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A SUBCLASS OF NOOR-TYPE HARMONIC p-VALENT FUNCTIONS BASED ON HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper, we introduce a new generalized Noor-type operator of harmonic p-valent functions associated with the Fox-Wright generalized hypergeometric functions (FWGH-functions). Furthermore, we consider a new subclass of complex-valued harmonic multivalent functions based on this new operator. Several geometric properties for this subclass are also discussed.

1. Introduction

Harmonic function has fruitful applications not only in applied mathematics, but also in physics, engineering. It appears in differential equations, such as harmonic differential equations, wave equations, and heat equations. In geometric function theory (GFT), the famed authors Clunie and Sheil-Small [11] launched the study of harmonic univalent functions in 1984. In their investigates, they provided a class $\mathcal{S}_{\mathcal{H}}$ of harmonic functions $\varphi = \phi + \overline{\psi}$ that are univalent, sense-preserving which is $|\phi'(z)| > |\psi'(z)|$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and normalized by the conditions $\varphi(0) = \varphi'(0) - 1 = 0$, where the regular(analytic) part ϕ and the co-regular part ψ are defined as follows:

$$\phi(z) = z + \sum_{\kappa=2}^{\infty} \mu_\kappa z^\kappa, \psi(z) = \sum_{\kappa=1}^{\infty} \nu_\kappa z^\kappa, \quad |\nu_1| < 1.$$

In addition, they studied its geometric properties, which involves coefficient bounds, growth and distortion formulas. Note that, class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ of regular univalent functions if the co-regular part ψ is zero.

Key words and phrases. Harmonic multivalent function, convolution product, Noor integral operator, Fox-Wright generalized hypergeometric function.

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In 2001, Ahuja and Jahangiri [2] defined a more general class $\mathcal{S}_{\mathcal{H}(p)}$ of harmonic $p-$valent (multivalent) functions, $\varphi = \phi + \overline{\psi}$ that are sense-preserving in $\mathbb{D}$, and ϕ and ψ are of the formula

$$\phi(z) = z^p + \sum_{\kappa=p+1}^{\infty} \mu_\kappa z^\kappa, \psi(z) = \sum_{\kappa=p}^{\infty} \nu_\kappa z^\kappa, \quad |\nu_p| < 1, p \in \mathbb{N} = \{1, 2, \dots\}. \tag{1.1}$$

Note that, class $\mathcal{S}_{\mathcal{H}(p)}$ reduces to the class $\mathcal{M}_p$ of normalized regular $p-$valent functions if the co-regular part ψ is zero. Consequently, the function $\varphi \in \mathcal{M}_p$ are expressed as:

$$\varphi(z) = z^p + \sum_{\kappa=p+1}^{\infty} \mu_\kappa z^\kappa. \tag{1.2}$$

Denoted by $\mathcal{NS}_{\mathcal{H}(p)}$ the subclass of $\mathcal{S}_{\mathcal{H}(p)}$ consisting of functions $\varphi = \phi + \overline{\psi}$ such that the regular functions ϕ and ψ are of the form

$$\phi(z) = z^p - \sum_{\kappa=p+1}^{\infty} |\mu_\kappa| z^\kappa, \psi(z) = -\sum_{\kappa=p}^{\infty} |\nu_\kappa| z^\kappa, \quad |\nu_p| < 1, p \in \mathbb{N} = \{1, 2, \dots\}. \tag{1.3}$$

Convolution (Hadamard) product is a mathematical operation on two regular functions φ_1 and φ_2 to yield a third regular function φ_3. It is used to define various subclasses and linear operators in GFT. This concept owes its origin to Hadamard in 1899 [22]. In the harmonic functions case, Clunie and Sheil-Small [11] studied and defined the following convolution product: for any two functions $\varphi_\imath \in \mathcal{S}_{\mathcal{H}}$ of the form

$$\varphi_\imath(z) = \phi_\imath(z) + \overline{\psi_\imath(z)} = z + \sum_{\kappa=2}^{\infty} \mu_{\kappa,\imath} z^\kappa + \overline{\sum_{\kappa=1}^{\infty} \nu_{\kappa,\imath} z^\kappa},$$

where $\imath = 1, 2$, $|\nu_{1,1}| < 1$, $|\nu_{1,2}| < 1$, their convolution is denoted by $\varphi_1 * \varphi_2$ and defined as

$$(\varphi_1 * \varphi_2)(z) = z + \sum_{\kappa=2}^{\infty} \mu_{\kappa,1}\, \mu_{\kappa,2}\, z^\kappa + \overline{\sum_{\kappa=1}^{\infty} \nu_{\kappa,1}\, \nu_{\kappa,2}\, z^\kappa}.$$

More generally, the convolution of two functions $\varphi_\imath \in \mathcal{S}_{\mathcal{H}(p)}$ is given by (see, [29]):

$$(\varphi_1 * \varphi_2)(z) = z^p + \sum_{\kappa=p+1}^{\infty} \mu_{\kappa,1}\, \mu_{\kappa,2}\, z^\kappa + \overline{\sum_{\kappa=p}^{\infty} \nu_{\kappa,1}\, \nu_{\kappa,2}\, z^\kappa}, \tag{1.4}$$

where

$$\varphi_\imath(z) = \phi_\imath(z) + \overline{\psi_\imath(z)} = z^p + \sum_{\kappa=p+1}^{\infty} \mu_{\kappa,\imath} z^\kappa + \overline{\sum_{\kappa=p}^{\infty} \nu_{\kappa,\imath} z^\kappa}, \quad \imath = 1, 2, |\nu_{p,1}| < 1, |\nu_{p,2}| < 1.$$

Operators Theory has a significant role in the study GFT. Actually, operators are utilized in defining new subclasses. The technique of convolution has a remarkable part in the evolution of this area. Numerous differential and integral operators (linear operators) can be established in terms of the convolution. In 1915, Alexander [4] introduced the first integral operator on class $\mathcal{A}$ that includes normalized regular functions. Later, several well-known integral operators are investigated by complex analysts, such as Libera [26], Bernardi [9], Miller, Mocanu and Reade [27,28], Pascu and Pescar [34], Ong et al. [33], Frasin [20], Frasin and Breaz [21], El-Ashwah, Aouf and El-Deeb [16], Deniz [13], Rahrovi [35], Al-Janaby and Ghanim [5], Al-Janaby, Ghanim, Darus [6], Al-Janaby [7] and others. The following are some important linear operators related to results in this study.

