$, $C_m *_e C_n$ and the graph obtained by duplicating an arbitrary vertex and edge of a cycle C_n admit an even vertex equitable even labeling.

Definition 1.1. [13] Let G be a graph with p vertices and q edges and let $A = \{0, 1, 2, \dots, \left\lceil \frac{q}{2} \right\rceil\}$. A vertex labeling $f : V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv. For $a \in A$, let $v_f(a)$ be the number of vertices v with $f(v) = a$. A graph G is said to be vertex equitable if there exists a vertex labeling f such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \dots, q$.

Definition 1.2. [12] Let G be a graph with p vertices and q edges and $A = \{0, 2, 4, \dots, q+1\}$ if q is odd or $A = \{0, 2, 4, \dots, q\}$ if q is even. A graph G is said to be an even vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \dots, 2q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits even vertex equitable even labeling is called an even vertex equitable even graph.

Definition 1.3. [6] An armed crown is a cycle attached with paths of equal lengths at each vertex of the cycle. It is denoted by $C_m \ominus P_n$, where P_n is a path of length $n-1$.

Definition 1.4. [14] The graph $C_n(Q_m)$ is defined as isomorphic quadrilateral snake attached with each vertex of cycle C_n. 'm' is the number of C_4 attached in the quadrilateral snake.

Definition 1.5. [1] Let G be a graph with fixed vertex v and let $[P_n; G]$ be the graph obtained from n copies of G by joining v_{ij} and v_{i+1j} by means of an edge, for some j and $1 \le i \le n-1$.

Definition 1.6. [6] Let G_1 and G_2 be two graphs. The identification of G_1 and G_2 is a graph obtained by identifying an edge of G_1, with an edge of G_2 and is denoted by $G_1 *_e G_2$.

Definition 1.7. [6] Let G be a graph and v be any vertex of G. A new vertex v' is said to be duplication of v if all the vertices which are adjacent to v are adjacent to v'. The graph obtained by duplication v is denoted by $D(G, v')$.

Definition 1.8. [6] Let G be a graph and e be any edge of G. A new edge e' is said to be duplication of an edge e if all the edges which are incident to e in G are incident to e'. The graph obtained by duplication e is denoted by $D(G, e')$.

2. Main Results

Theorem 2.1. *The graph $C_m \ominus P_n$ is an even vertex equitable even graph.*

Proof. Let $u_{11}, u_{21}, \dots, u_{m1}$ be the vertices of the cycle C_m and $u_{11}u_{12}\dots u_{1n}$, $u_{21}u_{22}u_{23}\dots u_{2n}, \dots, u_{m1}u_{m2}\dots u_{mn}$ be the vertices of the path P_n attached with u_{i1} by identifying u_{ij} with u_{i1} for $1 \le i \le m$, $1 \le j \le n$. Then $C_m \ominus P_n$ is of order mn and size mn.

Define $f : V(C_m \ominus P_n) \to A = \begin{cases} 0, 2, \dots, mn+1, & \text{if } mn \text{ is odd}, \\ 0, 2, \dots, mn, & \text{if } mn \text{ is even}, \end{cases}$

as follows.

Case 1. $m \equiv 0, 3 \pmod 4$.

For $1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor$,

$$f(u_{ij}) = \begin{cases} n(i-1)+j-1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \le j \le n, \\ n(i-1)+j, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \le j \le n, \end{cases}$$

$$f(u_{ij}) = \begin{cases} ni-j+1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \le j \le n, \\ ni-j, & \text{if } i \text{ is even, } j \text{ is even and } 1 \le j \le n. \end{cases}$$

For $\left\lfloor \frac{m}{2} \right\rfloor + 1 \leq i \leq m$,

$$f(u_{ij}) = \begin{cases} n(i-1)+j+1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\ n(i-1)+j, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n, \end{cases}$$

$$f(u_{ij}) = \begin{cases} ni-j+1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\ ni-j+2, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n. \end{cases}$$

Case 2. $m \equiv 1, 2 \pmod 4$ and n is odd.
For $1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil$,

$$f(u_{ij}) = \begin{cases} ni-j, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\ ni-j+1, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n, \end{cases}$$

$$f(u_{ij}) = \begin{cases} n(i-1)+j, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\ n(i-1)+j-1, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n. \end{cases}$$

