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FIXED POINT THEOREMS VIA WF-CONTRACTIONS

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Abstract. In this paper, we introduce a new class of contractions which remains a mixed type of weak and F-contractions but not any of them.

1. Introduction and Preliminaries

Investigating fixed point of a mapping continues to be an active topic of research in nonlinear analysis wherein Banach contraction principle remains the main tool as it offers an efficient and plain technique to compute such points. This vital principle has undergone considerable extensions and generalizations in various ways concerning two or three terms in the contraction inequality. One of the noteworthy generalization of this principle involving three terms was due to Alber and Guerre-Delabriere [1] which was refined later by Rhoades [17] and then generalized by Dutta and Choudhury [7].

Let Ψ be the set of all continuous and monotonically nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) = 0$ if and only if $t = 0$.

Theorem 1.1 ([7])**.** *Let (X, d) be a complete metric space and $f : X \to X$ a weak contractive mapping, i.e.,*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)),$$

for all $x, y \in X$, where $\psi, \varphi \in \Psi$. Then f has a unique fixed point.

Nowadays, there is a tradition of proving unified fixed point results employing an auxiliary function general enough yielding several contractions and henceforth several fixed point results in one go. In 1997, Popa [15] introduced the idea of implicit function which was well followed by [2,3,9,10,16]. Khojasteh et al. [12] introduced the idea of simulation function which is also designed to unify several contractions. For

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further work on simulation functions, one can consult [4,6,8,11,13,18] and some other ones. One of the recent widely discussed generalizations of Banach principle (utilizing auxiliary function) is due to Wardowski [19] wherein the author generalized Banach contraction principle by introducing a new type of contractions called F-contraction and proved that every such contraction defined on a complete metric space possesses a unique fixed point.

Definition 1.1 ([19])**.** A self-mapping f on a metric space (X, d) is said to be an *F-contraction* if there exists $\tau > 0$ such that

$$d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(d(x, y)), \quad \text{for all } x, y \in X, \tag{1.1}$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

F1: F is strictly increasing;

F2: for every sequence $\{s_n\}$ of positive real numbers,

$$\lim_{n\to\infty} s_n = 0 \Leftrightarrow \lim_{n\to\infty} F(s_n) = -\infty;$$

F3: there exists $k \in (0, 1)$ such that $\lim\limits_{s\to 0^+} s^k F(s) = 0$.

We denote by $\mathcal{F}$ the family of all functions F satisfying conditions (**F1**)-(**F3**). Some natural and known members of $\mathcal{F}$ are $F(s) = \ln s$, $F(s) = s + \ln s$ and $F(s) = \frac{-1}{\sqrt{s}}$.

2. WF-Contractions

Definition 2.1. A self-mapping f on a metric space (X, d) is said to be *WF-contraction* if there exist two functions $G, \delta : [0, \infty) \to [0, \infty)$ such that, for all $x, y \in X$ with $d(fx, fy) > 0$, we have

$$\delta(d(x, y)) + G(d(fx, fy)) \leq G(d(x, y)), \tag{2.1}$$

where G and δ satisfy the following conditions:

G1: G is strictly increasing;

G2: $\delta(t) > 0$ for all $t > 0$ and for every strictly decreasing sequence $\{s_n\}$ of positive real numbers,

$$\lim_{n\to\infty} \delta(s_n) = 0 \Rightarrow \lim_{n\to\infty} s_n = 0;$$

G3: there exists $k \in (0, 1)$ such that $\lim\limits_{s\to 0^+} s^k G(s) = 0$.

In the sequel, $\mathbb{G}$ denotes the family of all functions G meeting the requirements of Definition 2.1 while Δ stands for the set of all functions δ enjoying (**G2**). Some members of $\mathbb{G}$ are $G(s) = \ln(s + 1)$, $G(s) = s$, $G(s) = (s + 1) + \frac{1}{(s+1)}$ and $G(s) = \sqrt[n]{s}$, $n \in \mathbb{N}$.

Example 2.1. Let $X = [0, \infty)$ and f a self-mapping on X given by

$$f(x) = \begin{cases} \dfrac{x+2}{2}, & \text{for } x \leq 2, \\ 2, & \text{for } x \geq 2. \end{cases}$$

Then f satisfies (2.1) for $G(s) = s + \frac{1}{2(s+1)}$ and $\delta(t) = \frac{t}{8}$. Indeed, the following three cases arise.

Case 1. If $2 \leq x \leq y$, then $d(fx, fy) = 0$. However, inequality (2.1) becomes:

$$\frac{y-x}{8} + \frac{1}{2} \leq (y-x) + \frac{1}{2(y-x+1)},$$

which can be written as

$$\frac{1}{2} \leq \frac{7}{8}z + \frac{1}{2(z+1)}, \tag{2.2}$$

where $z = y - x \geq 0$. Observe that, the R.H.S of (2.2) is increasing mapping in z for $z \geq 0$ having the value $\frac{1}{2}$ at $z = 0$.