In 1975, Ruscheweyh [37] introduced the differential operator $D^{\tau}\varphi(z)$ so-called the Ruscheweyh differential operator as follows: for $\varphi \in \mathcal{A}$, $\tau > -1$ and $D^{\tau} : \mathcal{A} \to \mathcal{A}$ is given by

$$D^{\tau}\varphi(z) = \frac{z}{(1-z)^{\tau+1}} * \varphi(z) = z + \sum_{\kappa=2}^{\infty} \frac{(\tau+1)_{\kappa-1}}{(\kappa-1)!} \mu_{\kappa} z^{\kappa}, \tag{1.5}$$

where $(a)_{\kappa} = \frac{\Gamma(a+\kappa)}{\Gamma(a)}$ denotes the Pochhammer symbol. Note that $D^0\varphi(z) = \varphi(z)$ and $D^1\varphi(z) = z\varphi'(z)$.

Analogous manner to the Ruscheweyh operator, in 1999, the author Noor [31] presented an integral operator $I_{\tau}\varphi(z)$, namely Noor Integral of $\tau-$th order, as follows: for a function $\varphi \in \mathcal{A}$ and $\tau \in \mathbb{N}_0$, the Noor integral operator $I_{\tau}(z)$ is given by $I_{\tau} : \mathcal{A} \to \mathcal{A}$,

$$I_{\tau}\omega(z) = \varphi_{\tau}^{(-1)}(z) * \varphi(z) = \left[\frac{z}{(1-z)^{\tau+1}}\right]^{-1} * \varphi(z) = z + \sum_{\kappa=2}^{\infty} \frac{\kappa!}{(\tau+1)_{\kappa-1}} \mu_k \, z^{\kappa}, \tag{1.6}$$

such that $\varphi_{\tau}(z) * \varphi_{\tau}^{(-1)}(z) = \frac{z}{(1-z)^2}$. Note that $I_0\varphi(z) = z\varphi'(z)$, $I_1\varphi(z) = \varphi(z)$. This version of integral operator is a considerable gadget in imposing several subclasses of regular functions.

On the other hand, special functions have been applied in GFT. In 1984, de Branges [12] employed hypergeometric function in proving the prominent problem called Bieberbach's conjecture. Since then, the study of hypergeometric function and its generalizations have attracted the attention of many function theorists. The important role played by special functions is defining new operators. The generalized hypergeometric function known as Fox-Wright generalized hypergeometric function (FWGH-function) is defined as: (see for example [19,40] and [41])

$$\eta\mathcal{W}_{\delta}[(\rho_l, \mathcal{C}_l)_{1,\eta}; (\sigma_l, \mathcal{D}_l)_{1,\delta}; z] = \eta\mathcal{W}_{\delta}[(\rho_1, \mathcal{C}_1) \cdots (\rho_{\eta}, \mathcal{C}_{\eta}); (\sigma_1, \mathcal{D}_1) \cdots (\sigma_{\delta}, \mathcal{D}_{\delta}); z]$$

$$= \sum_{\kappa=0}^{\infty} \frac{\Gamma(\rho_1 + \kappa\mathcal{C}_1)\Gamma(\rho_2 + \kappa\mathcal{C}_2) \cdots \Gamma(\rho_{\eta} + \kappa\mathcal{C}_{\eta})}{\Gamma(\sigma_1 + \kappa\mathcal{D}_1)\Gamma(\sigma_2 + \kappa\mathcal{D}_2) \cdots \Gamma(\sigma_{\delta} + \kappa\mathcal{D}_{\delta})} \frac{z^{\kappa}}{\kappa!}$$

(1.7)

$$= \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{\eta} \Gamma(\rho_j + \kappa\mathcal{C}_j)}{\prod_{j=1}^{\delta} \Gamma(\sigma_j + \kappa\mathcal{D}_j)} \frac{z^{\kappa}}{\kappa!},$$

where $\mathcal{C}_\jmath > 0$, $\jmath = 1, 2, \ldots, \eta$, $\mathcal{B}_\jmath > 0$, $\jmath = 1, 2, \ldots, \delta$, $1 + \sum\limits_{\jmath=1}^{\eta} \mathcal{C}_\jmath - \sum\limits_{\jmath=1}^{\delta} \mathcal{D}_\jmath \geq 0$, $\rho_\jmath + \kappa\mathcal{C}_\jmath \neq 0, -1, \ldots$, $\jmath = 1, 2, \ldots, \eta, \kappa = 0, 1, \ldots$, $\sigma_\jmath + \kappa\mathcal{D}_\jmath \neq 0, -1, \ldots$, $\jmath = 1, 2, \ldots, \delta, \kappa = 0, 1, \ldots$ and $z \in \mathbb{C}$. The condition $1 + \sum\limits_{\jmath=1}^{\eta} \mathcal{C}_\jmath - \sum\limits_{\jmath=1}^{\delta} \mathcal{D}_\jmath \geq 0$ is essential so that the series in (1.7) is absolutely convergent for all $z \in \mathbb{C}$, and is an entire function of z (for details, see [25]). Special case of FWGH-function defined in (1.7), given as: if $\mathcal{C}_\jmath = 1$, $\jmath = 1, 2, \ldots, \eta$, $\mathcal{D}_\jmath = 1$, $\jmath = 1, 2, \ldots, \delta$, $\eta \leq \delta + 1$ and