For $i = \left\lceil \frac{m}{2} \right\rceil + 1$,

$$f(u_{ij}) = \begin{cases} n(i-1)+j+2, & \text{if } j \text{ is odd and } 1 \leq j \leq n, \\ n(i-1)+j-1, & \text{if } j \text{ is even and } 1 \leq j \leq n. \end{cases}$$

For $i = \left\lceil \frac{m}{2} \right\rceil + 2$,

$$f(u_{ij}) = \begin{cases} ni-j+2, & \text{if } j \text{ is odd and } 1 \leq j \leq n-1, \\ ni-j+1, & \text{if } j \text{ is even and } 1 \leq j \leq n-1, \\ ni-j, & \text{if } j = n. \end{cases}$$

For $\left\lceil \frac{m}{2} \right\rceil + 3 \leq i \leq m$,

$$f(u_{ij}) = \begin{cases} n(i-1)+j, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\ n(i-1)+j+1, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n, \end{cases}$$

$$f(u_{ij}) = \begin{cases} ni-j+2, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\ ni-j+1, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n. \end{cases}$$

Case 3. $m \equiv 2 \pmod 4$ and n is even.
For $1 \leq i \leq \frac{m}{2}$,

$$f(u_{ij}) = \begin{cases} n(i-1)+j-1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\ n(i-1)+j, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n, \end{cases}$$

$$f(u_{ij}) = \begin{cases} ni-j+1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\ ni-j, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n. \end{cases}$$

For $i = \frac{m}{2} + 1$,

$$f(u_{ij}) = \begin{cases} ni + 2j, & \text{if } j = 1, \\ ni - j + 2, & \text{if } j \text{ is even and } 2 \leq j \leq n, \\ ni - j + 1, & \text{if } j \text{ is odd and } 2 \leq j \leq n. \end{cases}$$

For $i = \frac{m}{2} + 2$,

$$f(u_{ij}) = \begin{cases} n(i-1) + j + 1, & \text{if } j \text{ is odd and } 1 \leq j \leq n-1, \\ n(i-1) + j + 2, & \text{if } j \text{ is even and } 1 \leq j \leq n-1, \\ \frac{nm}{2}, & \text{if } j = n. \end{cases}$$

For $\frac{m}{2} + 3 \leq i \leq m$,

$$f(u_{ij}) = \begin{cases} ni - j + 1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\ ni - j + 2, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n, \end{cases}$$

$$f(u_{ij}) = \begin{cases} n(i-1) + j + 1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\ n(i-1) + j, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n. \end{cases}$$

Case 4. $m \equiv 1 \pmod 4$ and n is even.

For $1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil$,

$$f(u_{ij}) = \begin{cases} ni - j + 1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\ ni - j, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n, \end{cases}$$

$$f(u_{ij}) = \begin{cases} n(i-1) + j - 1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\ n(i-1) + j, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n. \end{cases}$$

For $\left\lceil \frac{m}{2} \right\rceil + 1 \leq i \leq m$,

$$f(u_{ij}) = \begin{cases} n(i-1) + j + 1, & \text{if } i \text{ is even, } j \text{ is odd and } 1 \leq j \leq n, \\ n(i-1) + j, & \text{if } i \text{ is even, } j \text{ is even and } 1 \leq j \leq n, \end{cases}$$

$$f(u_{ij}) = \begin{cases} ni - j + 1, & \text{if } i \text{ is odd, } j \text{ is odd and } 1 \leq j \leq n, \\ ni - j + 2, & \text{if } i \text{ is odd, } j \text{ is even and } 1 \leq j \leq n. \end{cases}$$

In above four cases, it can be verified that the induced edge labels of $C_m \ominus P_n$ are $2, 4, \ldots, 2mn$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $C_m \ominus P_n$ is an even vertex equitable even graph. ☐

Theorem 2.2. *The graph $C_n(Q_m)$ is an even vertex equitable even graph if $n \equiv 0, 3 \pmod 4$.*

Proof. Let $G = C_n(Q_m)$. Let

$$V(G) = \{u_i : 1 \leq i \leq n\} \bigcup \{u_{ij}, v_{ij}, w_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\}$$

and $E(G) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_1u_n\} \bigcup \{u_iv_{i1}, u_iw_{i1} : 1 \leq i \leq n\} \bigcup \{v_{ij}u_{ij}, w_{ij}u_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\} \bigcup \{u_{ij}v_{ij+1}, u_{ij}w_{ij+1} : 1 \leq i \leq n; 1 \leq j \leq m-1\}$. Then G is of order $3mn+n$ and size $4mn+n$.