Case 2. If $2 \geq y \geq x$, then (2.1) becomes:

$$\frac{y-x}{8} + \frac{y-x}{2} + \frac{1}{(y-x)+2} \leq (y-x) + \frac{1}{2(y-x)+2},$$

which can be written as

$$0 \leq \frac{3}{8}z + \frac{1}{2z+2} - \frac{1}{z+2}, \tag{2.3}$$

where $z = y - x \geq 0$. Here, also, the R.H.S of (2.3) is increasing mapping in z for $z \geq 0$ with the value 0 at $z = 0$.

Case 3. If $x \leq 2 \leq y$, then (2.1) becomes:

$$\frac{y-x}{8} + \left(1 - \frac{x}{2}\right) + \frac{1}{2\left((1-\frac{x}{2})+1\right)} \leq (y-x) + \frac{1}{2((y-x)+1)}$$

or

$$\left(1 - \frac{x}{2}\right) + \frac{1}{4-x} \leq \frac{7}{8}(y-x) + \frac{1}{2(y-x)+2}$$

Let $2 - x = a$ and $y - 2 = b$. Then,

$$\frac{a}{2} + \frac{1}{2+a} \leq \frac{7}{8}(a+b) + \frac{1}{2(a+b)+2},$$

which is equivalent to

$$\frac{2b+a}{(2+a)(1+a+b)} \leq \frac{3a+7b}{4},$$

which is true if we expand it and remember that $a, b \geq 0$.

The following two remarks highlight the relation between WF-contractions and the weak and F-contractions.

Remark 2.1. Observe that ψ in Theorem 1.1 may not belong to $\mathbb{G}$ as it is not required to be strictly increasing. On the other hand, f in Example 2.1 is a WF-contraction for $G(s) = s + \frac{1}{2(s+1)}$ but not weak contraction as $G(0) \neq 0$. Consequently, the class of WF-contractions and the class of weak contractions are independent.

Remark 2.2. Notice that, $G(s) = s,\ s \in [0, \infty)$, is a member of $\mathbb{G}$ which is not in $\mathcal{F}$. On the other hand, $F \in \mathcal{F}$ given by $F(s) = \ln s$ is not in $\mathbb{G}$ (for $\delta \equiv \tau$).

Remark 2.3. Every WF-contraction mapping is a contractive mapping and hence continuous. This fact follows from (**G1**) and (2.1), i.e.,

$$d(fx, fy) < d(x, y), \quad \text{for all } x, y \in X, x \neq y.$$

Lemma 2.1. *Every WF-contraction mapping has at most one fixed point.*

Proof. If $x, y \in X$ are two distinct fixed points of f, then (2.1) gives rise $\delta(d(x, y)) \leq 0$, which is a contradiction as $\delta(t) > 0$ for all $t > 0$. □

Lemma 2.2. *Let (X, d) be a metric space space and $\{t_n\}$ a sequences of positive real numbers such that*

$$\delta(t_n) + G(t_{n+1}) \leq G(t_n), \tag{2.4}$$

for all n, where $G \in \mathbb{G}$ and $\delta \in \Delta$. Then the sequence $\{t_n\}$ is decreasing and $\sum_{i=0}^{\infty} \delta(t_i) < \infty$.

Proof. As $\delta(t) > 0$ for all $t > 0$, we have $G(t_{n+1}) < G(t_n)$ for all $n \in \mathbb{N}$. Since G is strictly increasing, we get $t_{n+1} < t_n$, for all $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} t_n = r$ for some $r \geq 0$. Then $G(r) \leq G(t_{n+1})$ for all $n \geq 0$. In view of (2.4), we have

$$\begin{aligned} G(t_{n+1}) &\leq G(t_n) - \delta(t_n) \\ &\leq G(t_{n-1}) - [\delta(t_n) + \delta(t_{n-1})] \\ &\ \vdots \\ &\leq G(t_0) - \sum_{i=0}^{n} \delta(t_i). \end{aligned} \tag{2.5}$$

Therefore, $\sum_{i=0}^{n} \delta(t_i) \leq G(t_0)$ for all $n \geq 0$. □

Now, we are equipped to state and prove our main result.