$$\Xi = \left(\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath)\right) \left(\prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath)\right)^{-1}, \tag{1.8}$$

then

$$\Xi \; \eta\mathcal{W}_\delta[(\rho_\jmath, 1)_{1,\eta}; (\sigma_\jmath, 1)_{1,\delta}; z] = \eta\mathcal{F}_\delta[(\rho_1, ...\rho_\eta; \sigma_1, ...\sigma_\delta; z],$$

where $\eta\mathcal{F}_\delta[(\rho_1, \ldots, \rho_\eta; \sigma_1, \ldots, \sigma_\delta; z]$ is a generalized hypergeometric function, [14]. Other special cases of FWGH-function were presented in [25].

In the well-known theory of regular univalent functions, there are numerous investigates on hypergeometric functions associated with classes of regular functions. In 2004, Ahuja and Silverman [1] discovered the corresponding connections between hypergeometric functions and harmonic univalent functions. Recently, the connections between WGHF and harmonic univalent functions were discussed by some authors, such that Murugusundaramoorthy and Raina [30], Sharma [39], Raina and Sharma [36], Ahuja and Sharma [3] and Hussain et al.[23]. In addition, several operators have been extended to harmonic functions by authors. For instance, Chandrashekar et al. [10], El-Ashwah, and Aouf [17] Yaşar and Yalçin [42], Seoudy [38], Al-Janaby [8] and others. Some previous studies that involving hypergeometric and FWGH functions are presented in this paper.

In 2004, Dziok and Raina [15] considered the linear operator $W(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}]$ by means of FWGH-function on $\mathcal{A}$ as:

$$W(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}]\varphi(z) = z + \sum_{\kappa=2}^{\infty} \Xi \, \vartheta_\kappa \, \mu_\kappa \, z^\kappa,$$

where

$$\vartheta_\kappa = \frac{\Gamma(\rho_1 + (\kappa-1)\mathcal{C}_1)\Gamma(\rho_2 + (\kappa-1)\mathcal{C}_2) \cdots \Gamma(\rho_\eta + (\kappa-1)\mathcal{C}_\eta)}{\Gamma(\sigma_1 + (\kappa-1)\mathcal{D}_1)\Gamma(\sigma_2 + (\kappa-1)\mathcal{D}_2) \cdots \Gamma(\sigma_\delta + (\kappa-1)\mathcal{D}_\delta)(\kappa-1)!},$$

and Ξ is defined in (1.8). Following that, in 2016, Hussain, Rasheed and Darus [23] introduced a new subclass of harmonic functions by using the extension of the above linear operator to harmonic functions. Also, they investigated various properties such as coefficient bounds, extreme points, and inclusion results and closed under an integral operator for this subclass.

In 2006, the author Noor [32] again imposed the integral operator $I_\tau(\zeta,\xi;\gamma)$ by employing the Gauss hypergeometric function as follows:

$$I_\tau(\zeta,\xi;\gamma)\varphi(z) = [z\mathcal{F}(\zeta,\xi;\zeta;z)]^{(-1)} * \varphi(z) = z + \sum_{\kappa=2}^{\infty} \frac{(\gamma)_{\kappa-1}(\tau+1)_{\kappa-1}}{(\zeta)_{\kappa-1}(\xi)_{\kappa-1}}\, \mu_\kappa z^\kappa, \tag{1.9}$$

where

$$[z\mathcal{F}(\zeta,\xi;\gamma;z)] * [z\mathcal{F}(\zeta,\xi;\gamma;z)]^{(-1)} = \frac{z}{(1-z)^{\tau+1}} = z + \sum_{\kappa=2}^{\infty} \frac{(\tau+1)_{\kappa-1}}{(\kappa-1)!} z^\kappa.$$

In 2008, Ibrahim and Darus [24] studied the following generalized integral operator $I_\tau[(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta};(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta}]$ associated with FWGH-function on $\mathcal{A}$, where

$$I_\tau[(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta};(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta}]\varphi(z) = z + \sum_{\kappa=2}^{\infty} \frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-1)\mathcal{D}_\jmath)}{\prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-1)\mathcal{C}_\jmath)} (\tau+1)_{\kappa-1}\, \mu_\kappa\, z^\kappa \tag{1.10}$$

and

$$\frac{\Gamma(\sigma_1)\cdots\Gamma(\sigma_\delta)}{\Gamma(\rho_1)\cdots\Gamma(\rho_\eta)} = 1.$$

Posterior, in 2016, the authors El-Ashwah and Hassan [18] established the linear operator $\Theta_\kappa[(\rho_\jmath,\mathcal{C}_\jmath)_{1,\tau};(\nu_\jmath,\mathcal{D}_\jmath)_{1,\varsigma}]$ on the class $\mathcal{M}_p$ of regular $p-$valent functions in $\mathbb{D}$ as:

$$\Theta_\kappa[(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta};(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta}]\varphi(z) = z^p + \sum_{\kappa=p+1}^{\infty} \Xi\, \vartheta_\kappa\, \rho_\kappa\, z^\kappa,$$

where

$$\vartheta_\kappa = \frac{\Gamma(\rho_1 + (\kappa-p)\mathcal{C}_1)\Gamma(\rho_2 + (\kappa-p)\mathcal{C}_2)\cdots\Gamma(\rho_\eta + (\kappa-p)\mathcal{C}_\eta)}{\Gamma(\sigma_1 + (\kappa-p)\mathcal{D}_1)\Gamma(\sigma_2 + (\kappa-p)\mathcal{D}_2)\cdots\Gamma(\sigma_\delta + (\kappa-p)\mathcal{D}_\delta)(\kappa-p)!},$$

and Ξ is defined in (1.8).