Case 1. $n \equiv 0 \pmod 4$.

Define $f : V(G) \to A = \{0, 2, \dots, 4mn+n\}$ as follows:

$$f(u_i) = \begin{cases} (4m+1)(i-1), & \text{if } i \text{ is odd and } 1 \leq i \leq \frac{n}{2}, \\ (4m+1)(i-1)+2, & \text{if } i \text{ is odd and } \frac{n}{2}+1 \leq i \leq n, \\ (4m+1)i, & \text{if } i \text{ is even and } 1 \leq i \leq n. \end{cases}$$

For $1 \leq j \leq m$,

$$f(u_{ij}) = \begin{cases} (4m+1)(i-1)+4j, & \text{if } i \text{ is odd and } 1 \leq i \leq \frac{n}{2}, \\ (4m+1)(i-1)+4j+2, & \text{if } i \text{ is odd and } \frac{n}{2}+1 \leq i \leq n, \\ (4m+1)i-4j, & \text{if } i \text{ is even and } 1 \leq i \leq n, \end{cases}$$

$$f(v_{ij}) = \begin{cases} (4m+1)(i-1)+4(j-1)+2, & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\ (4m+1)i-4j, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n}{2}, \\ (4m+1)i-4j+2, & \text{if } i \text{ is even and } \frac{n}{2}+1 \leq i \leq n, \end{cases}$$

$$f(w_{ij}) = \begin{cases} (4m+1)(i-1)+4j, & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\ (4m+1)i-4j+2, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n}{2}, \\ (4m+1)i-4j+4, & \text{if } i \text{ is even and } \frac{n}{2}+1 \leq i \leq n. \end{cases}$$

Case 2. $n \equiv 3 \pmod 4$.

Define $f : V(G) \to A = \{0, 2, \dots, 4mn+n+1\}$ as follows:

$$f(u_i) = \begin{cases} (4m+1)i-1, & \text{if } i \text{ is odd and } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ (4m+1)i+1, & \text{if } i \text{ is odd and } \left\lceil \frac{n}{2} \right\rceil \leq i \leq n, \\ (4m+1)(i-1)+1, & \text{if } i \text{ is even and } 1 \leq i \leq n. \end{cases}$$

For $1 \leq j \leq m$,

$$f(u_{ij}) = \begin{cases} (4m+1)i-4j-1, & \text{if } i \text{ is odd and } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ (4m+1)i-4j+1, & \text{if } i \text{ is odd and } \left\lceil \frac{n}{2} \right\rceil \leq i \leq n, \\ (4m+1)(i-1)+4j+1 & \text{if } i \text{ is even and } 1 \leq i \leq n, \end{cases}$$

$$f(v_{ij}) = \begin{cases} (4m+1)i-4(j-1)-1, & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\ (4m+1)(i-1)+4j-1, & \text{if } i \text{ is even and } 1 \leq i \leq n, \end{cases}$$

$$f(w_{ij}) = \begin{cases} (4m+1)i-4j+1, & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\ (4m+1)(i-1)+4(j-1)+1, & \text{if } i \text{ is even and } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ (4m+1)(i-1)+4j+1, & \text{if } i \text{ is even and } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

In both the cases, it can be verified that the induced edge labels of $C_n(Q_m)$ are $2, 4, \ldots, 8mn + 2n$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $C_n(Q_m)$ is an even vertex equitable even graph for $n \equiv 0, 3 \pmod 4$. $\square$

Theorem 2.3. *The graph* $\left[P_n; C_m^{(2)}\right]$ *is an even vertex equitable even graph if* $m \equiv 0 \pmod 4$.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of a path P_n and the vertex $u_i (1 \leq i \leq n)$ is attached with the center vertex of the i^{th} copy of $C_m^{(2)}$. Let $u_i = v_{i1}$ (center vertex in the i^{th} copy of C_n^2). Let v_{ij} and $v_{ij}^{'}$, for $1 \leq i \leq n$, $2 \leq j \leq m$, be the remaining vertices in the i^{th} copy of $C_m^{(2)}$. Then $\left[P_m; C_n^{(2)}\right]$ is of order $2mn - n$ and size $2mn + n - 1$.