Theorem 2.1. *Let (X, d) be a complete metric space and $f : X \to X$ a WF-contraction for some $G \in \mathbb{G}$ and $\delta \in \Delta$. Then f has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ in X by $x_{n+1} := fx_n$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Notice that, if $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then x_n is the required fixed point and we are done. Henceforth, we assume that such equality does not occur for all $n \in \mathbb{N}_0$. Denote $t_n = d(x_n, x_{n+1})$. On setting $x = x_n$ and $y = x_{n+1}$ in (2.1), we have

$$\delta(t_n) + G(t_{n+1}) \leq G(t_n). \tag{2.6}$$

In view of Lemma 2.2, $\sum_{i=0}^{\infty} \delta(t_i) < \infty$ so that $\lim_{n\to\infty} \delta(t_n) = 0$ and hence, in view of **(G2)**,

$$\lim_{n\to\infty} t_n = 0. \tag{2.7}$$

We assert that $\{x_n\}$ is a Cauchy sequence. From **(G3)**, there is $k \in (0,1)$ such that

$$\lim_{n\to\infty} t_n^k G(t_n) = 0. \tag{2.8}$$

Let $M = \min \delta(t_i),\ 0 \le i \le n$. In view of (2.5), we have

$$\begin{aligned} t_{n+1}^k\Big(G(t_{n+1}) - G(t_0)\Big) &\le t_{n+1}^k\Big(\Big[G(t_0) - \sum_{i=0}^{n}\delta(t_i)\Big] - G(t_0)\Big) \\ &= -t_{n+1}^k \sum_{i=0}^{n}\delta(t_i) \\ &\le -nt_{n+1}^k M \\ &\le 0. \end{aligned}$$

Letting $n \to \infty$ (in view of (2.7) and (2.8)) gives rise

$$\lim_{n\to\infty} nt_n^k = 0.$$

Therefore, there exists $n \in \mathbb{N}$ such that $nt_n^k \le 1$ for all $n \ge N$ so that

$$t_n \le \frac{1}{n^{1/k}}, \quad \text{for all } n \ge N. \tag{2.9}$$

Hence, for $m, n \in \mathbb{N}$ with $m > n \ge N$, we have

$$d(x_m, x_n) \le \sum_{i=n}^{m} t_i < \sum_{i=n}^{\infty} t_i \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}} < \infty.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. In view of Remark 2.3 and the completeness of X, we have

$$x = \lim_{n\to\infty} x_{n+1} = f(\lim_{n\to\infty} x_n) = fx.$$

Now, Lemma 2.1 concludes the proof. □

Remark 2.4. f in Example 2.1 is a WF-contraction. As X is complete, f has a unique fixed point (namely $x = 2$).

3. Consequences

Corollary 3.1 (Banach Contraction Principle)**.** *Every self-mapping f on a complete metric space (X, d) has a unique fixed point if it satisfies the following:*

$$d(fx, fy) \le \beta d(x, y), \quad \textit{for all } x, y \in X, \ \textit{where } \beta \in (0,1). \tag{3.1}$$

Proof. The result is a direct consequence of Theorem 2.1 by taking $G(s) = s$ and $\delta(s) = \lambda s$ where $\lambda = 1 - \beta$. □

Corollary 3.2. *Every self-mapping f on a complete metric space (X, d) has a unique fixed point if it satisfies the following: for all $x, y \in X$ with $d(fx, fy) > 0$, we have*

$$d(fx, fy) \le e^{-\tau}[d(x, y) + 1] - 1, \quad \textit{where } \tau > 0. \tag{3.2}$$

Proof. Follows from Theorem 2.1 by taking $G(s) = \ln(s + 1)$ and $\delta(s) \equiv \tau$. □

One can list further consequences by varying the functions G and δ suitably such as in above two corollaries.

4. Application

Finally, we discuss the application of fixed point methods to the following two-point boundary value problem of second order differential equation:

$$\begin{cases} x''(t) = u(t, x(t)), & t \in J = [0,1], \\ x(0) = x(1) = 0, \end{cases} \tag{4.1}$$

where $u : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the Green function $G(t,s)$ associated to (4.1) is given by

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t < s \le 1, \\ s(1-t), & 0 \le s < t \le 1. \end{cases}$$

Let $\mathcal{C}(J)$ denotes the space of all continuous functions defined on J. We know that $(\mathcal{C}(J), d)$ is a complete metric space (see [5, 14]) where

$$d(x,y) = \|u - v\|_\infty = \max_{t \in J} \left\{ |x(t) - y(t)| e^{-\tau t} \right\}, \quad \tau > 0. \tag{4.2}$$

Now, we prove the following result on the existence and uniqueness solution of the problem described by (4.1).

Theorem 4.1. *Problem* (4.1) *has at least one solution $x^* \in \mathcal{C}^2$ provided the following condition hold:*

$$\Big| G(t,s)u(s,x(s)) - G(t,s)u(s,y(s)) \Big| \le \tau e^{-2\tau} |x(s) - y(s)| - 1,$$

for all $t, s \in J$ and $x, y \in \mathcal{C}(J)$ where τ is a given positive number.