In this study, we continue our investigates in the theory of operators. Here we'll introduce a new generalized Noor-type operator of harmonic $p-$valent functions associated with FWGH-functions. We then define a new subclass and discuss several of its properties.

2. Imposed Operator $\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)$

This section proposes a new generalized Noor-type operator $\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)$ for harmonic p-valent functions based on FWGH-function in (1.7).

By giving an extension of the FWGH-function in (1.7)

$$\eta\mathcal{M}_\delta[(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta};(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta};z] := \Omega z^p \eta\mathcal{W}_\delta[(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta};(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta};z]$$

$$= z^p + \sum_{\kappa=p+1}^{\infty} \frac{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)} \cdot \frac{z^\kappa}{(\kappa-p)!}, \tag{2.1}$$

where

$$\Omega = \Big(\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath)\Big)\Big(\prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath)\Big)^{-1}. \tag{2.2}$$

We define a new generalization of the extended FWGH-function in (2.1) in terms of $\ell-$th convolution product as:

$$\begin{aligned} &\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z] \\ :=&\underbrace{\eta\mathcal{M}_\delta[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z] * \cdots * \eta\mathcal{M}_\delta[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]}_{\ell-\text{times}} \\ =&z^p + \sum_{\kappa=p+1}^{\infty} \left[\frac{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^\ell z^\kappa. \end{aligned} \tag{2.3}$$

Then we introduce a new function $\Big(\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]\Big)^{-1}$ as:

$$\begin{aligned} &\Big(\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]\Big)^{-1} \\ =&z^p + \sum_{\kappa=p+1}^{\infty} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)}\right]^\ell \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} z^\kappa, \end{aligned} \tag{2.4}$$

such that for $\tau > -p$

$$\begin{aligned} &\Big(\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]\Big) * \Big(\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]\Big)^{-1} \\ =&\frac{z^p}{(1-z)^{\tau+p}} = \sum_{\kappa=p}^{\infty} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} z^\kappa. \end{aligned}$$

Next, we consider the following linear operator: $\mathcal{J}_p^\ell[(\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; (\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}] : \mathcal{M}_p \to \mathcal{M}_p$, where

$$\begin{aligned} &\mathcal{J}_p^\ell[(\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; (\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}]\varphi(z) = \Big(\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]\Big)^{-1} * \varphi(z) \\ =&z^p + \sum_{\kappa=p+1}^{\infty} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)}\right]^\ell \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} \mu_\kappa z^\kappa. \end{aligned} \tag{2.5}$$

For brevity,

$$\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)=\mathcal{J}_p^{\ell}[(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta};(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta}]\varphi(z). \tag{2.6}$$

Remark 2.1. For suitably chosen parameters p, ℓ, δ, η, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{D}_1$, ρ_1, ρ_2 and σ_1, the generalized Noor-type operator $\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]$ (2.6) reduces to some of the above linear operators. Thus, we obtain the following special cases.

- For $p=1$, $\ell=1$, $\delta=1$, $\eta=2$, $\mathcal{C}_1=\mathcal{C}_2=\mathcal{D}_1=1$, and $\rho_1=\rho_2=\sigma_1=1$ in (2.6), we gain the Ruscheweyh differential operator given by (1.5).
- For $p=1$, $\ell=1$, $\delta=1$, $\eta=2$, $\mathcal{C}_1=\mathcal{C}_2=\mathcal{D}_1=1$, $\rho_1=\rho_2=1+\tau$ and $\sigma_1=1$, the operator (2.6) provides the Noor integral operator in (1.6).
- By taking $p=1$, $\ell=1$, $\delta=1$, $\eta=2$, $\mathcal{C}_1=\mathcal{C}_2=\mathcal{D}_1=1$, $\rho_1=\zeta$, $\rho_2=\xi$ and $\sigma_1=\gamma$ in (2.6), gives us an integral operator defined by (1.9).
- If $p=1$, $\ell=1$ and $\Omega=1$, we yield the linear operator given by (1.10).

The generalized Noor-type operator $\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)$ (2.6) when extended to harmonic $p-$valent function $\varphi=\phi+\overline{\psi}$ is defined by

$$\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)=\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\phi(z)+\overline{\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\psi(z)}, \tag{2.7}$$

where

$$\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\phi(z)=z^p+\sum_{\kappa=p+1}^{\infty}\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,\mu_\kappa\,z^\kappa$$

and

$$\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\psi(z)=\sum_{\kappa=p}^{\infty}\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,\nu_\kappa\,z^\kappa.$$

3. Geometric Results

This section introduces a certain subclass of harmonic $p-$valent functions which includes the generalized Noor-type operator $\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)$ extended to harmonic functions. This subclass is denoted by $\mathcal{H}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$. Further, coefficient bounds, growth formula, extreme points, convolution, convex combinations and class-preserving integral operator are also investigated for harmonic functions satisfying the subclass $\mathcal{H}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$.

Definition 3.1. A function $\varphi\in\mathcal{S}_{\mathcal{H}}$ is said to be in subclass $\mathcal{H}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ if it satisfies the following inequality:

$$\operatorname{Re}\left\{(1-\alpha)\,\frac{\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)}{z^p}+\alpha\,\frac{\left[\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)\right]'}{pz^{p-1}}\right\}\geq\frac{\beta}{p}, \tag{3.1}$$

where $\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)$ is defined by (2.7), $0 \leq \alpha \leq 1$ and $0 \leq \beta < p$.

Also, let $\mathcal{NH}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath]) = \mathcal{H}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath]) \bigcap \mathcal{NS}_{\mathcal{H}(p)}$.

A sufficient coefficient condition for function belonging to the class $\mathcal{H}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ is now derived.