Define $f : V(G) \to A = \begin{cases} 0, 2, \ldots, 2mn + n, & \text{if } 2mn + n - 1 \text{ is odd}, \\ 0, 2, \ldots, 2mn + n - 1 & \text{if } 2mn + n - 1 \text{ is even}, \end{cases}$

as follows:

$$f(v_{i1}) = \begin{cases} (2m+1)i - (m+1), & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\ (2m+1)i - m, & \text{if } i \text{ is even and } 1 \leq i \leq n, \end{cases}$$

For $1 \leq i \leq n$ and i is odd,

$$f(v_{ij}) = \begin{cases} (2m+1)i - (m+1) - j + 1, & \text{if } j \text{ is odd and } 2 \leq j \leq m, \\ (2m+1)i - (m+1) - j + 2, & \text{if } j \text{ is even and } 2 \leq j \leq \frac{m}{2}, \\ (2m+1)i - (m+1) - j, & \text{if } j \text{ is even and } \frac{m}{2} + 1 \leq j \leq m, \end{cases}$$

$$f(v_{ij}^{'}) = \begin{cases} (2m+1)i - (m+1) + j, & \text{if } j \text{ is even and } 2 \leq j \leq m, \\ (2m+1)i - (m+1) + j - 1, & \text{if } j \text{ is odd and } 2 \leq j \leq \frac{m}{2}, \\ (2m+1)i - (m+1) + j + 1, & \text{if } j \text{ is odd and } \frac{m}{2} + 1 \leq j \leq m. \end{cases}$$

For $1 \leq i \leq n$ and i is even,

$$f(v_{ij}) = \begin{cases} (2m+1)i - (m+1) - j + 1, & \text{if } j \text{ is even and } 2 \leq j \leq m, \\ (2m+1)i - (m+1) - j + 2, & \text{if } j \text{ is odd and } 2 \leq j \leq \frac{m}{2}, \\ (2m+1)i - (m+1) - j, & \text{if } j \text{ is odd and } \frac{m}{2} + 1 \leq j \leq m, \end{cases}$$

$$f(v_{ij}^{'}) = \begin{cases} (2m+1)i - (m+1) + j, & \text{if } j \text{ is odd and } 2 \leq j \leq m, \\ (2m+1)i - (m+1) + j - 1, & \text{if } j \text{ is even and } 2 \leq j \leq \frac{m}{2}, \\ (2m+1)i - (m+1) + j + 1, & \text{if } j \text{ is even and } \frac{m}{2} + 1 \leq j \leq m. \end{cases}$$

It can be verified that the induced edge labels of $\left[P_n; C_m^{(2)}\right]$ are $2, 4, \ldots, 4mn + 2n - 2$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $\left[P_n; C_m^{(2)}\right]$ is an even vertex equitable even graph for $m \equiv 0 \pmod 4$. $\square$

Theorem 2.4. *If* $G = C_m *_e C_n$ *is a graph obtained by identifying an edge of* C_m *with an edge of* C_n *then* G *is an even vertex equitable even graph.*

Proof. Let $v_1, v_2, \cdots, v_m$ be the vertices of C_m and $u_1, u_2, \ldots, u_n$ be the vertices of C_n. Let G be a graph obtained by identifying an edge v_1v_m of C_m with an edge u_nu_1 of C_n. Then G is of order $m+n-2$ and size $m+n-1$.

Define $f : V(G) \to A = \begin{cases} 0, 2, \ldots, m+n, & \text{if } m+n-1 \text{ is odd,} \\ 0, 2, \ldots, m+n-1, & \text{if } m+n-1 \text{ is even,} \end{cases}$

as follows.

Case 1. $m \equiv 0 \pmod 4$.

Subcase 1.1. $n \equiv 0 \pmod 4$.

$$f(v_1) = m,$$

$$f(v_i) = \begin{cases} i-2, & \text{if } i \text{ is even and } 2 \le i \le \frac{m}{2}, \\ i, & \text{if } i \text{ is even and } \frac{m}{2}+1 \le i \le m, \\ i-1, & \text{if } i \text{ is odd and } 2 \le i \le m, \end{cases}$$

$$f(u_i) = \begin{cases} (m+n)-i+2, & \text{if } i \text{ is even and } 2 \le i \le \frac{n}{2}, \\ (m+n)-i, & \text{if } i \text{ is even and } \frac{n}{2}+1 \le i \le n, \\ (m+n)-i+1, & \text{if } i \text{ is odd and } 2 \le i \le n. \end{cases}$$

Subcase 1.2. $n \equiv 3 \pmod 4$.