Proof. Observe that $x \in \mathcal{C}^2$ is a solution of the problem described by (4.1) if and only if $x \in \mathcal{C}$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s)u(s,x(s))ds, \quad \text{for all } t \in J. \tag{4.3}$$

Define a function $f : \mathcal{C}(J) \to \mathcal{C}(J)$ by

$$fx(t) = \int_0^1 G(t,s)u(s,x(s))ds, \quad \text{for all } t \in J. \tag{4.4}$$

Clearly, if $x \in \mathcal{C}(J)$ is a fixed point of f, then $x \in \mathcal{C}(J)$ is a solution of (4.3) and hence of (4.1). Let $x, y \in \mathcal{C}(J)$ then, by the hypothesis, we have

$$\begin{aligned}
|fx(t) - fy(t)| =& \Big| \int_0^1 G(t,s)u(s,x(s))ds - \int_0^1 G(t,s)u(s,y(s))ds \Big| \\
\leq& \int_0^1 \Big| G(t,s)u(s,x(s)) - G(t,s)u(s,y(s)) \Big| ds \\
\leq& \int_0^1 \Big[\tau e^{-2\tau} |y(s) - x(s)| e^{-\tau s} e^{\tau s} - 1 \Big] ds \\
=& \int_0^1 \tau e^{-2\tau} e^{\tau s} |y(s) - x(s)| e^{-\tau s} ds - 1 \\
\leq& \tau e^{-2\tau} d(x,y) \int_0^1 e^{\tau s} ds - 1 \\
\leq& e^{-\tau} d(x,y) - 1 \\
\leq& e^{-\tau} d(x,y) + e^{-\tau} - 1,
\end{aligned}$$

so that

$$|fx(t) - fy(t)| e^{-\tau t} \leq e^{-\tau} d(x,y) + e^{-\tau} - 1.$$

Thus, $d(fx, fy) \leq e^{-\tau} d(x,y) + e^{-\tau} - 1$ so that condition (3.2) is satisfied. Now, Corollary 3.2 ensures the existence of a unique solution of 4.1. $\square$

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References

[1] Y. I. Alber and S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert spaces*, in: *New Results in Operator Theory and Its Applications*, Springer, Verlag, Basel, 1997, 7–22.

[2] J. Ali and M. Imdad, *An implicit function implies several contraction conditions*, Sarajevo J. Math. **4** (2008), 269–285.

[3] I. Altun and D. Turkoglu, *Some fixed point theorems for weakly compatible mappings satisfying an implicit relation*, Taiwanese J. Math. **13** (2009), 1291–1304.

[4] H. Argoubi, B. Samet and C. Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order*, J. Nonlinear Sci. Appl. **8** (2015), 1082–1094.

[5] A. Augustynowicz, *Existence and uniqueness of solutions for partial differential-functional equations of the first order with deviating argument of the derivative of unknown function*, Serdica Math. J. **23** (1997), 203–210.

[6] M. Cvetkovic, E. Karapinar and V. Rakocevic, *Fixed point results for admissible z-contractions*, Fixed Point Theory **19** (2018), 515–526.

[7] P. Dutta and B. S. Choudhury, *A generalisation of contraction principle in metric spaces*, Fixed Point Theory Appl. (2008), Article ID 406368, 8 pages.

[8] R. Gubran, W. M. Alfaqih and M. Imdad, *Common fixed point results for alpha-admissible mappings via simulation function*, J. Anal. **25** (2017), 281–290.

[9] M. Imdad and J. Ali, *A general fixed point theorem in fuzzy metric spaces via an implicit function*, J. Appl. Math. Inform. **26** (2008), 591–603.
[10] M. Imdad, R. Gubran and M. Ahmadullah, *Using an implicit function to prove common fixed point theorems*, J. Adv. Math. Stud. **11**(3) (2018), 481–495.
[11] E. Karapınar, *Fixed points results via simulation functions*, Filomat **30** (2016), 2343–2350.
[12] F. Khojasteh, S. Shukla and S. Radenović, *A new approach to the study of fixed point theory for simulation functions*, Filomat **29** (2015), 1189–1194.
[13] A. Kostić, V. Rakočević and S. Radenović, *Best proximity points involving simulation functions with w_0-distance*, Rev. R. Acad. Cienc. Exactas Fís. Nat. (2018), 1–13.
[14] D. O'Regan and A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. **341** (2008), 1241–1252.
[15] V. Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cerc. St. Ser. Mat. Univ. Bacau **7** (1997), 127–133.
[16] V. Popa, *A general fixed point theorem for weakly compatible mappings in compact metric spaces*, Turkish J. Math. **25** (2001), 465–474.
[17] B. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. **47** (2001), 2683–2693.
[18] A.-F. Roldán-López-de Hierro, E. Karapınar, C. Roldán-López-de Hierro and J. Martínez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math **275** (2015), 345–355.
[19] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. (2012), Article ID 94, 6 pages.

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