Theorem 3.1. *Let $\varphi = \phi + \overline{\psi}$ given by (1.1). Then $\varphi \in \mathcal{H}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ if*

(3.2)
$$\begin{aligned}
&\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\mu_\kappa|\\
&+\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\nu_\kappa| \leq p-\beta,
\end{aligned}$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta < p$.

Proof. Using the fact that $\mathrm{Re}(\lambda) \geq 0$ if and only if $|1+\lambda| \geq |1-\lambda|$, it suffices to show that

$$|p-\beta+p\theta(z)| \geq |p+\beta-p\theta(z)|, \tag{3.3}$$

where

$$\theta(z) = (1-\alpha)\,\frac{\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)}{z^p} + \alpha\,\frac{\left[\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)\right]'}{pz^{p-1}}.$$

Substituting for ϕ and ψ in θ, we gain

(3.4)
$$\begin{aligned}
&|p-\beta+p\theta(z)|\\
\geq{}&2p-\beta-\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\mu_\kappa||z|^{\kappa-p}\\
&-\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\nu_\kappa||z|^{\kappa-p}
\end{aligned}$$

and

(3.5)

$$
\begin{aligned}
&|p+\beta-p\theta(z)| \\
\leq&\beta+\sum_{\kappa=p+1}^{\infty}[(\kappa-p)a+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa||z|^{\kappa-p} \\
&+\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa||z|^{\kappa-p}.
\end{aligned}
$$

These inequalities (3.4) and (3.5) in conjunction with (3.2) yields

$$
\begin{aligned}
&|p-\beta+p\theta(z)| \\
\geq&|p+\beta-p\theta(z)| \\
\geq&2\left[(p-\beta)-\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|\right. \\
&\left.-\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|\right]\geq 0.
\end{aligned}
$$

The harmonic function

(3.6)

$$
\begin{aligned}
\varphi(z)=&z^p+\sum_{\kappa=p+1}^{\infty}\left[\frac{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!}{(\tau+p)_{\kappa-p}\,[(\kappa-p)\alpha+p]}x_\kappa z^\kappa \\
&+\sum_{\kappa=p}^{\infty}\left[\frac{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!}{(\tau+p)_{\kappa-p}\,[(\kappa-p)\alpha+p]}\overline{y}_\kappa\,\overline{z}^\kappa,
\end{aligned}
$$

where $\sum_{\kappa=p+1}^{\infty}|x_\kappa|+\sum_{\kappa=p}^{\infty}|y_\kappa|=p-\beta$ shows that the coefficient bound given by (3.2) is sharp.

The functions of the from (3.6) are in subclass $\mathcal{H}_p^{\beta}(\ell,\, \eta,\, \delta)$ because in view of (3.2), we acquire

$$\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|$$
$$+\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|$$
$$\le\sum_{\kappa=p+1}^{\infty}|x_\kappa|+\sum_{\kappa=p}^{\infty}|y_\kappa|=p-\beta.$$

This completes the proof. □

Now, we yield the necessary and sufficient condition for the function $\varphi=\phi+\overline{\psi}$ given by (1.3) to be in $\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath;\rho_\jmath])$.

Theorem 3.2. *Let $\varphi=\phi+\overline{\psi}$ be given by* (1.3)*. Then $\varphi\in\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath;\rho_\jmath])$ if and only if the condition* (3.2) *is as follows:*

$$\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|$$
$$+\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|\le p-\beta,$$

where $0\le\alpha\le1$, $0\le\beta<p$.

Proof. In view of Theorem 3.1 and $\varphi\in\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath;\rho_\jmath])\subset\mathcal{H}_p^{\beta}(\alpha,\, [\sigma_\jmath;\rho_\jmath])$, we only need to prove the "only if" part of this theorem. Assume that $\varphi\in\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath;\rho_\jmath])$, then by virtue of (3.1), we get

(3.7)
$$\mathrm{Re}\left\{(p-\beta)-\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|\right.$$

$$\left.\times z^{\kappa-p} - \sum_{\kappa=p}^{\infty} [(\kappa-p)\alpha+p] \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} |\nu_\kappa| \overline{z}^{\kappa-p}\right\}$$

$$\geq 0.$$

This inequality (3.7) must hold for all values of z in $\mathbb{D}$. Upon choosing the values of z on the positive real axis, where $0 < |z| = r < 1$, (3.7) reduces to

$$(p-\beta) - \sum_{\kappa=p+1}^{\infty} [(\kappa-p)\alpha+p] \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} |\mu_\kappa| r^{\kappa-p}$$

$$- \sum_{\kappa=p}^{\infty} [(\kappa-p)\alpha+p] \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} |\nu_\kappa| r^{p-k} \geq 0.$$

Letting $r \to -1$ through real values, it follows that

(3.8)

$$(p-\beta) - \sum_{\kappa=p+1}^{\infty} [(\kappa-p)\alpha+p] \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} |\mu_\kappa|$$

$$- \sum_{\kappa=p}^{\infty} [(\kappa-p)\alpha+p] \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} |\nu_\kappa| \geq 0.$$

Thus, (3.8) yields (3.2). This completes the proof. □

The following theorem considers the growth bounds for the function φ that belongs to $\mathcal{NH}_p^{\beta}(\alpha, [\sigma_\jmath; \rho_\jmath])$.