Subcase 1.2.1. $n = 3$.

$$f(v_i) = \begin{cases} (m+n)-i+1, & \text{if } i \text{ is even and } 2 \le i \le m, \\ (m+n)-i+2, & \text{if } i \text{ is odd and } 2 \le i \le \frac{m}{2}+1, \\ (m+n)-i, & \text{if } i \text{ is odd and } \frac{m}{2}+2 \le i \le m, \end{cases}$$

$$f(u_1) = 4,$$

$$f(u_2) = 0,$$

$$f(u_3) = 2.$$

Subcase 1.2.2. $n > 3$.

$$f(v_1) = m,$$

$$f(v_2) = m,$$

$$f(v_i) = \begin{cases} i-3, & \text{if } i \text{ is odd and } 3 \le i \le \frac{m}{2}+1, \\ i-1, & \text{if } i \text{ is odd and } \frac{m}{2}+2 \le i \le m, \\ i-2, & \text{if } i \text{ is even and } 3 \le i \le m, \end{cases}$$

$$f(u_i) = \begin{cases} (m+n)-i+1, & \text{if } i \text{ is even and } 2 \le i \le n, \\ (m+n)-i+2, & \text{if } i \text{ is odd and } 2 \le i \le \left\lceil \frac{n}{2} \right\rceil + 1, \\ (m+n)-i, & \text{if } i \text{ is odd and } \left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n. \end{cases}$$

Subcase 1.3. $n \equiv 2 \pmod 4$.

$$f(v_1) = m,$$

$$\begin{aligned} f(v_2) =& m, \\ f(v_i) =& \begin{cases} i-3, & \text{if } i \text{ is odd and } 3 \le i \le \frac{m}{2}+1, \\ i-1, & \text{if } i \text{ is odd and } \frac{m}{2}+2 \le i \le m, \\ i-2, & \text{if } i \text{ is even and } 3 \le i \le m, \end{cases} \\ f(u_i) =& \begin{cases} (m+n)-i+2, & \text{if } i \text{ is even and } 2 \le i \le \frac{n}{2}+1, \\ (m+n)-i, & \text{if } i \text{ is even and } \frac{n}{2}+2 \le i \le n, \\ (m+n)-i+1, & \text{if } i \text{ is odd and } 2 \le i \le n. \end{cases} \end{aligned}$$

Subcase 1.4. $n \equiv 1 \pmod 4$.

$$\begin{aligned} f(v_1) =& m, \\ f(v_i) =& \begin{cases} i-2, & \text{if } i \text{ is even and } 2 \le i \le \frac{m}{2}, \\ i, & \text{if } i \text{ is even and } \frac{m}{2}+1 \le i \le m, \\ i-1 & \text{if } i \text{ is odd and } 2 \le i \le m, \end{cases} \\ f(u_i) =& \begin{cases} (m+n)-i+1, & \text{if } i \text{ is even and } 2 \le i \le n, \\ (m+n)-i+2, & \text{if } i \text{ is odd and } 2 \le i \le \left\lceil \frac{n}{2} \right\rceil, \\ (m+n)-i, & \text{if } i \text{ is odd and } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n. \end{cases} \end{aligned}$$

Case 2. $m \equiv 3 \pmod 4$.
Subcase 2.1. $n \equiv 3 \pmod 4$.

$$\begin{aligned} f(v_1) =& m+1, \\ f(v_i) =& \begin{cases} i-2, & \text{if } i \text{ is even and } 2 \le i \le \left\lceil \frac{m}{2} \right\rceil, \\ i, & \text{if } i \text{ is even and } \left\lceil \frac{m}{2} \right\rceil + 1 \le i \le m, \\ i-1, & \text{if } i \text{ is odd and } 2 \le i \le m, \end{cases} \\ f(u_i) =& \begin{cases} (m+n)-i+2, & \text{if } i \text{ is even and } 2 \le i \le \left\lceil \frac{n}{2} \right\rceil, \\ (m+n)-i, & \text{if } i \text{ is even and } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, \\ (m+n)-i+1, & \text{if } i \text{ is odd and } 2 \le i \le n. \end{cases} \end{aligned}$$

Subcase 2.2. $n \equiv 2 \pmod 4$ and $m > 3$.

$$\begin{aligned} f(v_1) =& m-1, \\ f(v_2) =& m+1, \\ f(v_i) =& \begin{cases} i-3, & \text{if } i \text{ is odd and } 3 \le i \le \left\lceil \frac{m}{2} \right\rceil + 1, \\ i-1, & \text{if } i \text{ is odd and } \left\lceil \frac{m}{2} \right\rceil + 2 \le i \le m, \\ i-2, & \text{if } i \text{ is even and } 3 \le i \le m, \end{cases} \end{aligned}$$

$$f(u_i) = \begin{cases} (m+n) - i + 2, & \text{if } i \text{ is odd and } 2 \leq i \leq n, \\ (m+n) - i + 1, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2} + 1, \\ (m+n) - i - 1, & \text{if } i \text{ is even and } \frac{n}{2} + 2 \leq i \leq n. \end{cases}$$

Subcase 2.3. $n \equiv 1 \pmod 4$ and $m > 3$.

$$\begin{aligned} f(v_1) =& m - 1, \\ f(v_2) =& m + 1, \end{aligned}$$

$$f(v_i) = \begin{cases} i - 3, & \text{if } i \text{ is odd and } 3 \leq i \leq \left\lceil \frac{m}{2} \right\rceil + 1, \\ i - 1, & \text{if } i \text{ is odd and } \left\lceil \frac{m}{2} \right\rceil + 2 \leq i \leq m, \\ i - 2, & \text{if } i \text{ is even and } 3 \leq i \leq m, \end{cases}$$

$$f(u_i) = \begin{cases} (m+n) - i + 2, & \text{if } i \text{ is even and } 2 \leq i \leq n, \\ (m+n) - i + 1, & \text{if } i \text{ is odd and } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ (m+n) - i - 1, & \text{if } i \text{ is odd and } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

Case 3. $m \equiv 2 \pmod 4$.
Subcase 3.1. $n \equiv 1 \pmod 4$.

$$f(v_1) = m + 2,$$

$$f(v_i) = \begin{cases} i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq m, \\ i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \left\lceil \frac{m}{2} \right\rceil + 1, \\ i, & \text{if } i \text{ is even and } \left\lceil \frac{m}{2} \right\rceil + 2 \leq i \leq m, \end{cases}$$

$$f(u_i) = \begin{cases} m + i + 1, & \text{if } i \text{ is odd and } 2 \leq i \leq n - 1, \\ m + i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ m + i, & \text{if } i \text{ is even and } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n - 1. \end{cases}$$

Subcase 3.2. $n \equiv 2 \pmod 4$.
Subcase 3.2.1. $m = n$.

$$f(v_1) = m + n,$$

$$f(v_i) = \begin{cases} i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq m, \\ i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq m, \end{cases}$$

$$f(u_i) = \begin{cases} m + i - 2, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2} - 1, \\ m + i, & \text{if } i \text{ is even and } \frac{n}{2} \leq i \leq n, \\ m + i - 1, & \text{if } i \text{ is odd and } 2 \leq i \leq n. \end{cases}$$

Subcase 3.2.2. $n > m$ and $m = 6$.

$$f(v_1) = m + n,$$

$$f(v_i) = \begin{cases} i-2, & \text{if } i \text{ is even and } 2 \leq i \leq m, \\ i-1, & \text{if } i \text{ is odd and } 2 \leq i \leq m, \end{cases}$$

$$f(u_i) = \begin{cases} m+i-2, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2}-1, \\ m+i, & \text{if } i \text{ is even and } \frac{n}{2} \leq i \leq n, \\ m+i-3, & \text{if } i \text{ is odd and } 2 \leq i \leq \frac{n}{2}-2, \\ m+i-1, & \text{if } i \text{ is odd and } \frac{n}{2}-1 \leq i \leq n. \end{cases}$$

Subcase 3.2.3. $n > m$ and $m > 6$.