Theorem 3.3. *Let $\varphi \in \mathcal{NH}_p^{\beta}(\alpha, [\sigma_\jmath; \rho_\jmath])$ and $r = |z| < 1$. Then*

$$|\varphi(z)| \leq (1+|\nu_p|)\ r^p + \left[\frac{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath+\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}\right]^{\ell} \frac{[p(1-|\nu_p|)-\beta]}{[\alpha+p](\tau+p)_1}\ r^{p+1}$$

and

$$|\varphi(z)| \geq (1+|\nu_p|)\ r^p - \left[\frac{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}\right]^{\ell}\frac{[p(1-|\nu_p|)-\beta]}{[\alpha+p](\tau+p)_1}\ r^{p+1}.$$

Proof. Let $\varphi \in \mathcal{NH}_p^{\beta}(\alpha,\ [\sigma_\jmath;\rho_\jmath])$. By taking the modulus value of φ and using Theorem 3.2, we have

$$\begin{aligned}
|\varphi(z)| \leq &(1+|\nu_p|)\, r^p + \sum_{\kappa=p+1}^{\infty}\ (|\mu_\kappa|+|\nu_\kappa|)\, r^\kappa \\
\leq &(1+|\nu_p|)\, r^p + r^{p+1}\sum_{\kappa=p+1}^{\infty}\ (|\mu_\kappa|+|\nu_\kappa|) \\
\leq &(1+|\nu_p|)\, r^p + \frac{r^{p+1}}{[\alpha+p](\tau+p)_1}\left[\frac{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}\right]^{\ell} \\
&\times\left(\sum_{\kappa=p+1}^{\infty}[\alpha+p](\tau+p)_1\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}\right]^{\ell}(|\mu_\kappa|+|\nu_\kappa|)\right) \\
\leq &(1+|\nu_p|)\, r^p + \frac{r^{p+1}}{[\alpha+p](\tau+p)_1}\left[\frac{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}\right]^{\ell}\left(\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\right. \\
&\left.\times\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\ (|\mu_\kappa|+|\nu_\kappa|)\right) \\
\leq &(1+|\nu_p|)\, r^p + \left[\frac{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}\right]^{\ell}\frac{[p(1-|\nu_p|)-\beta]}{[\alpha+p](\tau+p)_1}\, r^{p+1}.
\end{aligned}$$

Also,

$$\begin{aligned}
|\varphi(z)| \geq &(1+|\nu_p|)\, r^p - \sum_{\kappa=p+1}^{\infty}\ (|\mu_\kappa|+|\nu_\kappa|)\, r^\kappa \\
\geq &(1+|\nu_p|)\, r^p - \sum_{\kappa=p+1}^{\infty}\ (|\mu_\kappa|+|\nu_\kappa|)\, r^{p+1}
\end{aligned}$$

$$\geq (1+|\nu_p|)\, r^p - \frac{r^{p+1}}{[\alpha+p](\tau+p)_1} \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + \mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + \mathcal{D}_\jmath)} \right]^{\ell} \left(\sum_{\kappa=p+1}^{\infty} [\alpha+p](\tau+p)_1 \right.$$

$$\left. \times \left[\frac{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + \mathcal{D}_\jmath)}{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + \mathcal{C}_\jmath)} \right]^{\ell} (|\mu_\kappa| + |\nu_\kappa|) \right)$$

$$\geq (1+|\nu_p|)\, r^p - \frac{r^{p+1}}{[\alpha+p](\tau+p)_1} \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + \mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + \mathcal{D}_\jmath)} \right]^{\ell} \left(\sum_{\kappa=p+1}^{\infty} [(\kappa-p)\alpha+p] \right.$$

$$\left. \times \left[\frac{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} (|\mu_\kappa| + |\nu_\kappa|) \right)$$

$$\geq (1+|\nu_p|) r^p - \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + \mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + \mathcal{D}_\jmath)} \right]^{\ell} \frac{[p(1-|\nu_p|)-\beta]}{[\alpha+p](\tau+p)_1} r^{p+1}.$$

This completes the proof of Theorem 3.3. $\square$

The next theorem determines the extreme points of convex hulls of $\mathcal{NH}_p^{\beta}(\alpha, [\sigma_\jmath; \rho_\jmath])$ denoted by $\overline{\mathrm{co}}\mathcal{NH}_p^{\beta}(\alpha, [\sigma_\jmath; \rho_\jmath])$.

Theorem 3.4. *A function $\varphi \in \overline{\mathrm{co}}\mathcal{NH}_p^{\beta}(\alpha, [\sigma_\jmath; \rho_\jmath])$ if and only if*

$$\varphi(z) = \sum_{\kappa=p}^{\infty} \left(X_\kappa h_\kappa(z) + Y_\kappa g_\kappa(z) \right), \tag{3.9}$$

where

$$\begin{aligned} h_p(z) =& z^p, \\ h_\kappa(z) =& z^p - \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!} \right]^{\ell} \frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}} z^{\kappa}, \\ \kappa =& p+1, p+2, \ldots, \end{aligned}$$

$$\begin{aligned} g_\kappa(z) =& z^p - \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathfrak{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathfrak{D}_\jmath)(\kappa-p)!}\right]^\ell \frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}} \overline{z}^\kappa, \\ \kappa =& p, p+1, \dots, \\ \sum_{\kappa=p}^{\infty} (X_\kappa + Y_\kappa) =& 1,\ X_\kappa \geq 0 \ and\ \ Y_\kappa \geq 0. \end{aligned}$$