$$f(v_1) = m+n,$$
$$f(v_i) = \begin{cases} i-2, & \text{if } i \text{ is even and } 2 \leq i \leq m, \\ i-1, & \text{if } i \text{ is odd and } 2 \leq i \leq m, \end{cases}$$
$$f(u_3) = m,$$
$$f(u_i) = \begin{cases} m+i-2, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2}-1, \\ m+i, & \text{if } i \text{ is even and } \frac{n}{2} \leq i \leq n, \\ m+i-1 & \text{if } i \text{ is odd and } 5 \leq i \leq n. \end{cases}$$

Case 4. $m \equiv 1 \pmod 4$, $n \equiv 1 \pmod 4$ and $m, n > 5$.

$$f(v_1) = m-1,$$
$$f(v_2) = m-1,$$
$$f(v_3) = m+3,$$
$$f(v_i) = \begin{cases} i-3, & \text{if } i \text{ is odd and } 4 \leq i \leq m, \\ i-4, & \text{if } i \text{ is even and } 4 \leq i \leq \left\lceil \frac{m}{2} \right\rceil + 4, \\ i-2, & \text{if } i \text{ is even and } \left\lceil \frac{m}{2} \right\rceil + 5 \leq i \leq m, \end{cases}$$
$$f(u_i) = \begin{cases} (m+1)+i, & \text{if } i \text{ is even and } 2 \leq i \leq n-1, \\ m+i-2, & \text{if } i \text{ is odd and } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ m+i, & \text{if } i \text{ is odd and } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-1. \end{cases}$$

In above all cases, it can be verified that the induced edge labels of G are $2, 4, \ldots, 2m+2n-2$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence, the graph G is an even vertex equitable even graph. □

Theorem 2.5. *The graph obtained by duplicating an arbitrary vertex of a cycle C_n is an even vertex equitable even graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of cycle C_n. Let $G = D(C_n, u')$ be the graph obtained by duplicating an arbitrary vertex u of C_n. Without loss of generality take $u = u_1$ and the duplication of u_1 be u_1'. Then G is of order $n+1$ and size $n+2$.

Define $f : V(G) \to A = \begin{cases} 0, 2, \ldots, n+3, & \text{if } n+2 \text{ is odd,} \\ 0, 2, \ldots, n+2, & \text{if } \; n+2, \text{ is even,} \end{cases}$
as follows.

Case 1. $n \equiv 0 \pmod 4$ and $n > 4$.

$$\begin{aligned} f(u_1^{'}) =& n+2, \\ f(u_n) =& n+2, \\ f(u_i) =& \begin{cases} n-i+2, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n-4}{2}, \\ n-i, & \text{if } i \text{ is even and } \frac{n-4}{2}+1 \leq i \leq n-1, \\ n-i-1, & \text{if } i \text{ is odd and } 1 \leq i \leq n-1. \end{cases} \end{aligned}$$

Case 2. $n \equiv 1 \pmod 4$ and $n > 5$.

$$\begin{aligned} f(u_1^{'}) =& n+1, \\ f(u_1) =& n-1, \\ f(u_n) =& n+3, \\ f(u_i) =& \begin{cases} n-i+1, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n-5}{2}, \\ n-i-1, & \text{if } i \text{ is even and } \frac{n-5}{2}+1 \leq i \leq n-1, \\ n-i & \text{if } i \text{ is odd and } 2 \leq i \leq n-1. \end{cases} \end{aligned}$$

Case 3. $n \equiv 2 \pmod 4$.

$$\begin{aligned} f(u_1^{'}) =& n, \\ f(u_1) =& n+2, \\ f(u_n) =& n+2, \\ f(u_i) =& \begin{cases} n-i, & \text{if } i \text{ is even and } 2 \leq i \leq n-1, \\ n-i+1, & \text{if } \; i \text{ is odd and } 2 \leq i \leq \frac{n-4}{2}, \\ n-i-1, & \text{if } i \text{ is odd and } \frac{n-4}{2}+1 \leq i \leq n-1. \end{cases} \end{aligned}$$

Case 4. $n \equiv 3 \pmod 4$.

$$\begin{aligned} f(u_1^{'}) =& n+3, \\ f(u_n) =& n+1, \\ f(u_i) =& \begin{cases} n-i, & \text{if } i \text{ is odd and } 1 \leq i \leq n-1, \\ n-i+1, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n-3}{2}, \\ n-i-1, & \text{if } i \text{ is even and } \frac{n-3}{2}+1 \leq i \leq n-1. \end{cases} \end{aligned}$$