Proof. For a function φ of the form (3.9), we acquire

$$\begin{aligned} \varphi(z) =& \sum_{\kappa=p}^{\infty} (X_\kappa h_\kappa(z) + Y_\kappa g_\kappa(z)) \\ =& X_p\, h_p + \sum_{\kappa=p+1}^{\infty} X_\kappa h_\kappa(z) + \sum_{\kappa=p}^{\infty} Y_\kappa g_\kappa(z) \\ =& X_p\, z^p + \sum_{\kappa=p+1}^{\infty} X_\kappa\, z^p \\ & - \sum_{\kappa=p+1}^{\infty} \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathfrak{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathfrak{D}_\jmath)(\kappa-p)!}\right]^\ell \frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}} X_\kappa z^\kappa \\ & + \sum_{\kappa=p}^{\infty} Y_\kappa\, z^p - \sum_{\kappa=p}^{\infty} \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathfrak{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathfrak{D}_\jmath)(\kappa-p)!}\right]^\ell \\ & \times \frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}} Y_\kappa \overline{z}^\kappa \\ =& \sum_{\kappa=p}^{\infty} (X_\kappa + Y_\kappa)\, z^p \\ & - \sum_{\kappa=p+1}^{\infty} \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathfrak{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathfrak{D}_\jmath)(\kappa-p)!}\right]^\ell \frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}} X_\kappa z^\kappa \\ & - \sum_{\kappa=p}^{\infty} \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathfrak{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathfrak{D}_\jmath)(\kappa-p)!}\right]^\ell \frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}} Y_\kappa \overline{z}^\kappa \end{aligned}$$

$$
\begin{aligned}
=z^p - &\sum_{\kappa=p+1}^{\infty}\left[\frac{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}}X_\kappa z^\kappa\\
&-\sum_{\kappa=p}^{\infty}\left[\frac{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}}Y_\kappa \overline{z}^\kappa.
\end{aligned}
$$

Therefore, in view of Theorem 3.2, we gain

$$
\begin{aligned}
&\sum_{k=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\\
&\left[\left[\frac{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}}X_\kappa\right]\\
&+\sum_{k=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\\
&\left[\left[\frac{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}}Y_\kappa\right]\\
&\leq(p-\beta)\left(\sum_{\kappa=p}^{\infty}(X_\kappa+Y_\kappa)-X_p\right)=(p-\beta)\,(1-X_p)\leq p-\beta.
\end{aligned}
$$

Therefore, $\varphi\in\overline{\mathrm{co}}\mathcal{NH}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$.

Conversely, suppose that $\varphi\in\overline{\mathrm{co}}\mathcal{NH}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$. Set

$$
\begin{aligned}
X_\kappa=&((\kappa-p)\alpha+p)\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!\,(p-\beta)}\,|\mu_\kappa|,\\
\kappa=&p+1,p+2,\ldots,
\end{aligned}
$$

and

$$Y_\kappa =[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^\ell \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!\,(p-\beta)}\,|\nu_\kappa|,$$
$$\kappa =p,p+1,p+2,\ldots$$

On the basis of Theorem 3.2, we note that $0\leq X_\kappa\leq 1$, $\kappa=p+1,p+2,\ldots$ and $0\leq Y_\kappa\leq 1$, $\kappa=p,p+1,p+2,\ldots$ Let $X_p=1-\sum_{\kappa=p+1}^{\infty}X_\kappa+\sum_{\kappa=p}^{\infty}Y_\kappa$ and note that by Theorem 3.2, $X_p\geq 0$. Consequently, $\varphi(z)=\sum_{\kappa=p}^{\infty}(X_\kappa h_\kappa(z)+Y_\kappa g_\kappa(z))$ is obtained as required. □

Using convolution principle, we show the subclass $\mathcal{NH}_p^\beta(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ is closed under convolution.

Theorem 3.5. *For $0\leq\lambda\leq\beta<p$, let $\varphi\in\mathcal{NH}_p^\beta(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ and $\mathcal{F}\in\mathcal{NH}_p^\lambda(\alpha,\,[\sigma_\jmath;\rho_\jmath])$. Then $\varphi*\mathcal{F}\in\mathcal{NH}_p^\beta(\alpha,\,[\sigma_\jmath;\rho_\jmath])\subset\mathcal{NH}_p^\lambda(\alpha,\,[\sigma_\jmath;\rho_\jmath])$.*

Proof. Utilizing definition of convolution, let the harmonic function $\varphi(z)=z^p-\sum_{\kappa=p+1}^{\infty}|\mu_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|\nu_\kappa|\overline{z}^\kappa$ and $\mathcal{F}(z)=z^p-\sum_{\kappa=p+1}^{\infty}|A_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|B_\kappa|\overline{z}^\kappa$. Then, the convolution of φ and $\mathcal{F}$ is

$$(\varphi*\mathcal{F})(z)=z^p-\sum_{\kappa=p+1}^{\infty}|\mu_\kappa A_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|\nu_\kappa B_\kappa|\overline{z}^\kappa.$$

For $\mathcal{F}\in\mathcal{NH}_p^\lambda(\alpha,\,[\sigma_\jmath;\rho_\jmath])$, by Theorem 3.2, we conclude that $|A_\kappa|\leq 1$ and $|B_\kappa|\leq 1$. Now for the convolution $\varphi*\mathcal{F}$, we gain

$$\sum_{\kappa=p+1}^{\infty}\frac{[(\kappa-p)\alpha+p]}{(p-c)}\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^\ell\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\mu_\kappa|\,|A_\kappa|$$
$$+\sum_{\kappa=p}^{\infty}\frac{[(\kappa-p)\alpha+p]}{(p-c)}\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^\ell\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\nu_\kappa|\,|B_\kappa|$$
$$\leq\sum_{\kappa=p+1}^{\infty}\frac{[(\kappa-p)\alpha+p]}{(p-\beta)}\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^\ell\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\mu_\kappa|$$
$$+\sum_{\kappa=p}^{\infty}\frac{[(\kappa-p)\alpha+p]}{(p-\beta)}\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^\ell\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\nu_\kappa|\leq 1,$$

since $0 \leq \lambda \leq \beta < p$ and $\varphi \in \mathcal{NH}_p^{\beta}(\alpha, [\sigma_\jmath; \rho_\jmath])$. Therefore, $\varphi * \mathcal{F} \in \mathcal{NH}_p^{\beta}(\alpha, [\sigma_\jmath; \rho_\jmath]) \subset \mathcal{NH}_p^{\lambda}(\alpha, [\sigma_\jmath; \rho_\jmath])$. $\square$

In this theorem, we show that $\mathcal{NH}_p^{\beta}(\alpha, [\sigma_\jmath; \rho_\jmath])$ is closed under convex combination of its members. Let the functions $\varphi_\imath$ be defined, for $\imath = 1, 2, \ldots,$ by

$$\varphi_\imath(z) = z^p + \sum_{\kappa=p+1}^{\infty} |\mu_{\imath,\kappa}| z^\kappa - \sum_{\kappa=p}^{\infty} |\nu_{\imath,\kappa}| \overline{z}^\kappa. \tag{3.10}$$

Theorem 3.6. *Let the functions $\varphi_\imath$ given by* (3.10) *be in $\mathcal{NH}_p^{\beta}(\alpha, [\sigma_\jmath; \rho_\jmath])$ for every $\imath = 1, 2, \ldots$ Then, the function θ defined by*

$$\theta(z) = \sum_{\imath=1}^{\infty} c_\imath \omega_\imath(z), \quad 0 \leq c_\imath < 1, \tag{3.11}$$

is also in the subclass $\mathcal{NH}_p^{\beta}(\alpha, [\sigma_\jmath; \rho_\jmath])$, where $\sum_{\imath=1}^{\infty} c_\imath = 1$.

Proof. According to the definition of θ, we can write

$$\theta(z) = z^p + \sum_{\kappa=p+1}^{\infty} \left(\sum_{\imath=1}^{\infty} c_\imath |\mu_{\imath,\kappa}| \right) z^\kappa - \sum_{\kappa=p}^{\infty} \left(\sum_{\imath=1}^{\infty} c_\imath |\nu_{\imath,\kappa}| \right) \overline{z}^\kappa.$$

Then, by Theorem 3.2, we have

$$\sum_{\kappa=p+1}^{\infty} \frac{[(\kappa-p)\alpha+p]}{(p-\beta)} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} \left(\sum_{\imath=1}^{\infty} c_\imath |\mu_{\imath,\kappa}| \right)$$

$$+ \sum_{\kappa=p}^{\infty} \frac{[(\kappa-p)\alpha+p]}{(p-\beta)} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} \left(\sum_{\imath=1}^{\infty} c_\imath |\nu_{\imath,\kappa}| \right)$$

$$= \sum_{\imath=1}^{\infty} c_\imath \left(\sum_{\kappa=p+1}^{\infty} \frac{[(\kappa-p)\alpha+p]}{(p-\beta)} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} |\mu_{\imath,\kappa}| \right.$$

$$\left. + \sum_{\kappa=p}^{\infty} \frac{[(\kappa-p)\alpha+p]}{(p-\beta)} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} |\nu_{\imath,\kappa}| \right)$$

$$\leq \sum_{\imath=1}^{\infty} c_\imath = 1.$$

Hence, the proof is completed. $\square$

Finally, we discuss a closure property of subclass $\mathcal{NH}_p^\beta(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ under the generalized Bernardi-Libera-Livingston integral operator $\mathcal{F}$ which is given as (see [9]):

$$\mathcal{F}(z)=\frac{(\lambda+p)}{z^\lambda}\int_0^z t^{\lambda-1}\,\varphi(t)dt,\quad \lambda>-p.$$

Theorem 3.7. *Let $\varphi\in\mathcal{NH}_p^\beta(\alpha,\,[\sigma_\jmath;\rho_\jmath])$. Then $\mathcal{F}\in\mathcal{NH}_p^\beta(\alpha,\,[\sigma_\jmath;\rho_\jmath])$.*

Proof. Let

$$\varphi(z)=z^p-\sum_{\kappa=p+1}^{\infty}|\mu_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|\nu_\kappa|\overline{z}^\kappa.$$

From the representation of $\mathcal{F}$, it follows that

$$\begin{aligned}\mathcal{F}(z)&=\frac{\lambda+p}{z^\lambda}\int_0^z t^{\lambda-1}\left\{\phi(z)+\overline{\psi(z)}\right\}dt\\&=\frac{\lambda+p}{z^\lambda}\left\{\int_0^z t^{\lambda-1}\left(t^p-\sum_{\kappa=p+1}^{\infty}|\mu_\kappa|\,t^\kappa\right)dt-\overline{\int_0^z t^{\lambda-1}\left(\sum_{\kappa=p}^{\infty}|\nu_\kappa|\,t^\kappa\right)dt}\right\}\\&=z^p-\sum_{\kappa=p+1}^{\infty}A_\kappa z^\kappa-\sum_{\kappa=p}^{\infty}B_\kappa\overline{z}^\kappa,\end{aligned}$$

where

$$A_\kappa=\left(\frac{\lambda+p}{\lambda+\kappa}\right)|\mu_\kappa|\quad\text{and}\quad B_\kappa=\left(\frac{\lambda+p}{\lambda+\kappa}\right)|\nu_\kappa|.$$

Therefore, since $\varphi\in\mathcal{NH}_p^\beta(\alpha,\,[\sigma_\jmath;\rho_\jmath])$,

$$\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^\ell\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\left(\frac{\lambda+p}{\lambda+\kappa}\right)|\mu_\kappa|$$

$$+\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^\ell\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\left(\frac{\lambda+p}{\lambda+\kappa}\right)|\nu_\kappa|$$

$$\le\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^\ell\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|$$

$$+\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|\leq p-\beta.$$

By considering Theorem 3.2, we yield $\mathcal{F}(z)\in\mathcal{NH}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$. □

4. Conclusion

In this paper, we have introduced a new generalized Noor-type integral operator $\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]$ on the class of harmonic $p-$valent functions Correlating with FWGH-functions in the unit disc $\mathbb{D}$. A certain subclass including this new operator is studied. In addition, some outcomes are obtained by involving coefficient condition and by showing this significance condition for negative coefficient, growth bounds, extreme points, convolution property, convex linear combination and a class-preserving integral operator.

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