In above four cases, it can be verified that the induced edge labels of G are $2, 4, \ldots,$ $2n+4$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence, the graph obtained by duplicating an arbitrary vertex of a cycle C_n is an even vertex equitable even graph. $\square$

Theorem 2.6. *The graph obtained by duplicating an arbitrary edge of a cycle C_n is an even vertex equitable even graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of cycle C_n. Let $G = D(C_n, e^{'})$ be the graph obtained by duplicating an arbitrary edge e of C_n. Without loss of generality take $e = u_1u_2$ and the duplication of e be $e^{'} = u_1^{'}u_2^{'}$. Then G is of order $n+2$ and size $n+3$.

Define $f : V(G) \to A = \begin{cases} 0, 2, \ldots, n+4, & \text{if } n+3 \text{ is odd,} \\ 0, 2, \ldots, n+3, & \text{if } \ n+3, \text{is even,} \end{cases}$

as follows.

Case 1. $n \equiv 0 \pmod 4$.

Subcase 1.1. For $n = 4$,

$$\begin{aligned} f(u_1^{'}) =& 8, \\ f(u_2^{'}) =& 6, \\ f(u_1) =& 0, \\ f(u_2) =& 2, \\ f(u_3) =& f(u_4) = 4. \end{aligned}$$

Subcase 1.2. For $n > 4$,

$$\begin{aligned} f(u_1^{'}) =& n-2, \\ f(u_2^{'}) =& n, \\ f(u_1) =& n+2, \\ f(u_{n-1}) =& n, \\ f(u_n) =& n+4, \end{aligned}$$

$$f(u_i) = \begin{cases} n-i-1, & \text{if } i \text{ is odd and } 2 \le i \le n-2, \\ n-i, & \text{if } i \text{ is even and } 2 \le i \le \frac{n-2}{2}, \\ n-i-2, & \text{if } i \text{ is even and } \frac{n-2}{2}+1 \le i \le n-2. \end{cases}$$

Case 2. $n \equiv 1 \pmod 4$ and $n > 5$.

$$\begin{aligned} f(u_1^{'}) =& n-3, \\ f(u_2^{'}) =& f(u_{n-1}) = n+1, \\ f(u_1) =& f(u_n) = n+3, \end{aligned}$$

$$f(u_i) = \begin{cases} n-i+1, & \text{if } i \text{ is even and } 2 \le i \le \frac{n-3}{2}, \\ n-i-1, & \text{if } i \text{ is even and } \frac{n-3}{2}+1 \le i \le n-2, \\ n-i-2, & \text{if } i \text{ is odd and } 2 \le i \le n-2. \end{cases}$$

Case 3. $n \equiv 2 \pmod 4$.

$$f(u_1^{'}) = n+4,$$

$$\begin{aligned} f(u_2^{'}) =& n, \\ f(u_1) =& n, \\ f(u_{n-1}) =& n-2, \\ f(u_n) =& n+2, \end{aligned}$$

$$f(u_i) = \begin{cases} n-i, & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n-2}{2}, \\ n-i-2, & \text{if } i \text{ is even and } \frac{n-2}{2}+1 \leq i \leq n-2, \\ n-i-1, & \text{if } i \text{ is odd and } 2 \leq i \leq n-2. \end{cases}$$

Case 4. $n \equiv 3 \pmod 4$ and $n > 7$.

$$\begin{aligned} f(u_1^{'}) =& f(u_2^{'}) = n-1, \\ f(u_1) =& f(u_n) = n+3, \\ f(u_{n-1}) =& n+1, \end{aligned}$$

$$f(u_i) = \begin{cases} n-i-1, & \text{if } i \text{ is even and } 2 \leq i \leq n-2, \\ n-i, & \text{if } i \text{ is odd and } 2 \leq i \leq \frac{n-3}{2}, \\ n-i-2, & \text{if } i \text{ is odd and } \frac{n-3}{2}+1 \leq i \leq n-2. \end{cases}$$

In above four cases, it can be verified that the induced edge labels of G are $2, 4, \ldots,$ $2n+6$ and $|v_f(a)-v_f(b)| \leq 1$ for all $a, b \in A$. Hence, the graph obtained by duplicating an arbitrary edge of a cycle C_n is an even vertex equitable even graph. □

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